Exercise 12.4

Qu. 4

$$z = (3x^2 + y^2)^{1/2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(3x^2+y^2)^{-\frac{1}{2}}(6x) = \frac{3x}{\sqrt{3x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}(3x^2+y^2)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\sqrt{3x^2+y^2}(3) - 3x\cdot\frac{3x}{\sqrt{3x^2+y^2}}}{3x^2+y^2} = \frac{3y^2}{(3y^2+y^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\sqrt{x^2+y^2}(1) - y\cdot\frac{y}{\sqrt{x^2+y^2}}}{3x^2+y^2} = \frac{3x^2}{(3y^2+y^2)^{\frac{3}{2}}}$$

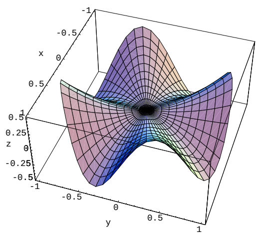
$$\therefore\ \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x} = -\frac{3xy}{(3y^2+y^2)^{\frac{3}{2}}}.$$

Qu. 16 Let $f(x,y) = (x^2-y^2)\frac{2xy}{x^2+y^2}$, then

$$f_x = \frac{4x^2y}{x^2+y^2} - \frac{2y(y^2-x^2)^2}{(x^2+y^2)^2}$$

$$f_y = -\frac{4xy^2}{x^2+y^2} + \frac{2x(x^2-y^2)^2}{(x^2+y^2)^2}$$

$$f_{xy} = \frac{2(x^6+9x^4y^2-9x^2y^4-y^6)}{(x^2+y^2)^3} = f_{yx}.$$



Surface of $f(x,y)$

For the value at $(0,0)$, we use

$$f_x(0,0) = \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0 = f_y(0,0)$$

$$f_{xy}(0,0) = \lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{-2\Delta y(\Delta y)^4}{\Delta y(\Delta y)^4} = -2$$

$$f_{yx}(0,0) = \lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\Delta x(\Delta x)^4}{\Delta x(\Delta x)^4} = 2.$$

This is not contradict Theorem 1 since the partials f_{xy} and f_{yx} are not continuous at $(0,0)$. (For instance, $f_{xy}(x,x) = 0$ while $f_{xy}(x,0) = 2$ for $x \neq 0$).

Qu. 18

$$u(x,y,t) = t^{-1}e^{-(x^2+y^2)/4t}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{t^2}e^{-(x^2+y^2)/4t} + \frac{x^2+y^2}{4t^3}e^{-(x^2+y^2)/4t}$$

$$\frac{\partial u}{\partial x} = -\frac{x}{2t^2}e^{-(x^2+y^2)/4t}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{2t^2}e^{-(x^2+y^2)/4t} + \frac{x^2}{4t^3}e^{-(x^2+y^2)/4t}$$

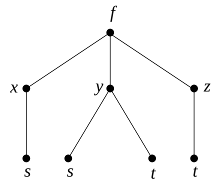
$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{2t^2}e^{-(x^2+y^2)/4t} + \frac{y^2}{4t^3}e^{-(x^2+y^2)/4t}$$

$$\therefore\ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Exercise 12.5

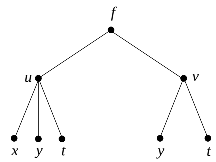
Qu. 2

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}.$$

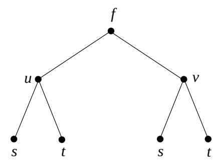


Qu. 12 Let $u = yf(x,t)$, $v = f(y,t)$, then

$$\begin{aligned}\frac{\partial}{\partial y} f(u,v) &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\ &= \frac{\partial f}{\partial u} f(x,t) + \frac{\partial f}{\partial v}\frac{\partial f}{\partial y}.\end{aligned}$$

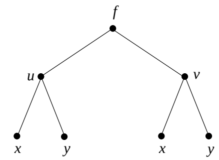


Qu. 20 Let $f = f(u,v)$, where $u(s,t) = s^2 - t$, $v(s,t) = s + t^2$



$$\begin{aligned}f_s &= f_u(2s) + f_v(1) = 2sf_u + f_v \\ \frac{\partial f_s}{\partial t} &= 2s\frac{\partial}{\partial t} f_u + \frac{\partial}{\partial t} f_v \\ &= 2s[\frac{\partial}{\partial u} f_u \cdot \frac{\partial u}{\partial t} + \frac{\partial}{\partial v} f_u \cdot \frac{\partial v}{\partial t}] + [\frac{\partial}{\partial u} f_v \cdot \frac{\partial u}{\partial t} + \frac{\partial}{\partial v} f_v \cdot \frac{\partial v}{\partial t}] \\ &= 2s[f_{uu}(-1) + f_{uv}(2t)] + f_{vu}(-1) + f_{vv}(2t) \\ &= -2sf_{uu} + (4st-1)f_{uv} + 2tf_{vv} \\ \frac{\partial}{\partial t} f_{st} &= -2s\frac{\partial}{\partial t} f_{uu} + 4sf_{uv} + (4st-1)\frac{\partial}{\partial t} f_{uv} + 2f_{vv} + 2t\frac{\partial}{\partial t} f_{vu} \\ &= -2s[\frac{\partial}{\partial u} f_{uu} \cdot \frac{\partial u}{\partial t} + \frac{\partial}{\partial v} f_{uu} \cdot \frac{\partial v}{\partial t}] + 4sf_{uv} + (4st-1)[\frac{\partial}{\partial u} f_{uv} \cdot \frac{\partial u}{\partial t} + \frac{\partial}{\partial v} f_{uv} \cdot \frac{\partial v}{\partial t}] \\ &\quad + 2f_{vv} + 2t[\frac{\partial}{\partial u} f_{vv} \cdot \frac{\partial u}{\partial t} + \frac{\partial}{\partial v} f_{vv} \cdot \frac{\partial v}{\partial t}] \\ &= -2s[f_{uuu}(-1) + f_{uuv}(2t)] + 4sf_{uv} + (4st-1)[f_{uvu}(-1) + f_{uvv}(-1)] \\ &\quad + 2f_{vv} + 2t[f_{vvu}(-1) + f_{vvv}(2t)] \\ &= 2sf_{uuu} + (1-8st)f_{uuv} + 4t(2st-1)f_{uvv} + 4t^2 f_{vvv} + 4sf_{uv} + 2f_{vv}.\end{aligned}$$

Qu. 21 Let $g(x,y) = f(u,v)$, where $u = u(x,y)$, $v = v(x,y)$. Then



$$\begin{aligned}g_x &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = f_u \cdot u_x + f_v \cdot v_x \\ g_y &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = f_u \cdot u_y + f_v \cdot v_y \\ g_{xx} &= \frac{\partial}{\partial x}(g_x) = \frac{\partial}{\partial x}[f_u \cdot u_x + f_v \cdot v_x] \\ &= \frac{\partial}{\partial x}(f_u)u_x + f_u u_{xx} + \frac{\partial}{\partial x}(f_v) \cdot v_x + f_v v_{xx} \\ &= \left[\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial x}\right] u_x + f_u u_{xx} + \left[\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial x}\right] v_x + f_v \cdot v_{xx} \\ &= (f_{uu}u_x + f_{uv}v_x)u_x + u_{xx}f_u + (f_{vu}u_x + f_{vv}v_x)v_x + f_v \cdot v_{xx} \\ &= u_{xx}f_u + v_{xx}f_v + (u_x)^2 f_{uu} + (v_x)^2 f_{vv} + u_x v_x f_{uv} + u_x v_x f_{vu} \\ g_{yy} &= \frac{\partial}{\partial y}(g_y) = \frac{\partial}{\partial y}[f_u u_y + f_v v_y] \\ &= u_{yy}f_u + v_{yy}f_v + (u_y)^2 f_{uu} + (v_y)^2 f_{vv} + u_y v_y f_{uv} + u_y v_y f_{vu}.\end{aligned}$$

Therefore

$$\begin{aligned}g_{xx} + g_{yy} &= (u_{xx} + u_{yy})f_u + (v_{xx} + v_{yy})f_v + [(u_x)^2 + (u_y)^2]f_{uu} \\ &\quad + [(v_x)^2 + (v_y)^2]f_{vv} + (u_x v_x + u_y v_y)f_{uv} + (u_x v_x + u_y v_y)f_{vu} \\ &= 0 \cdot f_u + 0 \cdot f_v + [(u_x)^2 + (v_x)^2]f_{uu} + [(v_x)^2 + (u_x)^2]f_{vv} \\ &\quad + [u_x(-u_y) + u_y(u_x)]f_{uv} + [u_x(-u_y) + u_y(u_x)]f_{vu} \\ &= [(u_x)^2 + (v_x)^2](f_{uu} + f_{vv}) + 0 \cdot f_{uv} + 0 \cdot f_{vu} \\ &= 0.\end{aligned}$$

$\therefore g$ is harmonic.

Note that u and v are harmonic.

Exercise 12.6

Qu. 6

$$f(x,y) = xe^{y+x^2} \qquad f(2,-4) = 2$$
$$f_x = e^{y+x^2}(1+2x^2) \qquad f_x(2,-4) = 9$$
$$f_y = xe^{y+x^2} \qquad f_y(2,-4) = 2$$

$$\begin{aligned} f(x+\triangle x, y+\triangle y) &= f(x,y) + \triangle x f_x(x,y) + \triangle y f_y(x,y) \\ \therefore\ f(2.05,-3.92) &= f(2,-4) + 0.05 f_x(2,-4) + 0.08 f_y(2,-4) \\ &= 2 + 0.45 + 0.16 \\ &= 2.61. \end{aligned}$$

Qu. 12

$$\begin{aligned} w &= x^2y^3/z^4 \\ w_x &= 2xy^3/z^4 = 2w/x \\ w_y &= x^2(3y^2)/z^4 = 3w/y \\ w_z &= -4x^2y^3/z^5 = -4w/z \\ dw &= w_x dx + w_y dy + w_z dz \\ \frac{dw}{w} &= 2\frac{dx}{x} + 3\frac{dy}{y} - 4\frac{dz}{z}. \end{aligned}$$

Since x increases by 1% then $\frac{dx}{x} = \frac{1}{100}$, similarly, $\frac{dy}{y} = \frac{2}{100}$ and $\frac{dz}{z} = \frac{3}{100}$. Therefore

$$\frac{\triangle w}{w} \simeq \frac{dw}{w} = \frac{2+6-12}{100} = -\frac{4}{100}$$

$\therefore w$ decreases by about 4%.

Qu. 17 If f is differentiable at (a,b), then

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{h^2+k^2}} = 0$$

Since the denominator of this fraction approach 0, the numerator must also approach 0 (faster!!) or the fraction would not have a limit. Since the terms $hf_x(a,b)$ and $kf_y(a,b)$ both approach 0, we must have

$$\lim_{(h,k)\to(0,0)} [f(a+h,b+k) - f(a,b)] = 0$$
$$\text{i.e.} \quad \lim_{(h,k)\to(0,0)} f(a+h,b+k) = f(a,b)$$

Thus f is continuous at (a,b).

Qu. 18 Let $g(t) = f(a+tb, b+tk) = f(x(t), y(t))$, where $x(t) = a+th$, $y(t) = b+tk$, then

$$\begin{aligned} \frac{dg}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= h\, f_x + k\, f_y. \end{aligned}$$

If h and k are small enough that $(a+h, b+k)$ belongs to the disk referred to in the statement of the problem then we can apply the (one-variable) mean-value Theorem to $g(t)$ on $[0,1]$ and obtain

$$g(1) = g(0) + g'(\theta),$$

$\because\ g'(\theta) = \frac{g(1)-g(0)}{1-0}$ for some θ satisfying $0 < \theta < 1$, i.e.

$$f(a+h,b+k) = f(a,b) + h\, f_x(a+\theta h, b+\theta k) + k\, f_y(a+\theta h, b+\theta k).$$

Exercise 12.7

Qu. 6

$$f(x,y) = 2xy \big/ (x^2+y^2)$$

$$f_x(x,y) = [(x^2+y^2)(2y) - 2xy(2x)] \big/ (x^2+y^2)^2 = 2y(y^2-x^2)\big/(x^2+y^2)^2$$

$$f_y(x,y) = 2x(y^2-x^2)\big/(x^2+y^2)^2$$

$$\nabla f(x,y) = \frac{2(y^2-x^2)}{(x^2+y^2)^2}\,(y\,\mathbf{i} - x\,\mathbf{j})$$

(a) $\nabla f(0,2) = \mathbf{i}$.

(b) Let $F(x,y,z) = f(x,y) - z = 0$. This is a level surface in 3D, hence

$$\nabla F = (f_x, f_y, -1).$$

At (0,2,0), $\nabla F(0,2,0) = (1,0,-1) = \mathbf{n}$

$\therefore$ The tangent plane is

$$(\mathbf{r} - \mathbf{r}_0)\cdot \mathbf{n} = 0$$

$$(x-0, y-2, z-0)\cdot(1,0,-1) = 0$$

$$x - z = 0.$$

(c) Let $f(x,y) = c$. This is a level curve in 2D, hence

$$\nabla f = (f_x, f_y)$$

At (0,2), $\nabla f(0,2) = \mathbf{i} = \mathbf{n}$, $\therefore$ parallel vector $\mathbf{v}$ to the required line is $\mathbf{j}$

$\therefore$ The required line:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$= (0,2) + t(0,1) = (0, 2+t)$$

i.e. $x = 0$.

Qu. 10

$$f(x,y) = 3x - 4y$$

$$\nabla f = (f_x, f_y) = (3,-4), \qquad \nabla f(0,2) = (3,-4)$$

$$\therefore \quad D_{-\mathbf{i}}\, f(0,2) = -\mathbf{i}\cdot(3\,\mathbf{i} - 4\,\mathbf{j}) = -3.$$

Qu. 14 First, we try to find $\nabla\,\|\mathbf{r}\|$

$$[\nabla\,\|\mathbf{r}\|]_i = \partial_i (r_j r_j)^{\frac{1}{2}} = \frac{1}{2}(r_j r_j)^{-\frac{1}{2}}\partial_i(r_j r_j)$$

$$= \frac{1}{2}\frac{1}{\|\mathbf{r}\|}(r_j\partial_i r_j + r_j \partial_i r_j) = \frac{1}{2}\frac{1}{\|\mathbf{r}\|}2(r_j\delta_{ij}) = \frac{1}{\|\mathbf{r}\|}r_i.$$

$$\therefore \quad \nabla\,\|\mathbf{r}\| = \frac{\mathbf{r}}{\|\mathbf{r}\|}$$

$$\therefore \quad \nabla \ln\|\mathbf{r}\| = \frac{1}{\|\mathbf{r}\|}\nabla\,\|\mathbf{r}\| = \frac{\mathbf{r}}{\|\mathbf{r}\|^2}.$$

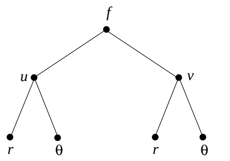
Qu. 16 $x = r\cos\theta$, $y = r\sin\theta$ and $f = f(x,y) = f(r,\theta)$

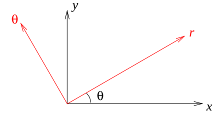
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$= \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}.$$





Also note that

$$\widehat{\mathbf{r}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{r} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$\widehat{\boldsymbol{\theta}} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{r} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

Note that $\widehat{\mathbf{r}} \perp \widehat{\boldsymbol{\theta}}$, therefore

$$\frac{\partial f}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\widehat{\boldsymbol{\theta}}$$

$$= \left(\cos^2\theta\frac{\partial f}{\partial x} + \sin\theta\cos\theta\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(\cos\theta\sin\theta\frac{\partial f}{\partial x} + \sin^2\theta\frac{\partial f}{\partial y}\right)\mathbf{j}$$

$$+ \left(\sin^2\theta\frac{\partial f}{\partial x} - \sin\theta\cos\theta\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(-\cos\theta\sin\theta\frac{\partial f}{\partial x} + \cos^2\theta\frac{\partial f}{\partial y}\right)\mathbf{j}$$

$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$= \nabla f.$$

Qu. 18 $f(x,y,z) = x^2 + y^2 - z^2$, $\nabla f = (2x, 2y, -2z)$ $\therefore \nabla f(a,b,c) = (2a, 2b, -2c)$.

The maximum rate of change of f at (a,b,c) is in the direction of $\nabla f(a,b,c)$ and is equal to $\|\nabla f(a,b,c)\|$.

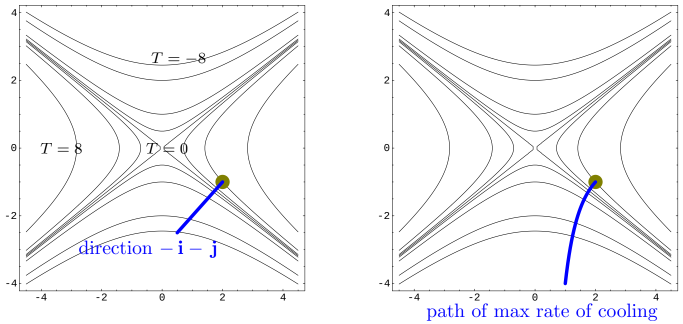
Let $\widehat{\mathbf{u}}$ be a unit vector making an angle θ with $\nabla f(a,b,c)$, then

$$\frac{1}{2}\,\|\nabla f(a,b,c)\| = \widehat{\mathbf{u}}\cdot\nabla f(a,b,c) = \|\nabla f(a,b,c)\|\cos\theta$$

$$\text{i.e.}\quad \cos\theta = \frac{1}{2} \quad\Rightarrow\quad \theta = 60^0.$$

$\therefore$ At (a,b,c), f increases at half its maximal rate in all directions making 60^0 angle with the direction $(a,b,-c)$.

Qu. 21 (a)



(b) $\nabla T = 2x\,\mathbf{i} - 4y\,\mathbf{j}$, $\quad \nabla T(2,-1) = 4\,\mathbf{i} + 4\,\mathbf{j} = 4(\mathbf{i}+\mathbf{j})$.

An ant at $(2,-1)$ should move in the direction of $-\nabla T(2,-1)$, that is, in the direction $-\mathbf{i}-\mathbf{j}$, in order to cool off as rapidly as possible.

(c) Since $\dfrac{dT}{ds} = D_{\mathbf{u}}T = \nabla T \cdot \widehat{\mathbf{u}}$, required rate $= |\nabla T(2,-1)|\,k = 4\sqrt{2}k$ degree/unit time.

(d) temp. change at rate

$$\frac{-\,\mathbf{i}-2\,\mathbf{j}}{\sqrt{5}} \cdot (4\,\mathbf{i} + 4\,\mathbf{j})k = -\frac{12}{\sqrt{5}}k$$

decreasing at rate $\dfrac{12}{\sqrt{5}}k$ degree/unit time.

(e) Let the required curve be $(x(t), y(t))$, this curve is everywhere tangent to $\nabla T(x,y)$. Thus

$$\frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} = \lambda(2x\,\mathbf{i} - 4y\,\mathbf{j}), \quad \text{i.e.} \quad \frac{dy}{dt} = \lambda 2x, \quad \frac{dy}{dt} = -\lambda 4y$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2y}{x} \quad \Rightarrow \quad yx^2 = c.$$

This curve passes through $(2,-1)$, we have $yx^2 = -4$, i.e. $y = -\dfrac{4}{x^2}$.

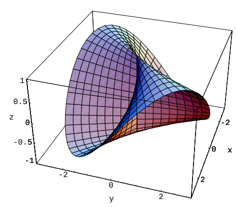
Qu. 26 Let $F_1(x,y,z) = x^2 + y^2 - 2 = 0$ and $F_2(x,y,z) = y^2 + z^2 - 2 = 0$, both of them are level surfaces.

$\mathbf{n}_1 = \nabla F_1 = (2x, 2y, 0)$ and $\mathbf{n}_2 = \nabla F_2 = (0, 2y, 2z)$. At $(1,-1,1)$,

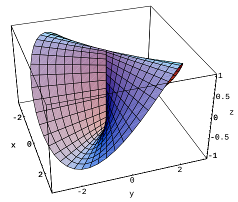
$$\mathbf{n}_1 = (2,-2,0) \quad \text{and} \quad \mathbf{n}_2 = (0,-2,2).$$

A vector tangent to the curve of intersection of the two surfaces at $(1,-1,1)$ must be perpendicular to both these normals, $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = (-1,-1,-1)$.

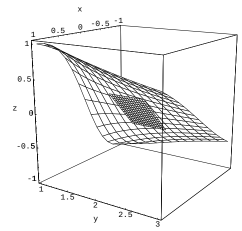
$\therefore$ The vector $k(1,1,1)$, where k is any scalar, is tangent to the curve at the point $(1,-1,1)$.



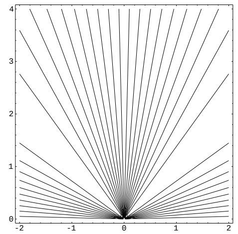
Ex. 12.7, Qu. 6a
$f(x,y) = 2xy/(x^2+y^2)$



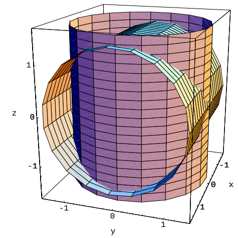
Ex. 12.7, Qu. 6a
Different angle of $f(x,y)$



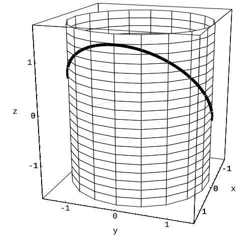
Ex. 12.7, Qu. 6b
Tangent plane at $(0,2,0)$



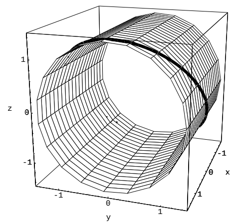
Ex. 12.7, Qu. 6c
Contour plot of $f(x,y)$



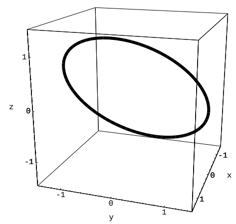
Ex. 12.7, Qu. 26
Two cylinders



Ex. 12.7, Qu. 26
Cylinder $x^2+y^2=2$



Ex. 12.7, Qu. 26
Cylinder $y^2+z^2=2$



Ex. 12.7, Qu. 26
Curve of intersection