Extreme Values

Definition Let f be a function of two variables, R a set contained in the domain of f, and (x_0, y_0) a point in R. Then f has a maximum value (respectively, a minimum value) on R at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ (respectively, $f(x, y) \geq f(x_0, y_0)$) $\forall (x, y)$ in R. If R is the domain of f, we say that f has a maximum value (respectively, a minimum value) at (x_0, y_0).

Definition $f(x, y)$ has a relative maximum value (respectively, a relative minimum value) at (x_0, y_0) if there is an open disk D centered at (x_0, y_0) and contained in the domain of f such that $f(x_0, y_0)$ is the maximum value (respectively, the minimum value) on D.

Extremum $\equiv$ maximum or minimum.

Theorem If $f(x, y)$ has a relative extreme value at (x_0, y_0) and its first partials exist at (x_0, y_0), then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

or equivalently, $\vec{\nabla} f(x_0, y_0) = \vec{0}$.

<u>Definition</u> A point (x_0, y_0) in the interior of the domain of f is a <u>critical point</u> of f if <u>either</u>

(i) $f_x(x_0, y_0) = f_y(x_0, y_0) = 0,$ or

(ii) any of the first partial derivatives does not exist.

The above theorem can be restated as: f has relative extreme values only at critical points in its domain.

<u>Example</u> Let $f(x, y) = 3 - x^2 + 2x - y^2 - 4y$. Find all critical points of f. $(1, -2)$

<u>Example</u> Consider the critical points of $f(x, y) = |x| + y^2$. $y-axis$

<u>Example</u> Let $f(x, y) = y^2 - x^2$. Show that the origin is the only critical point but $f(0, 0)$ is not an extremum.

<u>Definition</u> If f is a function for which $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, we say that f has a <u>saddle point at (x_0, y_0)</u> if $\exists$ a disk centered at (x_0, y_0) such that f assumes its maximum value on one diameter of the disk *only at* (x_0, y_0) and assumes its minimum value on another diameter of the disk *only at* (x_0, y_0).

The Second Partials Test

Theorem Assume that f has a critical points at (x_0, y_0) and that f has continuous second partial derivatives in a disk centered at (x_0, y_0). Let

$$D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$$

a. If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ (or $f_{yy}(x_0, y_0) < 0$), then f has a relative maximum value at (x_0, y_0).

b. If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ (or $f_{yy}(x_0, y_0) > 0$), then f has a relative minimum value at (x_0, y_0).

c. If $D(x_0, y_0) < 0$, then f has a saddle point at (x_0, y_0).

d. If $D(x_0, y_0) = 0$, the nature of the critical point is unknown.

Taylor series expansion is only for a special case

$g(m)=f_{xx}+2f_{xy}m+f_{yy}m^2$ *can be a parabola or a straight line.*

Example Let $f(x, y) = x^2 - 2xy + \frac{1}{3}y^3 - 3y$. Determine locations of relative extrema and saddle points. $D=4(y-1)$

(3,3) *rel. min.*, $(-1,-1)$ *saddle pt.*

Example Consider $f(x, y) = x^3$ and $g(x, y) = x^4 + y^4$.

Recall the following theorem:

The Maximum-Minimum Theorem

Theorem Let R be a close, bounded set in the plane, and let f be continuous on R. Then f has both a maximum value and a minimum value on R.

Procedure to find the maximum & minimum:

1. Find the critical points of f in the interior of R, and compute the values of f at these points.

2. Find the extreme values of f on the boundary of R.

3. The maximum value of f on R will be the largest of the values computed in steps 1 and 2, and the minimum value of f on R will be the smallest of those values.

Example Find the maximum value of $f(x, y) = xy$ on the close triangle bounded by the lines: $x = 0$, $y = 0$, and $x + 2y = 2$.

$f(1,1/2)=1/2$

Optimization Subject to Constraints

Example (for motivation) Suppose heavy-duty tape is to be applied on the bottom and side edges of a rectangular carton. If 288 cm of tape are available, find the maximum volume of the carton. $48 \times 48 \times 24$

Lagrange Multipliers

Theorem Let f, g be differentiable at (x_0, y_0). Let C be a smooth curve defined by the constraint $g(x, y) = c$, and $\vec{\nabla} g \neq \vec{0}$ at any point on the curve. If f has an extreme value on C at (x_0, y_0), then there is a number λ such that

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0).$$

pf: Let r(t) be a smooth parameterization of g(x...)=c, then df(r(t0))/dt=0.

The three-variable case is similar. The constraint defines a surface. The equation containing gradients looks the same.

The method of determining extreme values by means of Lagrange multipliers proceeds as follows:

1. Assume that f has an extreme value on the curve $g(x, y) = c$.

2. Solve the equations

$$\begin{cases} \vec{\bigtriangledown} f(x,y) = \lambda \vec{\bigtriangledown} g(x,y) \\ g(x,y) = c \end{cases}$$

Note λ must be put on the side with $\vec{\nabla} g$, and $\lambda = 0$ is possible.

3. Calculate the value of f at each point (x, y) that arises in step 2. If f has a maximum value on the curve $g(x, y) = c$, it will be the largest of the values computed; Similarly, the minimum can be found.

4. To find the extrema on a closed and bound region, include values at interior critical points for comparison.

Example Let $f(x, y) = 3x^2 + 2y^2 - 4y + 1$. Find the extreme values of f on the close disk $x^2 + y^2 \leq 16$.

boundary: $f(0,\pm 4)=17,\ 49 \quad f(\pm\sqrt{12},-2)=53\ (max)$

interior: $f(0,1)=-1\ (min)$

— Problem Set 6 —