Double Integrals in Polar Coordinates

Simple Regions

Suppose that $h_1(\theta)$ and $h_2(\theta)$ are continuous on an interval $[\alpha, \beta]$, and that

$$0 \leq h_1(\theta) \leq h_2(\theta) \quad \forall \theta \in [\alpha, \beta]$$

Let R be the closed region in the (r, θ) plane bounded by the lines $\theta = \alpha$ and $\theta = \beta$ and by the polar graphs of $r = h_1(\theta)$ and $r = h_2(\theta)$. We say that R is the (simple) region between the polar graphs of h_1 and h_2 on $[\alpha, \beta]$.

Theorem Let R be the region between the graphs of continuous functions h_1 and h_2 on $[\alpha, \beta]$. If f is continuous on R, then

$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta$$

Consider integrating rf over a rectangular region of (r, theta)

<u>Example</u> Suppose R is the region bounded by the circles $r = 1$ and $r = 2$ and the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$ (R is a quarter-ring). Express

$$\iint\limits_R (3x + 8y^2)dA$$

as an integral in polar coordinates and evaluate the integral. $\sin^2\theta = (1 - \cos 2\theta)/2$; $7 + 15\pi/2$

<u>Example</u> Let D be the solid region bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by the xy plane. Find the volume of D.

8π

Surface Area of a Graph

First, consider the relationship between the area of a tilted rectangle ($S = a\,b$) and the area of its projection ($A = a\,b\cos\theta$) on a plane.

$$S = a\,b = \frac{A}{\cos\theta}$$

Let $\hat{n}$ be the upward pointing (z component positive) unit normal vector to the rectangle. If the plane is the xy-plane, then

$$\cos\theta = \hat{n}\cdot\vec{k} \quad \text{and} \quad S = \frac{A}{\hat{n}\cdot\vec{k}} = \frac{A}{|\hat{n}\cdot\vec{k}|}.$$

Let Σ be the graph of a two-variable function f on a region R, then the surface area of Σ is

$$\iint_{\Sigma} dS = \iint_{R} \sqrt{f_x^2 + f_y^2 + 1}\;\, dA.$$

Pf: A normal to the graph at $(x, y, f(x, y))$ is $f_x\vec{i} + f_y\vec{j} - \vec{k}$. The upward pointing unit normal to $\sum$ is

$$\hat{n} = \frac{-f_x\vec{i} - f_y\vec{j} + \vec{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$$

Therefore $\hat{n}\cdot\vec{k} = \dfrac{1}{\sqrt{f_x^2 + f_y^2 + 1}}$, and

$$\sum \triangle S = \sum \frac{\triangle A}{\hat{n}\cdot\vec{k}} = \sum \sqrt{f_x^2 + f_y^2 + 1}\,\triangle A.$$

<u>Note</u> The following expression works for $\hat{l} = \vec{i}$, $\vec{j}$, or $\vec{k}$ (different projections), and for either direction of $\hat{n}$:

$$dA = |\hat{n}\cdot\hat{l}|\,dS.$$

The formula

$$\iint\limits_{\Sigma} dS = \iint\limits_{R} \frac{dA}{|\hat{n}\cdot\hat{l}|}$$

can be used for other projections.

<u>Example</u> Let R be the rectangular region $[0,3]\times[0,2]$ and $f(x,y) = \frac{2}{3}x^{3/2}$. Find the surface area of the graph of f over R. $28/3$

<u>Example</u> Find the surface area of the portion of the plane $x + 2y + 3z = 6$ inside the cylinder $x^2 - 4x + y^2 = 0$. $4\sqrt{14}\pi/3$

Surface Integrals

$$\sum_{k=1}^{n} g(x_k, y_k, z_k) \triangle s_k \to \iint_{\Sigma} g dS$$

Let Σ be the graph of a function f having continuous partial derivatives and defined on a region R in the xy plane that is composed of a finite number of vertically or horizontally simple regions. Let g be continuous on Σ. The surface integral of g over Σ is

$$\iint_{\Sigma} g(x, y, z) dS$$

$$= \iint_{R} g(x, y, f(x, y)) \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Example Evaluate $\iint_{\Sigma} z^2 dS$ where Σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ for which $1 \leq x^2 + y^2 \leq 4$. $15\pi/\sqrt{2}$

—— Problem Set 8 ——