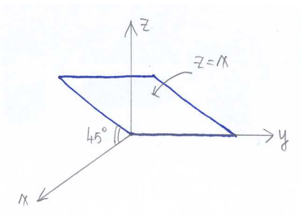
Exercise 10.1

Qu. 6 Let $A = (1, 2, 3)$, $B = (4, 0, 5)$ and $C = (3, 6, 4)$, then

$$\|\mathbf{AB}\| = \sqrt{3^2 + (-2)^2 + 2^2} = \sqrt{17}$$
$$\|\mathbf{AC}\| = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{21}$$
$$\|\mathbf{BC}\| = \sqrt{(-1)^2 + 6^2 + (-1)^2} = \sqrt{38}.$$

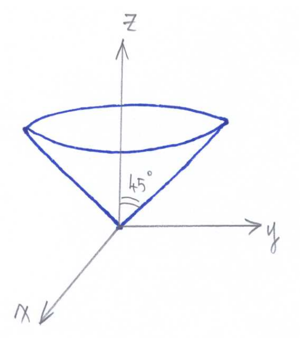
Since $\|\mathbf{AB}\|^2 + \|\mathbf{AC}\|^2 = 17 + 21 = 38 = \|\mathbf{BC}\|^2$, the triangle ABC has a right angle at A.

Qu. 14 $z = x$ is a plane containing the y-axis and making 45^o angles with the positive directions of the x- and z-axes.



Qu. 22 $z \geqslant \sqrt{x^2 + y^2}$ represents every point whose distance above the xy-plane is not less than its horizontal distance from the z-axis. It therefore consists of all points inside and on a circular cone with axis along the positive z-axis, vertex at the origin, and semi-vertical angle 45^o.

Alternatively, the question will be much easier if we change the equation $z \geqslant \sqrt{x^2 + y^2}$ in terms of cylindrical coord. Why!! (see §10.6)



Exercise 10.2

Qu. 2 If $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 2\,\mathbf{k}$, then

(a) $\mathbf{u} + \mathbf{v} = \mathbf{i} + 2\,\mathbf{k}$

$\mathbf{u} - \mathbf{v} = \mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k}$

$2\mathbf{u} - 3\mathbf{v} = 2\,\mathbf{i} - 5\,\mathbf{j} - 6\,\mathbf{k}$.

(b) $\|\mathbf{u}\| = \sqrt{2}$

$\|\mathbf{v}\| = \sqrt{5}$.

(c) $\widehat{\mathbf{u}} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$

$\widehat{\mathbf{v}} = (\mathbf{j} + 2\,\mathbf{k})/\sqrt{5}$.

(d) $\mathbf{u} \cdot \mathbf{v} = 0 - 1 + 0 = -1$.

(e) $\theta = \cos^{-1}(\mathbf{u} \cdot \mathbf{v}/\|\mathbf{u}\|\|\mathbf{v}\|) = \cos^{-1}(1/\sqrt{10}) \simeq 108.4^o$.

(f) The scalar projection of $\mathbf{u}$ on $\mathbf{v} = \mathbf{u} \cdot \widehat{\mathbf{v}} = -1/\sqrt{5}$.

(g) The vector projection of $\mathbf{v}$ along $\mathbf{u} = (\mathbf{v} \cdot \widehat{\mathbf{u}}) \cdot \widehat{\mathbf{u}} = -(\mathbf{i} - \mathbf{j})/2$.

Qu. 10 $\mathbf{v}_{\text{water}} = 3\,\mathbf{i}$, i.e., the water flow from west to east.

If you row through the water with speed 5 in the direction making angle θ west of north, then your velocity relative to the water will be

$$\mathbf{u} = -5\sin\theta\,\mathbf{i} + 5\cos\theta\,\mathbf{j}.$$

Therefore, your velocity relative to the land will be

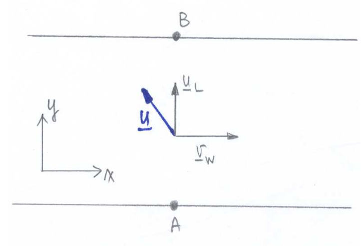
$$\begin{aligned}\mathbf{u}_L &= \mathbf{u} + \mathbf{v}_{\text{water}} \\ &= (3 - 5\sin\theta)\,\mathbf{i} + 5\cos\theta\,\mathbf{j}.\end{aligned}$$

To row directly from A to B ($\mathbf{j}$ direction only) choose θ so that

$$3 - 5\sin\theta = 0 \qquad \Rightarrow \qquad \theta = 36.87^o,$$

then $\mathbf{u}_L = 4\mathbf{j}$.

To row from A to B, head in the direction 36.87^o west of north. The 0.5km crossing will be $0.5/4 = 0.125$ of an hour $= 7.5$ minutes.



Qu. 16 If $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}$, then $\cos\alpha = \dfrac{\mathbf{u}\bullet\mathbf{i}}{\|\mathbf{u}\|} = \dfrac{u_1}{\|\mathbf{u}\|}$.

Similarly, $\cos\beta = \dfrac{u_2}{\|\mathbf{u}\|}$ and $\cos\gamma = \dfrac{u_3}{\|\mathbf{u}\|}$.

Thus, the unit vector in the direction of $\mathbf{u}$ is

$$\widehat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k},$$

and so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \|\widehat{\mathbf{u}}\|^2 = 1$.

Qu. 19 Since $\mathbf{r} - \mathbf{r}_1 = \lambda\mathbf{r}_1 + (1-\lambda)\mathbf{r}_2 - \mathbf{r}_1 = (1-\lambda)(\mathbf{r}_1 - \mathbf{r}_2)$, therefore $\mathbf{r} - \mathbf{r}_1$ is parallel to $\mathbf{r}_1 - \mathbf{r}_2$, that is, parallel to the line P_1P_2. Since P_1 is on that line, so must P be on it. If $\lambda = \dfrac{1}{2}$, then $\mathbf{r} = \dfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$, so P is midway between P_1 and P_2.

If $\lambda = \dfrac{2}{3}$, then $\mathbf{r} = \dfrac{2}{3}\mathbf{r}_1 + \dfrac{1}{3}\mathbf{r}_2$, so P is two-thirds of the way from P_2 towards P_1 along the line.

If $\lambda = -1$, then $\mathbf{r} = -\mathbf{r}_1 + 2\mathbf{r}_2 = \mathbf{r}_2 + (\mathbf{r}_2 - \mathbf{r}_1)$, so P is such that P_2 bisects the segment P_1P.

If $\lambda = 2$, then $\mathbf{r} = 2\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_1 + (\mathbf{r}_1 - \mathbf{r}_2)$, so P is such that P_1 bisects the segment P_2P.

Qu. 20 If $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{a}\cdot\mathbf{r} = 0$ implies that the position vector $\mathbf{r}$ is perpendicular to $\mathbf{a}$, i.e.

$$\begin{aligned}(a_1, a_2, a_3)\cdot(x, y, z) &= 0\\ a_1x + a_2y + a_3z &= 0.\end{aligned}$$

Thus the equation is satisfied by all points on the plane through the origin that is normal to $\mathbf{a}$.

Qu. 24 Note that $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 3$, a vector $\mathbf{r} = (x, y, z)$ will make equal angle with all three if it has equal dot products with all three, that is, if

$$\begin{cases}\mathbf{u}\cdot\mathbf{r} = \mathbf{v}\cdot\mathbf{r}\\ \mathbf{u}\cdot\mathbf{r} = \mathbf{w}\cdot\mathbf{r}\end{cases}$$

$$\begin{cases}2x + y - 2z = x + 2y - 2z\\ 2x + y - 2z = 2x - 2y + z\end{cases}$$

$$\begin{cases}x = y\\ y = z\end{cases}$$

i.e. $x = y = z$. Two unit vectors satisfying this condition are

$$\mathbf{r} = \pm\frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Qu. 27 (a)

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2\end{aligned}$$

(b) If θ is angle between $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta \leqslant 1$, so

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \leqslant \|\mathbf{u}\|\|\mathbf{v}\|.$$

(c)

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}\\ &\leqslant \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\|\\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

Thus $\|\mathbf{u}+\mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$ (take +ve root only, why!)

Qu. 29 $\mathbf{u} = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$, $v = \dfrac{4}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{j}$, $\mathbf{w} = \mathbf{k}$.

(a) $\|\mathbf{u}\| = \sqrt{\dfrac{9}{25} + \dfrac{16}{25}} = 1$, $\|\mathbf{v}\| = \sqrt{\dfrac{16}{25} + \dfrac{9}{25}} = 1$, $\|w\| = 1$, $\mathbf{u}\bullet\mathbf{v} = \dfrac{12}{25} - \dfrac{12}{25} = 0$, $\mathbf{u}\bullet\mathbf{w} = 0$, $\mathbf{v}\bullet\mathbf{w} = 0$.

(b) If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then

$$\begin{aligned}&(\mathbf{r}\bullet\mathbf{u})\mathbf{u} + (\mathbf{r}\bullet\mathbf{v})\mathbf{v} + (\mathbf{r}\bullet\mathbf{w})\mathbf{w}\\ &= \left(\frac{3}{5}x + \frac{4}{5}y\right)\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) + \left(\frac{4}{5}x - \frac{3}{5}y\right)\left(\frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}\right) + z\,\mathbf{k}\\ &= \frac{9x + 16x}{25}\mathbf{i} + \frac{16y + 9y}{25}\mathbf{j} + z\,\mathbf{k}\\ &= x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = \mathbf{r}.\end{aligned}$$

Qu. 33 Let $\|\mathbf{a}\|^2 - 4rst = K^2$, where $K > 0$. Now

$$\begin{aligned}\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a} &= (r\,\mathbf{x} + s\,\mathbf{y})\cdot(r\,\mathbf{x} + s\,\mathbf{y})\\ &= (r^2\|\mathbf{x}\|^2 + s^2\|\mathbf{y}\|^2 + 2rs\,\mathbf{x}\cdot\mathbf{y}\end{aligned}$$

$$\begin{aligned}K^2 &= \|\mathbf{a}\|^2 - 4rs\,\mathbf{x}\cdot\mathbf{y}\\ &= r^2\|\mathbf{x}\|^2 + s^2\|\mathbf{y}\|^2 - 2rs\,\mathbf{y}\cdot\mathbf{y}\\ &= \|r\,\mathbf{x} - s\,\mathbf{y}\|^2.\end{aligned}$$

Therefore $r\mathbf{x} - s\mathbf{y} = K\widehat{\mathbf{u}}$ for some unit vector $\widehat{\mathbf{u}}$. (1)

Since $r\,\mathbf{x} + s\,\mathbf{y} = \mathbf{a}$, we have (2)

$(1) + (2)$ $\quad 2r\,\mathbf{x} = \mathbf{a} + K\widehat{\mathbf{u}}$

$(2) - (1)$ $\quad 2s\,\mathbf{y} = \mathbf{a} - K\widehat{\mathbf{u}}$.

Thus

$$\mathbf{x} = \frac{\mathbf{a} + K\widehat{\mathbf{u}}}{2r}$$
$$\mathbf{y} = \frac{a - K\widehat{\mathbf{u}}}{2s},$$

where $K = \sqrt{\|\mathbf{a}\|^2 - 4rst}$ and $\widehat{\mathbf{u}}$ is any unit vector.

Note that the solution is *not* unique.

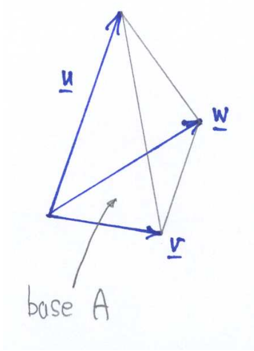
Exercise 10.3

Qu. 14 Base area A$=\frac{1}{2}\|\mathbf{v} \times \mathbf{w}\|$

The altitude h of the tetrahedron is

$$\begin{aligned} h &= |\mathbf{u} \cdot \widehat{\mathbf{v} \times \mathbf{w}}| \\ &= \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|} \end{aligned}$$

$$\begin{aligned} \therefore \quad V &= \frac{1}{3} Ah \\ &= \frac{1}{3}\frac{1}{2}\|\mathbf{v} \times \mathbf{w}\| \cdot \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\|\mathbf{v} \times \mathbf{w}\|} \\ &= \frac{1}{6}|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \\ &= \frac{1}{6}\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \end{aligned}$$



Qu. 20 If $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ but $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$, i.e. $\mathbf{v}$ is not parallel with $\mathbf{w}$. Therefore, $\mathbf{v}$ and $\mathbf{w}$ form the base vectors in the vw-plane, i.e. any vector in the vw-plane can be represented as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

Moreover, since $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0 \quad \Rightarrow \quad \mathbf{u} \perp \mathbf{v} \times \mathbf{w}$,

i.e. $\mathbf{u}$ must be on the vw-plane. Therefore

$$\mathbf{u} = \lambda \mathbf{v} + \mu \mathbf{w}.$$

Qu. 26 Let $\mathbf{x} = (x, y, z)$, then

$$\begin{aligned} (-\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) \times \mathbf{x} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ x & y & z \end{vmatrix} \\ &= (2z - 3y)\,\mathbf{i} + (3x + z)\,\mathbf{j} - (y + 2x)\,\mathbf{k} \\ &= \mathbf{i} + 5\,\mathbf{j} - 3\,\mathbf{k} \end{aligned}$$

$$\therefore \quad \begin{cases} 2z - 3y &= 1 & (1) \\ 3x + z &= 5 & (2) \\ y + 2x &= 3 & (3) \end{cases}$$

From (1) and (2), we have $y + 2x = 3$, which is the same as (3), so the system is under-determined.

Let $x = t$, $y = 3 - 2t$, $z = 5 - 3t$

$$\therefore \quad \mathbf{x} = t\,\mathbf{i} + (3 - 2t)\,\mathbf{j} + (5 - 3t)\,\mathbf{k}$$

for any real number t.

Qu. 28 The equation $\mathbf{a}\times\mathbf{x}=\mathbf{b}$ can be solved for $\mathbf{x}$ if and only if $\mathbf{a}\bullet\mathbf{b}=0$. (The "only if", why!!). For the "if" part, observe that if $\mathbf{a}\bullet\mathbf{b}=0$ and $\mathbf{x}_0=(\mathbf{b}\times\mathbf{a})/|\mathbf{a}|^2$, then,

$$\mathbf{a}\times\mathbf{x}_0=\frac{1}{|a|^2}\mathbf{a}\times(\mathbf{b}\times\mathbf{a})=\frac{(\mathbf{a}\bullet\mathbf{a})\mathbf{b}-(\mathbf{a}\bullet\mathbf{b})\mathbf{a}}{\|a\|^2}=\mathbf{b}.$$

The solution $\mathbf{x}_0$ is not unique because any multiple of $\mathbf{a}$ can be added to it and the result iwll still be a solution. If $\mathbf{x}=\mathbf{x}_0+t\mathbf{a}$, then

$$\mathbf{a}\times\mathbf{x}=\mathbf{a}\times\mathbf{x}_0+t\,\mathbf{a}\times\mathbf{a}=\mathbf{b}+\mathbf{0}=\mathbf{b}.$$

Exercise 10.4

Qu. 8 (i) Find the pencil of planes: Since $\mathbf{r}_0=(-2,0,-1)$ does not lie on $x-4y+2z=-5$, the required plane will have an equation of the form

$$2x+3y-z+\lambda(x-4y+2z+5)=0$$

for some λ. This plane passes through the point $(-2,0,-1)$ if

$$-4+1+\lambda(y-z-3)=0 \qquad \Rightarrow \qquad \lambda=3.$$

$\therefore$ The required plane is $5x-9y+5z=-15$.

(ii) Find three points on the required plane

$$2x+3y-z=0 \qquad (1)$$
$$x-4y+2z=-5 \qquad (2)$$

$2(1)+(2) \qquad \Rightarrow \qquad 5x+2y=-5.$

$\therefore$ Let $x=1$, then $y=-5$ and $z=-13$ (point P_1).

Also let $x=-1$, then $y=0$ and $z=-2$ (point P_2),

together with the given point $P_3=(-2,0,-1)$, we have three points, therefore the required plane is uniquely determined.

The normal vector of the plane

$$\mathbf{n}=\mathbf{P}_1\mathbf{P}_2\times\mathbf{P}_1\mathbf{P}_3=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2&5&11\\-3&5&12\end{vmatrix}=(5,-9,5).$$

$\therefore$ The required plane is

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0$$
$$(5,-9,5)\cdot(x+2,y,z+1)=0$$
$$\therefore\quad 5x-9y+5z=-15.$$

Qu. 18 A line parallel to $x+y=0$ and to $x-y+2z=0$ is parallel to the cross product of the normal vectors to these two planes, that is, to the vector

$$\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\1&-1&2\end{vmatrix}$$
$$=2(\mathbf{i}-\mathbf{j}-\mathbf{k}).$$

$\therefore$ The required equation is (in vector form)

$$\mathbf{r}=\mathbf{r}_0+\mathbf{v}t=(2+t)\,\mathbf{i}-(1+t)\,\mathbf{j}-(1+t)\,\mathbf{k}.$$

In scalar parametric form

$$x=2+t, \qquad y=-(1+t), \qquad z=-(1+t).$$

or in standard form

$$x-2=-(y+1)=-(z+1).$$

Qu. 28 First, we find the equation of the line as in Qu. 18

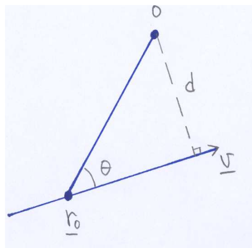
$$\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\2&-1&-5\end{vmatrix}=(-4,7,-3).$$

We need a point on this line: set $z=0$, then we have

$$\begin{cases}x+y=0\\2x-y=1\end{cases} \qquad \Rightarrow \qquad \begin{cases}x=\frac{1}{3}\\y=-\frac{1}{3}\end{cases}$$

$$\therefore\quad \mathbf{r}_0=\left(\frac{1}{3},-\frac{1}{3},0\right).$$

$\therefore$ The required distance is



$$\begin{aligned}d&=\|\mathbf{r}_0\|\sin\theta\\&=\|\mathbf{r}_0\|\,\|\hat{\mathbf{v}}\|\sin\theta\\&=\|\mathbf{r}_0\times\hat{\mathbf{v}}\|\\&=\frac{1}{\sqrt{74}}\left\|\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\\frac{1}{3}&-\frac{1}{3}&0\\-4&7&-3\end{vmatrix}\right\|\\&=\frac{1}{\sqrt{74}}\|\mathbf{i}+\mathbf{j}+\mathbf{k}\|\\&=\sqrt{\frac{3}{74}}.\end{aligned}$$