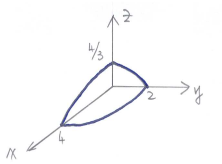
Exercise 10.5

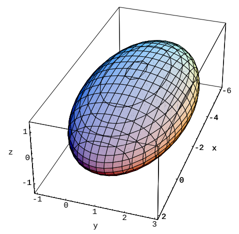
Qu. 4

$$x^2 + 4y^2 + 9z^2 + 4x - 8y = 8$$
$$(x+2)^2 + 4(y-1)^2 + 9z^2 = 16$$
$$\frac{(x+2)^2}{4^2} + \frac{(y-1)^2}{2^2} + \frac{z^2}{(4/3)^2} = 1$$

This is an ellipsoid with centre $(-2, 1, 0)$ and semi-axis 4, 2 and 4/3.

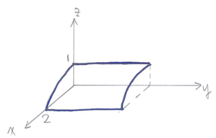
(see also page 8)

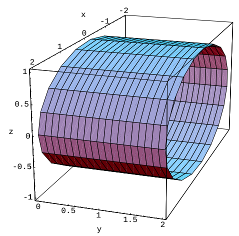




This surface is symmetric about the xy-plane, xz-plane and yz-plane

Qu. 10 $x^2 + 4z^2 = 4$. This equation is independent of y. Therefore $\frac{x^2}{2^2} + z^2 = 1$ represents an elliptic cylinder with axis along the y-axis.





This surface is symmetric about the xy-plane and yz-plane
(see also page 8)

Exercise 12.1

Qu. 4

$$f(x,y) = \frac{xy}{x^2 - y^2}$$

This function is defined except when $x^2 - y^2 = 0$, i.e. the domain consists of all points not on the lines $x = \pm y$.

i.e.

$$D = \{(x,y) \,\big|\, x \in \mathbb{R},\ y \in \mathbb{R}, \ \ \text{except} \ \ x = \pm y\}$$
$$\text{range} = \{(-\infty, \infty)\}.$$

(see also page 8)

Qu. 10 $f(x,y,z) = \frac{\exp xyz}{\sqrt{xyz}}$. This is a surface in 4D.

This function is defined as long as $xyz > 0$, that is all points in the four octants.

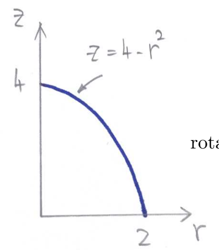
(a) $x > 0,\ y > 0,\ z > 0$,
(b) $x > 0,\ y < 0,\ z < 0$,
(c) $x < 0,\ y < 0,\ z > 0$,
(d) $x < 0,\ y > 0,\ z < 0$.

range $= \{(\sqrt{2e}, \infty)\}$.

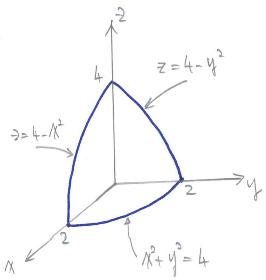
Qu. 14 $z = f(x,y) = 4 - x^2 - y^2$

In cylindrical coord. $f(r,\theta) = 4 - r^2$, independent of θ.

(see also page 8)



rotate the curve by 90^o
$\Rightarrow$



Qu. 24 $f(x,y) = \dfrac{y}{x^2+y^2} = c$

This is the family $x^2 + \left(y - \dfrac{1}{2c}\right)^2 = \dfrac{1}{4c^2}$ of circles passing through the origin and having centres on the y-axis. The origin itself is, however not on any of the level curves!! (see also page 8)

Qu. 36 $f(x,y,z) = c = px + qy + rz$ where p, q, r are constants since the level surface of $f(x,y,z)$ is a plane. Then when

$$y = 0, \quad z = 0 \quad \Rightarrow \quad px = c$$
$$x = \frac{c}{p} = c^3 \quad \Rightarrow \quad p = \frac{1}{c^2}$$

Similarly,

$$q = \frac{1}{2c^2}, \qquad r = \frac{1}{3c^2}$$
$$\therefore \frac{1}{c^2}x + \frac{1}{2c^2}y + \frac{1}{3c^2}z = c \quad \Rightarrow \quad c = \left(x + \frac{y}{2} + \frac{z}{3}\right)^{\frac{1}{3}}$$
$$\therefore \quad f(x,y,z) = \left(x + \frac{y}{2} + \frac{z}{3}\right)^{\frac{1}{3}} \quad \text{(4D surface)}.$$

Qu. 42 The "level hyper-surface" $f(x,y,z,t) = c > 0$ is the "4-sphere" of radius $\sqrt{c}$ centred at the origin in $\mathbb{R}^4$.

Exercise 12.2

Qu. 6 Let $x = r\cos\theta(r)$, $y = 1 + r\sin\theta(r)$, then

$$\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2} = \lim_{r\to 0} \frac{r^4\cos^2\theta(r)\sin^2\theta(r)}{r^2}$$
$$= \lim_{r\to 0} r^2\cos^2\theta(r)\sin^2\theta(r)$$
$$\leqslant \lim_{r\to 0} r^2 \to 0^+.$$

Also note that $r^2\cos^2\theta(r)\sin^2\theta(r) > 0^+$

$$\therefore \lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2} = 0.$$

Alternatively, note also that

$$0 \leqslant \left|\frac{x^2(y-1)^2}{x^2+(y-1)^2}\right| \leqslant x^2$$

and $x^2 \to 0$ as $(x,y)\to(0,1)$. Hence the result!

(see also page 8)

Qu. 12 Along the y-axis, $x = 0$, $y \neq 0$, then

$$\frac{x^2y^2}{2x^4+y^4} = 0.$$

Along the line $x = y \neq 0$, then

$$\frac{x^2y^2}{2x^4+y^4} = \frac{x^4}{2x^4+x^4} = \frac{1}{3}.$$

$\therefore$ Two different limits along two different paths, therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4} \qquad \text{does not exist.}$$

(see also page 8)

Qu. 13 Domain of $f(x,y)$ is $\{(x,y) \mid x\in\mathbb{R},\, y\in\mathbb{R} \text{ except } x = y = 0\}$

i.e. $f(x,y)$ is a continuous function on the whole xy-plane except at (0,0). Also note that

$$f(x,y) = \frac{x^2+y^2-x^3y^3}{x^2+y^2} = 1 - \frac{x^3y^3}{x^2+y^2}$$
$$\left|\frac{x^3y^3}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2}\right| |xy^3| \leqslant |xy^3| \to 0^+$$

as $(x, y) \to (0, 0)$. Thus

$$\lim_{(x,y)\to(0,0)} f(x, y) = 1 - 0 = 1$$

Therefore if we define $f(0, 0) = 1$, then

$$\lim_{(x,y)\to(0,0)} f(x, y) = 1 = f(0, 0)$$

$\therefore$ $f(x, y)$ is a continuous function everywhere (see also page 9).

Qu 17 $f_{\mathbf{u}}(t) = f(a + tu, b + tv)$, where $\mathbf{u} = u\,\mathbf{i} + v\,\mathbf{j}$ is a unit vector.

$f(x, y)$ may not be continuous at (a, b) even if $f_{\mathbf{u}}(t)$ is continuous at $t = 0$ for every unit vector $\mathbf{u}$. A counter-example is the function f of Ex. 2.1 of the lecture notes.

Here $a = b = 0$. The condition that each $f_{\mathbf{u}}$ should be continuous is the condition that f should be continuous on each straight line through $(0, 0)$, which it is if we extend the domain of f to include $(0, 0)$ by defining $f(0, 0) = 0$.

We showed that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along every straight line. However, we also showed that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

On the other hand, if $f(x, y)$ is continuous at (a, b), then $f(x, y) \to f(a, b)$ if (x, y) approaches (a, b) in any way, in particular, along the line through (a, b) parallel to $\mathbf{u}$. Thus all such function $f_{\mathbf{u}}(t)$ must be continuous at $t = 0$.

Qu. 18

$$\begin{aligned} \left|\frac{x^m y^n}{(x^2 + y^2)^p}\right| &= \left|\frac{r^m \cos^m\theta \cdot r^n \sin^m\theta}{r^{2p}}\right| \\ &= \left|r^{m+n-2p} \cos^m\theta \sin^n\theta\right| \\ &\leqslant \left|r^{m+n-2p}\right| \to 0^+ \end{aligned}$$

as $r \to 0^+$, if $m + n - 2p > 0$.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^m y^n}{(x^2 + y^2)^{2p}} = 0 \qquad \text{provided} \qquad m + n > 2p.$$

Alternatively, since $|x| \leqslant \sqrt{x^2 + y^2}$, $y \leqslant \sqrt{x^2 + y^2}$, we have

$$\left|\frac{x^m y^n}{(x^2 + y^2)^p}\right| \leqslant \frac{(x^2 + y^2)^{(m+n)/2}}{(x^2 + y^2)^p} = (x^2 + y^2)^{-p+(m+n)/2}.$$

Same conclusion.