MATH2023 Multivariable Calculus 2013

From the textbook Calculus of Several Variables (5th) by R. Adams, Addison Wesley.

Homework 8

(*Total: 12 questions*)

Ex. 15.1

7 Sketch the plane vector field $\mathbf{F}(x,y) = \nabla \ln(x^2+y^2)$ and determine its field lines.

9 Describe the streamlines of the velocity fields $\mathbf{v}(x,y,z) = y\,\mathbf{i} - y\,\mathbf{j} - y\,\mathbf{k}$.

16 Describe the streamlines of the velocity fields $\mathbf{v}(x,y) = x\,\mathbf{i} + (x+y)\,\mathbf{j}$. (Hint: let $y = xv(x)$.)

Ex. 15.3

2 Let C be the conical helix with parametric equations $x = t\cos t$, $y = t\sin t$, $z = t$, $(0 \leqslant t \leqslant 2\pi)$. Find $\int_C z\,ds$.

8 Find $\int_C \sqrt{1+4x^2z^2}\,ds$, where C is the curve of intersection of the surfaces $x^2+z^2=1$ and $y = x^2$.

15 Find $\int_C \frac{ds}{(2y^2+1)^{3/2}}$, where C is the parabola $z^2 = x^2+y^2$, $x+z=1$.

Ex. 15.4

3 Evaluate the line integral of the tangential component of the vector field

$$\mathbf{F}(x,y,z) = y\,\mathbf{i} + z\,\mathbf{j} - x\,\mathbf{k}$$

along the straight line from $(0,0,0)$ to $(1,1,1)$.

5 Evaluate the line integral of the tangential component of the vector field $\mathbf{F}(x,y,z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ from $(-1,0,0)$ to $(1,0,0)$ along either direction of the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $z=y$.

11 Determine the values of A and B for which the vector field

$$\mathbf{F} = Ax\ln z\,\mathbf{i} + By^2z\,\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k}$$

is conservative. If C is the straight line from $(1,1,1)$ to $(2,1,2)$, find

$$\int_C 2x\ln z\,dx + 2y^2z\,dy + y^3\,dz.$$

13 If C is the intersection of $z = \ln(1+x)$ and $y = x$ from $(0,0,0)$ to $(1,1,\ln 2)$, evaluate

$$\int_C (2x\sin(\pi y) - e^z)\,dx + (\pi x^2\cos(\pi y) - 3e^z)\,dy - xe^z\,dz.$$

14 Is each of the following sets a domain? a connected domain? a simply connected domain?

(a) the set of points (x,y) in the plane such that $x > 0$ and $y \geqslant 0$

(b) the set of points (x,y) in the plane such that $x = 0$ and $y \geqslant 0$

(c) the set of points (x,y) in the plane such that $x \neq 0$ and $y > 0$

(d) the set of points (x,y,z) in 3-space such that $x^2 > 1$

(e) the set of points (x,y,z) in 3-space such that $x^2+y^2 > 1$

(f) the set of points (x,y,z) in 3-space such that $x^2+y^2+z^2 > 1$

22 Evaluate $\frac{1}{2\pi}\oint_C \frac{-y\,dx + x\,dy}{x^2+y^2}$

(a) counterclockwise around the circle $x^2+y^2=a^2$,

(b) clockwise around the square with vertices $(-1,-1)$, $(-1,1)$, $(1,1)$, and $(1,-1)$,

(c) counterclockwise around the boundary of the region $1 \leqslant x^2+y^2 \leqslant 2$, $y \geqslant 0$.

Homework 9

(*Total: 9 questions*)

Ex. 15.2

5 Determine whether the vector field

$$\mathbf{F}(x,y,z) = (2xy - z^2)\,\mathbf{i} + (2yz + x^2)\,\mathbf{j} - (2zx - y^2)\,\mathbf{k}$$

is conservative and find a potential if it is conservative.

7 Find the three-dimensional vector field with potential $\phi(\mathbf{r}) = \frac{1}{\|\mathbf{r} - \mathbf{r}_0\|^2}$.

9 Show that the vector field

$$\mathbf{F}(x,y,z) = \frac{2x}{z}\,\mathbf{i} + \frac{2y}{z}\,\mathbf{j} - \frac{x^2+y^2}{z^2}\,\mathbf{k}$$

is conservative, and find its potential. Describe the equipotential surfaces. Find the field lines of $\mathbf{F}$.

Ex. 15.5

10 Find the area of the part of the cylinder $x^2+z^2=a^2$ that lies inside the cylinder $y^2+z^2=a^2$.

14 Find $\iint_S y\,dS$, where S is the part of the cone $z=\sqrt{2(x^2+y^2)}$ that lies below the plane $z=1+y$.

Ex. 15.6

1 Find the flux of $\mathbf{F}=x\,\mathbf{i}+z\,\mathbf{j}$ out of the tetrahedron bounded by the coordinate planes and the plane $x+2y+3z=6$.

6 Find the flux of $\mathbf{F}=x\,\mathbf{i}+x\,\mathbf{j}+\mathbf{k}$ upward through the part of the surface $z=x^2-y^2$ lying inside the cylinder $x^2+y^2=a^2$.

10 Find the flux of $\mathbf{F}=2x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ upward through the surface $\mathbf{r}=u^2v\,\mathbf{i}+uv^2\,\mathbf{j}+v^3\,\mathbf{k}$, $(0\leqslant u\leqslant 1,\ 0\leqslant v\leqslant 1)$.

15 Define the flux of a *plane* vector field across a piecewise smooth *curve*. Find the flux of $\mathbf{F}=x\,\mathbf{i}+y\,\mathbf{j}$ outward across

(a) the circle $x^2+y^2=a^2$.

(b) the boundary of the square $-1\leqslant x,\ y\leqslant 1$.

Homework 10

(Total: 9 questions)

Ex. 16.4

4 Use the Divergence Theorem to calculate the flux of the vector field

$$\mathbf{F}=x^3\,\mathbf{i}+3yz^2\,\mathbf{j}+(3y^2z+x^2)\,\mathbf{k}$$

out of the sphere S with equation $x^2+y^2+z^2=a^2$, where $a>0$.

8 Evaluate the flux of $\mathbf{F}=x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k}$ outward across the boundary of the solid cylinder $x^2+y^2\leqslant 2y$, $0\leqslant z\leqslant 4$.

11 A conical domain with vertex $(0,0,b)$ and axis along the z-axis has as base a disk of radius a in the xy-plane. Find the flux of

$$\mathbf{F}=(x+y^2)\,\mathbf{i}+(3x^2y+y^3-x^3)\,\mathbf{j}+(z+1)\,\mathbf{k}$$

upward through the conical part of the surface of the domain.

23 If $\mathbf{F}$ is a smooth vector field on D, show that

$$\iiint_D \phi\,\nabla\cdot\mathbf{F}\,dV+\iiint_D \nabla\phi\cdot\mathbf{F}\,dV=\oiint_S \phi\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dS.$$

24 If $\nabla^2\phi=0$ in D and $\phi(x,y,z)=0$ on S, show that $\phi(x,y,z)=0$ in D.

Ex. 16.5

2 Evaluate $\oint_C y\,dx-x\,dy+z^2\,dz$ around the curve C of intersection of the cylinders $z=y^2$ and $x^2+y^2=4$, oriented counterclockwise as seen from a point high on the z-axis.

3 Evaluate $\iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$, where S is the hemisphere $x^2+y^2+z^2=a^2$, $z\geqslant 0$ with outward normal, and $\mathbf{F}=3y\,\mathbf{i}-2xz\,\mathbf{j}+(x^2-y^2)\,\mathbf{k}$.

8 Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}=ye^x\,\mathbf{i}+(x+e^x)\,\mathbf{j}+z^2\,\mathbf{k}$ and C is the curve

$$\mathbf{r}=(1+\cos t)\,\mathbf{i}+(1+\sin t)\,\mathbf{j}+(1-\sin t-\cos t)\,\mathbf{k},$$

where $0\leqslant t\leqslant 2\pi$.

9 Let C_1 be a straight line joining $(-1,0,0)$ to $(1,0,0)$ and let C_2 be the semicircle $x^2+y^2=1$, $z=0$, $y\geqslant 0$. Let S be a smooth surface joining C_1 to C_2 having upward normal, and let

$$\mathbf{F}=(\alpha x^2-z)\,\mathbf{i}+(xy+y^3+z)\,\mathbf{j}+\beta y^2(z+1)\,\mathbf{k}.$$

Find the values of α and β for which $\mathrm{I}=\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ is independent of the choice of S, and find the value of I for these values of α and β.