Triple Integrals

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\triangle V_k \to \iiint\limits_D f(x,y,z)dV$$

Theorem Let D be the solid (3-dimensional) region between the graphs of two continuous functions F_1 and F_2 on a vertically or horizontally simple region R in the xy plane, and let f be continuous on D. Then

$$\iiint\limits_D f(x,y,z)dV = \iint\limits_R \left(\int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)dz \right) dA.$$

Example Let D be the solid rectangular region $[2, \frac{5}{2}] \times [0,\pi] \times [0,2]$ and $f(x,y,z) = zx\sin xy$. Evaluate

$$\iiint\limits_D f\ dV$$

$1 - \frac{2}{\pi}$

Triple Integrals in Cylindrical Coordinates

Cylindrical Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

$$dV = dzrdrd\theta$$

This coordinate system is best for problems with axial symmetry (e.g. the region is bounded by a cylinder, and the integrand can be converted to depend on r and z only).

Example Evaluate the mass of a cylindrical rod with radius a, length l, and density $f(x, y, z) = 1 + z$ where z is the distance from one end of the rod. $\pi a^2 l(1+l/2)$

Example Let D be the solid region bounded above by the plane $y + z = 4$, below by the xy plane, and on the sides by the cylinder $x^2 + y^2 = 16$. Evaluate

$$\iiint_D \sqrt{x^2 + y^2}\, dV$$

$4^4 2\pi/3$

Triple Integrals in Spherical Coordinates

Spherical Coordinates

$$\begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

This coordinate system is suitable for problems in which the solid region D can be described in terms of the coordinates in a simple way (especially if the problem has spherical symmetry).

Note that in some texts, the symbols ϕ and θ are switched. dV can be written as $\rho^2 d\rho d\Omega$ where $d\Omega = \sin\phi d\phi d\theta$ is the differential *solid angle.*

Example Evaluate the mass of a solid ball with radius a and density $1 + k\rho^2$. $4\pi/3a^3(1+3ka^2/5)$

Example Let D be the solid region between the spheres $\rho = 1$ and $\rho = 2$, and inside the cone $\phi = \pi/4$. Evaluate

$$\iiint_D z^2 dV$$

$\pi(31/15)(2-1/\sqrt{2})$

Change of Variables in Multiple Integrals

Single integral

A substitution of x by $g(u)$ converts an integral as following

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)du.$$

Double integral

Consider a *transformation* T from the uv-plane to the xy-plane with the parametric equations

$$x = x(u, v) \quad \& \quad y = y(u, v).$$

With $v = v_0$ held fixed, the curve $(x(u, v_0), y(u, v_0))$ describes a "coordinate line" of constant v on the xy-plane.

Example Consider polar coordinates in 2D.

A short *secant vector* of this line with initial point at (u_0, v_0) is
$\vec{a} = (x(u_0+\Delta u, v_0)-x(u_0, v_0), y(u_0+\Delta u, v_0)-y(u_0, v_0))$
$\approx (\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u)$.
Similarly, a short secant vector of the v coordinate line can be approximated by $\vec{b} \approx (\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v)$.

The area of the parallelogram bounded between the two vectors can be obtained as the magnitude of their cross product, which gives

$$\Delta A \approx |J(u,v)|\Delta u \Delta v$$

where

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is called the <u>Jacobian</u> of the transformation T. $|J(u,v)|$ is the absolute value of $J(u,v)$.

Therefore, if the transformation $x = x(u,v),\ y = y(u,v)$ maps the region S in the uv-plane onto the region R in the xy-plane, and if the Jacobian $\partial(x,y)/\partial(u,v)$ is nonzero and does not change sign on S, then

$$\iint_R f(x,y)dA_{xy} = \iint_S f(x(u,v),y(u,v))|J(u,v)|dA_{uv}$$

in which the subscripts are attached to identify the associated variables and

$$S = T^{-1}(R) = \{(u,v)|(x,y) = T(u,v),\ (x,y) \in R\}.$$

Example Evaluate

$$\iint_R e^{xy} dA$$

where R is the region in the first quadrant enclosed by the lines $y = \frac{1}{2}x$ and $y = x$ and the hyperbolas $y = 1/x$ and $y = 2/x$.

$\frac{1}{2}(e^2 - e) \ln 2$

Triple integral

If T is the transformation from uvw-space to xyz-space defined by the equations

$$x = x(u, v, w), \ y = y(u, v, w), \ z = z(u, v, w),$$

then the volume of the parallelepiped bounded by short secant vectors $\vec{a}, \ \vec{b}, \ \vec{c}$ of the coordinate lines can be obtained by the magnitude of the triple product $|(\vec{a} \times \vec{b}) \cdot \vec{c}|$, thus $\Delta V \approx |J(u, v, w)| \Delta u \Delta v \Delta w$.

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

is the Jacobian of T.

An integral is transformed as

$$\iiint_D f(x, y, z) dV_{xyz} =$$

$$\iiint_{T^{-1}(D)} f(x(u, v, w), y(u, v, w), z(u, v, w))|J(u, v, w)|dV_{uvw}$$

Example Show that the Jacobian of the transformation from cylindrical coordinates to rectangular coordinates is r.

—— Problem Set 9 ——