Double Integrals

*Motivation*

Double integral as volume: Consider a region $R$ in the $xy$-plane, and a function $f$ that is nonnegative and continuous on $R$. What is the volume of the solid region between the graph of $f$ and $R$? $_{sketch}$

There are two steps in the consideration:

1. Area of the base region

First, consider a *convex* region $R$ on the $xy$-plane (so that a straight line segment connecting any two points in $R$ lies inside $R$). How to find its area?

We use the standard *divide and add* strategy to estimate the area. Apply a rectangular mesh (over $[a, b] \times [c, d]$) composed of uniform sub-rectangles $R_k$ to cover the region $R$. Each sub-rectangle has an area

$$\Delta A_k = \Delta x \Delta y = \left(\frac{b-a}{M}\right)\left(\frac{d-c}{N}\right).$$

Count and add the areas of the sub-rectangles inside $R$:

$$\sum_{R_i \subset R} \triangle A_i.$$

This will give an approximate value, but the approximation will become better as the mesh is refined (i.e. $\Delta x,\ \Delta y \to 0$).

2. The solid region between the graph of $f$ and $R$

The volume underneath the graph of $f(x, y)$ over the region $R$ can be approximated by the sum

$$\sum_{R_i \subset R} f(\xi_i, \eta_i) \triangle A_i \qquad \text{(the \underline{Riemann sum} for } f \text{ on } R\text{).}$$

where $(\xi_i, \eta_i) \in R_i$ (note the non-uniqueness of choice). $f(\xi_i, \eta_i)$ represents an approximate height of the slender rectangular sub-solid over $R_i$.

Next, try to take the limit

$$\lim_{\Delta x,\ \Delta y \to 0} \sum_{R_i \subset R} f(\xi_i, \eta_i) \triangle A_i.$$

If this limit exists, it is the volume of the solid region.

The limit of the Riemann sum, denoted by

$$\iint_R f(x, y)\, dA,$$

is called the double integral of $f$ over $R$. This definition is used even when $f$ is negative somewhere and the 'volume' interpretation is not used.

Procedure to compute the volume/double integral

First, consider a rectangular region $R = [a, b] \times [c, d]$. Divide the region into $M \times N$ subrectangles ($M$, $N$ are integers) with sides $\Delta x = (b-a)/M$ and $\Delta y = (d-c)/N$. The interval $[a, b]$ is partitioned into M equal subintervals $[x_0, x_1]$, ..., $[x_{M-1}, x_M]$, and the interval $[c, d]$ is partitioned into $N$ equal subintervals $[y_0, y_1]$, ..., $[y_{N-1}, y_N]$.

The Riemann sum can now be rewritten as a double sum

$$\sum_{j=1}^{N} \sum_{i=1}^{M} f(\xi_i, \eta_j) \Delta x \Delta y.$$

where $\xi_i$ is a point in the subinterval $[x_{i-1}, x_i]$ and $\eta_j$ is a point in $[y_{j-1}, y_j]$. For example, if one picks the point to be the upper-right corner of the subrectangle, $(\xi_i, \eta_j) = (x_i, y_j)$. Now try to take the limit of this sum for $\Delta x \to 0, \Delta y \to 0$.

Theorem (Fubini's Theorem) If $f$ is continuous, this limit always exists. The result is

$$\int_c^d \left( \int_a^b f(x, y) dx \right) dy.$$

Next, consider slightly more complex regions.

Vertically and Horizontally Simple Regions

Definition 1. A plane region $R$ is vertically simple if there are two continuous functions $g_1$ and $g_2$ on an interval $[a, b]$ such that $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$ and such that $R$ is the region between the graphs of $g_1$ and $g_2$ on $[a, b]$.

2. A plane region $R$ is horizontally simple if there are two continuous functions $h_1$ and $h_2$ on an interval $[c, d]$ such that $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$ and such that $R$ is the region between the graphs of $h_1$ and $h_2$ on $[c, d]$.

3. A plane region $R$ is simple if it is both vertically simple and horizontally simple.

All these regions can be approximated as composed of rectangular slices.

Cover the region with a larger rectangle. The double sum for case 1 can be written as

$$\sum_{i=1}^{M} \sum_{j=N_1(i)}^{N_2(i)} f(x_i, y_j)\Delta y \Delta x \qquad \text{where } N_1,\ N_2 \text{ depend on } i.$$

## Evaluation of Double Integrals

Theorem Let $f$ be continuous on a region $R$ in the $xy$ plane.

1. If $R$ is the vertically simple region between the graphs of $g_1$ and $g_2$ on $[a, b]$, then $f$ is integrable on $R$, and

$$\iint\limits_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

(called iterated integral)

2. If $R$ is the horizontally simple region between the graphs of $h_1$ and $h_2$ on $[c, d]$, then

$$\iint\limits_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

3. If $R$ is simple, then

$$\iint\limits_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f dxdy.$$

<u>Example</u> Let $f(x, y) = 1, \quad R = [a, b] \times [c, d]$. Evaluate

$$\iint\limits_R f dA$$

<u>Note</u> The area of a plane region is given by

$$\iint\limits_R 1 dA$$

<u>Example</u> Let $R = [0, 1] \times [2, 3]$. Evaluate

$$\iint\limits_R x^2 y dA \qquad 5/6$$

<u>Example</u> Let $f(x, y) = 1 - 2y$ and $R$ be the triangular region between the graph of $y = 1 - x$ and the $x$ axis on $[-1, 1]$. Find $\iint\limits_R f dA$. $\quad -2/3$

<u>Example</u> Evaluate $\iint\limits_R f dA$ of the above example by reversing the order of integration. Then compare results.

<u>Example</u> Evaluate $\displaystyle\int_0^1 \int_0^y x\sqrt{y^2 - x^2} dx dy$ . $\quad 1/12$

What about reversing the order of integration?

$$\int \sqrt{u^2 - a^2}\,du = (u/2)\sqrt{u^2 - a^2} - (u^2/2)\ln|u + \sqrt{u^2 - a^2}| + C$$

—— Problem Set 7 ——