Exercise 13.1

Qu. 4

$$f(x,y) = x^4 + y^4 - 4xy$$

$$f_x = 4x^3 - 4y, \quad f_y = 4y^3 - 4x, \quad f_{xx} = 12x^2, \quad f_{xy} = -4 \quad \text{and} \quad f_{yy} = 12y^2.$$

For critical points: $f_x = f_y = 0$, i.e.

$$\begin{cases} y = x^3 \\ y^3 = x \end{cases}$$

$$\Rightarrow \quad x^9 = x$$
$$x(x^8 - 1) = 0$$
$$x(x^4+1)(x^2+1)(x^2-1) = 0$$
$$\Rightarrow \quad x = -1, 0, 1$$
$$y = -1, 0, 1.$$

Points	f_{xx}	f_{yy}	f_{xy}	D	Type
$(-1,-1)$	12	12	-4	$128 > 0$	min
$(0,0)$	0	0	-4	$-16 < 0$	saddle
$(1,1)$	12	12	-4	$128 > 0$	min

(see page 6)

Qu. 20

$$f(x,y) = x + 8y + \frac{1}{xy}$$

$$f_x = 1 - \frac{1}{x^2y}, \quad f_y = 8 - \frac{1}{xy^2}, \quad f_{xx} = \frac{2}{x^3y}, \quad f_{yy} = \frac{2}{xy^3} \quad \text{and} \quad f_{xy} = \frac{1}{x^2y^2}.$$

For critical points, $f_x = f_y = 0$, i.e.

$$\begin{cases} x^2y = 1 \\ 8xy^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = \frac{1}{4} \end{cases}$$

At $\left(2, \frac{1}{4}\right)$, $f_{xx} > 0$, $D = \frac{3}{x^4y^4} > 0$, hence $\left(2, \frac{1}{4}\right)$ is a relative minimum point.

Also note that.

(i) As the point (x,y) approaches to the x-axis, i.e. $y \to 0^+$, $x > 0^+$, $f(x,y) \to +\infty$

(ii) As the point (x,y) approaches to the y-axis, i.e. $x \to 0^+$, $y > 0^+$, $f(x,y) \to +\infty$

(iii) As the point (x,y) tends to infinity, .i.e. $x^2 + y^2 \to \infty$, $f(x,y) \to +\infty$

$\therefore \left(2, \frac{1}{4}\right)$ must be an absolute minimum point.

(see page 6)

Qu. 27

$$f(x,y) = (y - x^2)(y - 3x^2) = y^2 - 4x^2y + 3x^4$$
$$f_x = -8xy + 12x^3 = 4x(3x^2 - 2y)$$
$$f_y = 2y - 4x^2.$$

For critical points, $f_x = f_y = 0$, i.e.

$$\begin{cases} x(3x^2 - 2y) = 0 \\ y = 2x^2 \end{cases} \quad \Rightarrow \quad x = 0 \quad \text{or} \quad y = \frac{3}{2}x^2$$

$\Rightarrow \quad x = 0, \quad y = 0$ is the only solution (why!!)

$\therefore$ $(0,0)$ is a critical point of f.

(i) Let

$$g(x) = f(x, kx) = k^2x^2 - 4kx^3 + 3x^4$$
$$\text{Then} \quad g'(x) = 2kx - 12kx^2 + 12x^3$$
$$g''(x) = 2k^2 - 24kx + 36x^2.$$

For critical points, $g'(x) = 0$

$$x(2k - 12kx + 12x^2) = 0$$
$$\Rightarrow \quad x = 0 \quad \text{or} \quad k - 6kx + 6x^2 = 0 \quad \text{(no need to consider this case)}$$

At $x = 0$, $g''(0) = 2k^2 > 0$ if $k \neq 0$, thus $g(x) = f(x, kx)$ has a local minimum at $x = 0$ if $k \neq 0$.

Also need to consider: along y-axis, $f(0,y) = y^2$ and along x-axis, $f(x,0) = 3x^4$, both of these functions have a local minimum at (0,0).

$\therefore$ f has a local minimum at (0,0) when restricted to any straight line through the origin.

(ii) Note that on the curve $y = kx^2$, we have

$$f(x, kx^2) = (kx^2 - x^2)(kx^2 - 3x^2) = (k-1)(k-3)x^4$$
$$f(x, kx^2) < 0 \quad \text{if} \quad (k-1)(k-3) < 0, \quad \text{i.e. } 1 < k < 3$$
$$\text{i.e.} \quad f(x, kx^2) = -cx^4 \quad \text{with} \quad c > 0 \quad \text{if} \quad 1 < k < 3.$$

This function has a local maximum value at (0,0).

Therefore f does not have an (unrestricted) local minimum value at (0,0).

Note that

$$f_{xx} = -8y + 36x^2$$
$$f_{yy} = 2$$
$$f_{xy} = -8x.$$

At (0,0), $f_{xx} = 0$, $D = 0$. Thus the second derivative test is indeterminate at the origin.

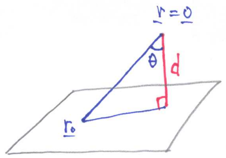
*Discuss why parts (i) and (ii) do not contradict one another.

Exercise 13.3

Qu. 3

(a) The point $\mathbf{r}_0 = (3,0,0)$ is on the given plane

$$\begin{aligned} d &= \|\mathbf{r}-\mathbf{r}_0\|\,|\cos\theta|\cdot\|\widehat{\mathbf{n}}\| \\ &= |(\mathbf{r}-\mathbf{r}_0)\cdot\widehat{\mathbf{n}}| \\ &= \left|(3,0,0)\cdot\frac{1}{3}(1,2,2)\right| \\ &= 1. \end{aligned}$$



Alternatively, let (x,y,z) be the point on the given plane closest to $(0,0,0)$. The vector (1,2,2) is normal to the plane, so must be parallel to the vector (x,y,z) from $\mathbf{0}$ to (x,y,z). Thus

$$(x,y,z) = \lambda(1,2,2) \quad \text{for some scalar } \lambda.$$

Since the point (x,y,z) is on the given plane, i.e.

$$\begin{aligned} t+4t+4t = 3 \quad &\Rightarrow \quad t = \frac{1}{3} \\ \therefore (x,y,z) &= \frac{1}{3}(1,2,2) \\ \therefore d &= \frac{1}{3}\sqrt{1+4+4} = 1. \end{aligned}$$

(b) Let (x,y,z) be the point on the given plane closest to $\mathbf{0}$, so the problem becomes: minimize

$$s(x,y,z) = x^2+y^2+z^2.$$

Since $x+2y+2z=3$, we have $x = 3-2y-2z$

$$\therefore\ s = s(y,z) = (3-2y-2z)^2+y^2+z^2$$

For critical points, $s_y = s_z = 0$

$$\begin{aligned} s_y &= -12+10y+8z = 0 \\ s_z &= -12+8y+10z = 0 \\ \Rightarrow \quad y = z &= \frac{2}{3}, \quad x = \frac{1}{3} \end{aligned}$$

$\therefore$ The distance is 1 unit as in part (a).

(c) Same as in part (b), but now the problem becomes:
Minimize $s = x^2+y^2+z^2$ subject to $x+2y+2z = 3 = g(x,y,z)$

Using Lagrangian multipliers, to find the critical points, we have

$$\begin{cases} \nabla s = \lambda\nabla g \\ g(x,y,z) = 3 \end{cases}$$

$$\therefore \begin{cases} 2x = \lambda \\ 2y = 2\lambda \\ 2z = 2\lambda \\ x+2y+2z = 3 \end{cases} \quad \Rightarrow \quad \begin{cases} y = z = \lambda \\ x = \dfrac{\lambda}{2} \\ \lambda = \dfrac{2}{3} \end{cases}$$

So the critical point is once again $\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$, whose distance from the origin is 1 unit.

Qu. 22 Let $f(x,y,z) = x^2+y^2+z^2$, $g_1(x,y,z) = x^2+y^2-z^2 = 0$ and $g_2(x,y,z) = x-2z = 3$, then from $\nabla f = \lambda_1\nabla g_1 + \lambda_2\nabla g_2$, then

$$\begin{aligned} 2x &= \lambda_1(2x)+\lambda_2 \\ 2y &= \lambda_1(2y) \\ 2z &= \lambda_1(-2z)+\lambda_2(-2) \\ x^2+y^2-z^2 &= 0 \\ x-2z &= 3. \end{aligned}$$

Solving the above system of equations, we get $(1,0,-1)$ and $(3,0,-3)$, so $f_{\min}(1,0,-1) = 2$ and $f_{\max}(3,0,-3) = 18$.

Qu. 22 Let $f(x,y,z) = xy + z^2$ on $B = \{(x,y,z) \mid x^2+y^2+z^2 \leqslant 1\}$

First we want to find critical points in B, i.e. $\nabla f = \mathbf{0}$.

$$\begin{cases} f_x = y = 0 \\ f_y = x = 0 \\ f_z = 2z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0. \end{cases}$$

$(0,0,0)$ is in B and $f(0,0,0) = 0$.

Now find critical points on the boundary of B, that is, on the sphere $x^2+y^2+z^2 = 1$. The problem becomes: find critical points of $f(x,y,z)$ subject to $g(x,y,z) = x^2+y^2+z^2 = 1$.

Using Lagrangian multipliers, we have

$$\nabla f = \lambda \nabla g,$$

$$\text{i.e.} \quad y = \lambda 2x \tag{1}$$

$$x = \lambda 2y \tag{2}$$

$$2z = \lambda 2z \tag{3}$$

$$x^2+y^2+z^2 = 1. \tag{4}$$

From(3), we have $\lambda = 1$ or $z = 0$

Case (I): if $\lambda = 1$, (1) and (2) imply that $x = y = 0$ and from (4),

$$z = \pm 1 \quad , \quad f(0,0,\pm 1) = 1.$$

Case (II): if $z = 0$, from (1) and (2) imply that $x^2 = y^2$ and from (4), we have $x^2 = y^2 = \dfrac{1}{2}$, i.e. we have four points

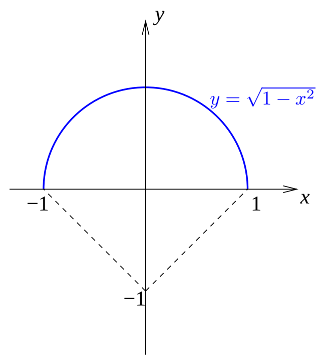
$$f(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 0) = \frac{1}{2}$$

$$\text{or} \quad f(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 0) = -\frac{1}{2}$$

$$\therefore \text{Maximum} \quad f(0,0,\pm 1) = 1$$

$$\text{Minimum} \quad f(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, 0) = -\frac{1}{2}.$$

Qu. 26 As can be seen in the figure, the minimum distance from $(0,-1)$ to points of the semicircle $y = \sqrt{1-x^2}$ is $\sqrt{2}$, the closest points to $(0,-1)$ on the semicircle being $(\pm 1, 0)$.



Try to use Lagrange multiplier: Min $D = d^2 = f(x,y) = (x-0)^2 + (y-(-1))^2 = x^2 + (y+1)^2$ subject to $g(x,y) = y - \sqrt{1-x^2} = 0$. We have

$$\nabla f = \lambda \nabla g \qquad (!!)$$

$$(2x, 2(y+1)) = \lambda \left(\frac{x}{\sqrt{1-x^2}}, 1 \right)$$

i.e.

$$2x = \frac{\lambda x}{\sqrt{1-x^2}} \quad \Rightarrow \quad \lambda = \sqrt{1-x^2} \tag{1}$$

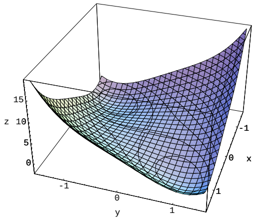
$$2(y+1) = \lambda \tag{2}$$

$$y = \sqrt{1-x^2} \tag{3}$$

i.e. Lagrange multiplier method failed!!

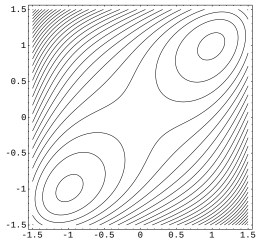
Why failed?

$\because$ The level curve $f(x,y) = 2$ is not tangent to the semi-circle at $(\pm 1, 0)$. This could only have happened because $(\pm 1, 0)$ are **end points** of the semicircle.



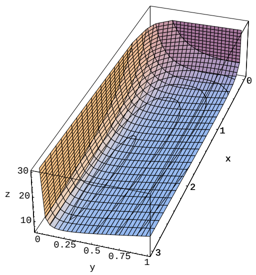
Ex. 13.1, Qu 4

$f(x,y) = x^4 + y^4 - 4xy$



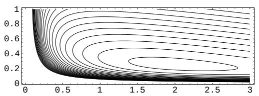
Ex. 13.1, Qu 4

Contour plot of $f(x,y)$



Ex. 13.1, Qu 20

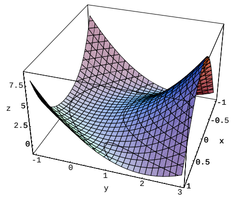
$f(x,y) = x + 8y + 1/xy$



Ex. 13.1, Qu 20

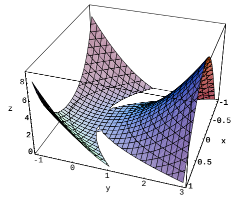
Contour plot of $f(x,y)$

Homework 5 **Math2023**



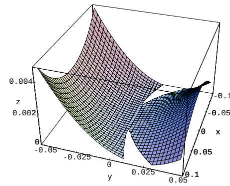
Ex. 13.1, Qu 27

$f(x,y) = (y - x^2)(y - 3x^2)$



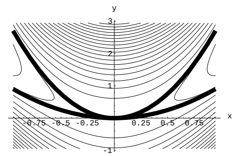
Ex. 13.1, Qu 27

Only plotted $f(x,y) \geqslant 0$



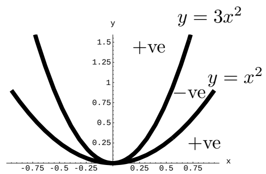
Ex. 13.1, Qu 27

Zoom in around the point $(0,0)$

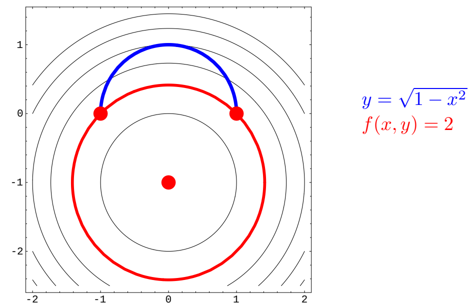


Ex. 13.1, Qu 27

Contour plot of $f(x,y)$



Ex. 13.1, Qu 27



Ex. 13.3, Qu 26