Exercise 11.1

Qu. 12 Position: $\mathbf{r}(t) = at\cos\omega t\,\mathbf{i} + at\sin\omega t\,\mathbf{j} + b\ln t\,\mathbf{k}$

Velocity: $\mathbf{v} = \mathbf{r}'(t) = a(\cos\omega t - \omega t\sin\omega t)\,\mathbf{i} + a(\sin\omega t + \omega t\cos\omega t)\,\mathbf{j} + (b/t)\,\mathbf{k}$

Acceleration: $\mathbf{a} = \mathbf{r}''(t) = -a\omega(2\sin\omega t + \omega t\cos\omega t)\,\mathbf{i} + a\omega(2\cos\omega t - \omega t\sin\omega t)\,\mathbf{j} - (b/t^2)\,\mathbf{k}$

Speed: $\|\mathbf{v}\| = \left[a^2(1+\omega^2t^2) + b^2/t^2\right]^{1/2}$

Let

$$x = at\cos\omega t \qquad (1)$$

$$y = at\sin\omega t \qquad (2)$$

$$z = b\ln t \quad \Rightarrow \quad t = e^{z/b} \qquad (3)$$

From (1) and (2)

$$\frac{x^2}{a^2t^2} + \frac{y^2}{a^2t^2} = 1$$

$$x^2 + y^2 = a^2t^2 = a^2e^{2z/b}$$

$\therefore$ Path: a spiral on the surface (a"cone" - see page 6)

$$x^2 + y^2 = a^2e^{2z/b}.$$

Qu. 24 Note that if

$$\begin{aligned} \mathbf{r}\cdot\mathbf{r}' &= 0 \\ \frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) &= 0 \\ \|\mathbf{r}\|^2 &= \text{const.} \\ \Rightarrow \qquad \|\mathbf{r}\| &= \text{const.} = a \end{aligned}$$

i.e. $\mathbf{r}(t)$ lies on a sphere centred at the origin with radius a.

Qu. 29

$$\frac{d}{dt}[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})] = \mathbf{u}'\cdot(\mathbf{v}\times\mathbf{w}) + \mathbf{u}\cdot(\mathbf{v}'\times\mathbf{w}) + \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}').$$

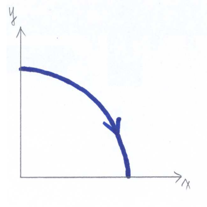
Qu. 32

$$\begin{aligned} \frac{d}{dt}[(\mathbf{u}\times\mathbf{u}')\cdot(\mathbf{u}'\times\mathbf{u}'')] &= (\mathbf{u}\times\mathbf{u}')\cdot\frac{d}{dt}(\mathbf{u}'\times\mathbf{u}'') + [\frac{d}{dt}(\mathbf{u}\times\mathbf{u}')]\cdot(\mathbf{u}'\times\mathbf{u}'') \\ &= (\mathbf{u}\times\mathbf{u}')\cdot[\mathbf{u}''\times\mathbf{u}'' + \mathbf{u}'\times\mathbf{u}'''] + [\mathbf{u}'\times\mathbf{u}' + \mathbf{u}\times\mathbf{u}'']\cdot(\mathbf{u}'\times\mathbf{u}'') \\ &= (\mathbf{u}\times\mathbf{u}'')\cdot(\mathbf{u}'\times\mathbf{u}'') + (\mathbf{u}\times\mathbf{u}')\cdot(\mathbf{u}'\times\mathbf{u}'''). \end{aligned}$$

Exercise 11.3

Qu. 2 On the first quadrant part of the circle $x^2+y^2=a^2$, we have $y=\sqrt{a^2-x^2}$, $\quad 0\leqslant x\leqslant a$. The required parametrization is

$$\mathbf{r} = \mathbf{r}(x) = x\,\mathbf{i} + \sqrt{a^2-x^2}\,\mathbf{j}, \qquad (0\leqslant x\leqslant a)$$



Qu. 4 In terms of the parameter t,

$$\begin{cases} x = a\sin t \\ y = a\cos t \end{cases} \qquad 0\leqslant t\leqslant\frac{\pi}{2}$$

$$s = \int_0^t a\,d\tau = at \quad \Rightarrow \quad t = \frac{s}{a}$$

$$\therefore \quad \mathbf{r}(s) = a\sin\frac{s}{a}\,\mathbf{i} + a\cos\frac{s}{a}\,\mathbf{j}, \quad \text{where} \quad 0\leqslant\frac{s}{a}\leqslant\frac{\pi}{2}.$$

Qu. 16

$$\begin{aligned} x &= a\cos t\sin t = \frac{a}{2}\sin 2t \\ y &= a\sin^2 t = \frac{a}{2}(1-\cos 2t) \\ z &= bt \end{aligned}$$

$\therefore \quad x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}$, note also that as t increases, z increases too.

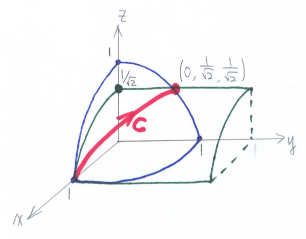
$\therefore \quad$ The curve is a circular helix lying on the cylinder

$$x^2 + \left(y-\frac{a}{2}\right)^2 = \frac{a^2}{4} \qquad \text{(see page 6)}$$

It's length, from $t=0$ to $t=T$, is

$$\begin{aligned} L &= \int_0^T \sqrt{a^2\cos^2 2t + a^2\sin^2 2t + b^2}\,dt \\ &= T\sqrt{a^2+b^2}\ \text{(units)} \end{aligned}$$

Qu. 18 (see page 6 also)



One-eighth of the curve C lies in the first octant.

$$x^2 + y^2 + z^2 = 1 \quad \text{(sphere)} \tag{1}$$

$$x^2 + 2z^2 = 1 \quad \text{(elliptic cylinder)} \tag{2}$$

From (2), Let

$$\begin{cases} x = \cos t \\ z = \dfrac{1}{\sqrt{2}} \sin t \end{cases} \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

From (1),

$$\begin{aligned} y &= \sqrt{1 - \cos^2 t - \frac{1}{2}\sin^2 t} \\ &= \frac{1}{\sqrt{2}} \sin t \qquad \text{(take +ve root only).} \end{aligned}$$

Note that the first octant part of C lies in the plane $y = z$, it must be a quarter of a circle of radius 1 (why!). Thus the length of all of C is

$$8 \times \frac{\pi}{2} = 4\pi \text{ (units).}$$

If you wish to use an integral, the length is

$$\begin{aligned} L &= 8 \int_0^{\frac{\pi}{2}} \sqrt{\sin^2 t + \frac{1}{2}\cos^2 t + \frac{1}{2}\cos^2 t}\, dt \\ &= 8 \int_0^{\frac{\pi}{2}} dt \\ &= 4\pi \text{ (units).} \end{aligned}$$

Qu. 27 Since $\mathbf{r}_1(t) = \mathbf{r}_2(u(t))$, where u is a function from $[a, b]$ to $[c, d]$, having $u(a) = c$ and $u(b) = d$. We assume u is differentiable. Since u is one-to-one and orientation-preserving.

$$\frac{du}{dt} \geqslant 0 \quad \text{on} \quad [a, b].$$

By the Chain rule:

$$\frac{d}{dt}\mathbf{r}_1(t) = \frac{d}{du}\mathbf{r}_2(u)\frac{du}{dt}$$

and so

$$\begin{aligned} \int_a^b \left\| \frac{d}{dt}\mathbf{r}_1(t) \right\| dt &= \int_a^b \left\| \frac{d}{du}\mathbf{r}_2(u) \right\| \frac{du}{dt}\, dt \\ &= \int_c^d \left\| \frac{d}{du}\mathbf{r}_2(u) \right\| du. \end{aligned}$$

Extra questions

Qu. 1 In cylindrical coord.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{dx}{dt} = \cos\theta \dfrac{dr}{dt} - r\sin\theta \dfrac{d\theta}{dt} \\ \dfrac{dy}{dt} = \sin\theta \dfrac{dr}{dt} + r\cos\theta \dfrac{d\theta}{dt} \\ \dfrac{dz}{dt} = \dfrac{dz}{dt} \end{cases}$$

$$\begin{aligned} \|\mathbf{r}'(t)\|^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \\ &= \cos^2\theta \left(\frac{dr}{dt}\right)^2 - 2r\cos\theta\sin\theta \frac{dr}{dt}\frac{d\theta}{dt} + r^2\sin^2\theta \left(\frac{d\theta}{dt}\right)^2 \\ &\quad + \sin^2\theta \left(\frac{dr}{dt}\right)^2 + 2r\cos\theta\sin\theta \frac{dr}{dt}\frac{d\theta}{dt} + r^2\cos^2\theta \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \\ &= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \end{aligned}$$

Hence the answer.

Qu. 2 In spherical coord.

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \qquad z = \rho\cos\phi$$

therefore,

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d\rho}{dt}\sin\phi\cos\theta + \rho\cos\phi\cos\theta\frac{d\phi}{dt} - \rho\sin\phi\sin\theta\frac{d\theta}{dt} \\
\frac{dy}{dt} &= \frac{d\rho}{dt}\sin\phi\sin\theta + \rho\cos\phi\sin\theta\frac{d\phi}{dt} + \rho\sin\phi\cos\theta\frac{d\theta}{dt} \\
\frac{dz}{dt} &= \frac{d\rho}{dt}\cos\phi - \rho\sin\phi\frac{d\phi}{dt} \\
\|\mathbf{r}'(t)\|^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \\
&= \left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2
\end{aligned}$$

Hence the answer.

Qu. 3

$$\begin{aligned}
\mathbf{r} &= a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j} + b\cos 2t\,\mathbf{k}, \quad 0 \leqslant t \leqslant \frac{\pi}{2} \\
\mathbf{v} &= -3a\cos^2 t\,\sin t\,\mathbf{i} + 3a\sin^2 t\,\cos t\,\mathbf{j} - 4b\sin t\,\cos t\,\mathbf{k} \\
v &= \|\mathbf{v}\| = \sqrt{9a^2+16b^2}\sin t\,\cos t \\
s &= \int_0^t \sqrt{9a^2+16b^2}\sin u\,\cos u\,du \\
&= \frac{1}{2}\sqrt{9a^2+16b^2}\,\sin^2 t = K\sin^2 t \\
&\qquad \text{where} \quad K = \frac{1}{2}\sqrt{9a^2+16b^2}
\end{aligned}$$

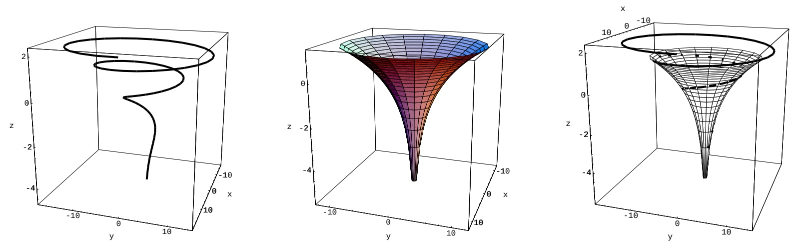
Therefore $\sin t = \sqrt{\dfrac{s}{K}}$, $\cos t = \sqrt{1 - \dfrac{s}{K}}$, $\cos 2t = 1 - 2\sin^2 t = 1 - \dfrac{2s}{K}$.

The required parameterization is

$$\mathbf{r} = a\left(1 - \frac{s}{K}\right)^{3/2}\mathbf{j} + a\left(\frac{s}{K}\right)^{3/2}\mathbf{j} + b\left(1 - \frac{2s}{K}\right)\mathbf{k}$$

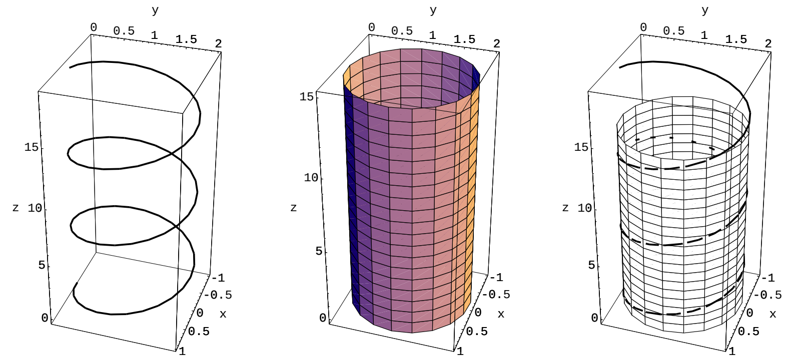
for $0 \leqslant s \leqslant K$, where $K = \dfrac{1}{2}\sqrt{9a^2+16b^2}$.

Ex 11.1 Qu. 12



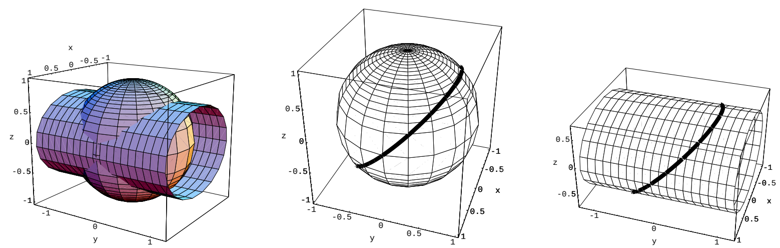
Taking $a = 2\pi$ and $b = 2$ — The surface $x^2 + y^2 = a^2 e^{2z/b}$ — The surface and the curve

Ex 11.3 Qu. 16



Taking $a = 2$ and $b = 2$ — The surface $x^2 + (y - a/2)^2 = a^2/4$

Ex 11.3 Qu. 18



The surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + 2z^2 = 1$