MATH2023 Multivariable Calculus 2013

From the textbook Calculus of Several Variables (5th) by R. Adams, Addison Wesley.

Homework 6

(*Total: 17 questions*)

Ex. 14.1

17 Evaluate the given double integral by inspection

$$\iint_{x^2+y^2\leqslant 1}(4x^2y^3 - x + 5)\,dA.$$

22 Evaluate the given double integral by inspection

$$\iint_R \sqrt{b^2-y^2}\,dA,$$

where R is the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$.

Ex. 14.2

12 Calculate the iterated integral $\iint_T \sqrt{a^2-y^2}\,dA$, where T is the triangle with vertices $(0,0)$, $(a,0)$, and (a,a).

18 Sketch the domain of integration and evaluate the iterated integral.

$$\int_0^1\int_x^{x^{\frac{1}{3}}} \sqrt{1-y^4}\,dydx.$$

22 Find the volume of the solid which is under $z = 1 - y^2$ and above $z = x^2$.

30 Let $F'(x) = f(x)$ and $G'(x) = g(x)$ on the interval $a \leqslant x \leqslant b$. Let T be the triangle with vertices (a,a), (b,a), and (b,b). By iterating $\iint_T f(x)g(y)\,dA$ in both directions, show that

$$\int_a^b f(x)G(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b g(y)F(y)\,dy.$$

(This is an alternative derivation of the formula for integration by parts.)

Qu. $\displaystyle\int_{-2}^3\int_0^1 |x|\sin \pi y\,dydx.$

Ex. 14.3

4 Determine the integral converges or diverges. Try to evaluate it if it converges.

$\displaystyle\iint_T \frac{1}{x\sqrt{y}}\,dA$ over the triangle T with vertices $(0,0)$, $(1,1)$ and $(1,2)$.

5 Determine the integral converges or not. Try to evaluate it if it exists.

$\displaystyle\iint_Q \frac{x^2+y^2}{(1+x^2)(1+y^2)}\,dA$, where Q is the first quadrant of the xy-plane.

21 Evaluate both iterations of the improper integral

$$\iint_S \frac{x-y}{(x+y)^3}\,dA,$$

where S is the square $0 < x < 1$, $0 < y < 1$. Show that the above improper double integral does not exist, by considering

$$\iint_T \frac{x-y}{(x+y)^3}\,dA,$$

where T is that part of the square S lying under the line $x = y$.

30. (**Another proof of equality of mixed partials**) Suppose that $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are continuous in a neighbourhood of the point(a,b). Without the equality of these mixed partial derivatives, show that

$$\iint_R f_{xy}(x,y)\,dA = \iint_R f_{yx}(x,y)\ dA,$$

where R is the rectangle with vertices (a,b), $(a+h,b)$, $(a,b+k)$, and $(a+h,b+k)$ and h^2+k^2 is sufficiently small. Now use the result of Exercise 29 to show that $f_{xy}(a,b) = f_{yx}(a,b)$. (This reproves Theorem 1 of Section 12.4 (or see below: the mean-value theorem). However, in that theorem we only assumed continuity of the mixed partials *at* (a,b). Here, we assume the continuity *at all points sufficiently near* (a,b).)

A mean-value theorem for double integrals

If the function $f(x,y)$ is continuous on a closed, bounded, connected set D in the xy-plane, then there exists a point (x_0,y_0) in D such that

$$\iint_D f(x,y)\,dA = f(x_0,y_0)\times(\text{area of } D).$$

Ex. 14.4

11 Evaluate $\iint_S (x+y)\,dA$, where S is the region in the first quadrant lying inside the disk $x^2+y^2 \leqslant a^2$ and under the line $y=\sqrt{3}x$.

14 Evaluate $\iint_{x^2+y^2\leqslant 1} \ln(x^2+y^2)\,dA$.

22 Find the volume lying inside both the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$.

26 Find the volume of the region lying inside the circular cylinder $x^2+y^2=2y$ and inside the parabolic cylinder $z^2=y$.

37 (**The gamma function**) The error function, $\mathrm{Erf}(x)$, is defined for $x \geqslant 0$ by

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

Show that $[\mathrm{Erf}(x)]^2 = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \left(1-e^{-x^2/\cos^2\theta}\right) d\theta$. Hence deduce that

$$\mathrm{Erf}(x) \geqslant \sqrt{1-e^{-x^2}}.$$

Qu Find the volume lying outside the cone $z^2=x^2+y^2$ and inside the sphere $x^2+(y-a)^2+z^2=a^2$.

* **At least try to do the underlined ones, the others are recommended exercises.**