Exercise 16.4

Qu. 4 If $\mathbf{F} = x^3\,\mathbf{i} + 3yz^2\,\mathbf{j} + (3y^2z + x^2)\,\mathbf{k}$, then

$$\nabla\cdot\mathbf{F} = 3x^2 + 3z^2 + 3y^2, \text{ and}$$

$$\begin{aligned}
\oiint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \iiint_G \nabla\cdot\mathbf{F}\,dV \qquad \text{(Divergence Theorem)}\\
&= 3\iiint_G (x^2+y^2+z^2)\,dV\\
&= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\
&= \frac{12}{5}\pi a^5.
\end{aligned}$$

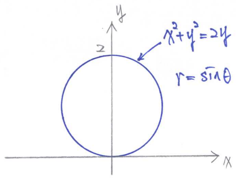
Qu. 8 Since the cylinder is a closed surface, so we can use the divergence theorem to do this question

$$\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$$

$$\nabla\cdot\mathbf{F} = 2x + 2y + 2z$$

$$\begin{aligned}
\therefore \oiint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \iiint_G \nabla\cdot\mathbf{F}\,dV \qquad \text{(where } G \text{ is the cylinder)}\\
&= 2\iiint_G (x+y+z)\,dV
\end{aligned}$$

$$\begin{aligned}
\therefore \iiint_G x\,dV &= \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4} r\cos\theta\cdot r\,dz\,dr\,d\theta\\
&= 4\int_0^{\pi}\int_0^{2\sin\theta} r^2\cos\theta\,dr\,d\theta\\
&= 4\times\frac{8}{3}\int_0^{\pi}\cos\theta\sin^3\theta\,d\theta\\
&= \frac{32}{3}\sin^4\theta\Big|_0^{\pi} = 0
\end{aligned}$$



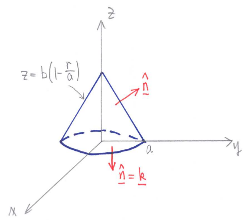
$$\begin{aligned}
\iiint_G y\,dV &= \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4} r\sin\theta\cdot r\,dz\,dr\,d\theta\\
&= 4\int_0^{\pi}\int_0^{2\sin\theta} r^2\sin\theta\,dr\,d\theta\\
&= \frac{32}{3}\int_0^{\pi}\sin^4\theta\,d\theta\\
&= \frac{32}{3}\frac{1}{4}\left(\frac{3}{2} - 2\cos 2\theta + \frac{1}{4}\cos 4\theta\right)\Big|_0^{\pi} = 4\pi
\end{aligned}$$

$$\begin{aligned}
\iiint_G z\,dV &= \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{4} z\cdot r\,dz\,dr\,d\theta\\
&= 8\int_0^{\pi}\int_0^{2\sin\theta} r\,dr\,d\theta\\
&= 16\int_0^{\pi}\sin^2\theta\,d\theta\\
&= 16\times\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi} = 8\pi
\end{aligned}$$

$$\therefore \oiint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = 2(0 + 4\pi + 8\pi) = 24\pi.$$

Qu. 11 This is a closed surface, we can use Divergence Theorem to do this question.

$$\mathbf{F} = (x+y^2)\,\mathbf{i} + (3x^2y + y^3 - x^3)\,\mathbf{j} + (z+1)\,\mathbf{k}$$
$$\nabla\cdot\mathbf{F} = 1 + 3(x^2+y^2) + 1 = 2 + 3(x^2+y^2).$$



Let G be the conical domain, S its conical surface, and B its base disk, as shown in the figure. We have

$$\begin{aligned}
\oiint_{S+B} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \iiint_G \nabla\cdot\mathbf{F}\,dV \\
&= \iiint_G [2 + 3(x^2+y^2)]\,dV \\
&= \int_0^{2\pi}\int_0^a\int_0^{b(1-\frac{r}{a})} (2+3r^2)r\,dz\,dr\,d\theta \qquad \text{(Cylindrical coord)} \\
&= 2\pi b\int_0^a r(2+3r^2)(1-\frac{r}{a})\,dr \\
&= 2\pi b\int_0^a (2r + 3r^3 - \frac{2r^2}{a} - \frac{3r^4}{a})\,dr \\
&= \frac{2\pi a^2 b}{3} + \frac{3\pi a^4 b}{10}.
\end{aligned}$$

Qu. 23 Note that $\nabla\cdot(\phi\mathbf{F}) = \phi\nabla\cdot\mathbf{F} + \nabla\phi\cdot\mathbf{F}$, thus

$$\begin{aligned}
\iiint_D \phi\nabla\cdot\mathbf{F}\,dV + \iiint_D \nabla\phi\cdot\mathbf{F}\,dV &= \iiint_D \nabla\cdot(\phi\mathbf{F})\,dV \\
&= \oiint_S \phi\mathbf{F}\cdot\widehat{\mathbf{n}}\,dS \qquad \text{(Divergence Theorem)}
\end{aligned}$$

Qu. 24 If $\mathbf{F} = \nabla\phi$ in Qu. 23, then $\nabla\cdot\mathbf{F} = \nabla^2\phi$ and

$$\iiint_D \phi\nabla^2\phi\,dV + \oiint_D \|\nabla\phi\|^2\,dV = \oiint_S \phi\nabla\phi\cdot\widehat{\mathbf{n}}\,dS.$$

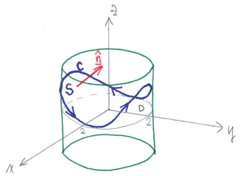
If $\nabla^2\phi = 0$ in D and $\phi = 0$ on S, then

$$\iiint_D \|\nabla\phi\|^2\,dV = 0.$$

Since ϕ is assumed to be smooth, $\nabla\phi = 0$ throughout D, and therefore ϕ is constant on each connected component of D. Since $\phi = 0$ on S, these constants must all be zero, and $\phi = 0$ on D.

Exercise 16.5

Qu. 2 Let S be the part of the surface $z = y^2$ lying inside the cylinder $x^2+y^2 = 4$, and having upward normal $\widehat{\mathbf{n}}$. Then C is the oriented boundary of S. Let D be the disk $x^2+y^2 \leqslant 4$, $z = 0$, that is, the projection of S onto the xy-plane. If $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j} + z^2\,\mathbf{k}$, then

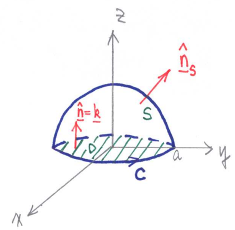


$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & z^2 \end{vmatrix} = -2\,\mathbf{k}.$$

Let $f(x,y,z) = z - y^2 = 0$, this is a level surface in 3D, so $\nabla f = (0,-1,1) = \mathbf{n}$, so

$$\begin{aligned}\oint_C y\,dx - x\,dy + z^2\,dz &= \oint_C \mathbf{F}\cdot\,d\mathbf{r} \\ &= \iint_S (\nabla\times\mathbf{F})\cdot\widehat{\mathbf{n}}\,dS \qquad \text{(Stoke's Theorem)} \\ &= \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dA \\ &= \iint_D (-2\,\mathbf{k})\cdot(-\mathbf{j}+\mathbf{k})\,dA \\ &= \iint_D (-2)\,dA \\ &= -2\pi(2)^2 \\ &= -8\pi.\end{aligned}$$

Qu. 3 Let C be the circle $x^2+y^2 = a^2$, $z = 0$, oriented counter-clockwise as seen from the positive z-axis. Let D be the disk bounded by C, with normal $\widehat{\mathbf{n}} = \mathbf{k}$.

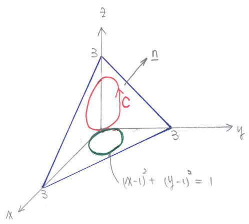


$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & -2xz & x^2-y^2 \end{vmatrix} \\ &= 2(x-y)\,\mathbf{i} + 2x\,\mathbf{j} - (2z+3)\,\mathbf{k}.\end{aligned}$$

$$\begin{aligned}\iint_S (\nabla\times\mathbf{F})\cdot\widehat{\mathbf{n}}_S\,dS &= \oint_C \mathbf{F}\cdot\,d\mathbf{r} \qquad \text{(Stoke's Theorem)} \\ &= \iint_D (\nabla\times\mathbf{F})\cdot\,\mathbf{k}\,dA \qquad \text{(Green's Theorem)} \\ &= -\iint_D 3\,dA \\ &= -3\pi a^2.\end{aligned}$$

Qu. 8 Note that $\mathbf{r}(0) = 2\,\mathbf{i} + \mathbf{j} = \mathbf{r}(2\pi)$, therefore $\mathbf{r}(t)$ is a closed space curve for $0 \leqslant t \leqslant 2\pi$. So we can use Stoke's Theorem.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ye^x & x + e^x & z^2 \end{vmatrix} = \mathbf{k}.$$



Note also that

$$x = 1 + \cos t$$
$$y = 1 + \sin t$$
$$z = 1 - \sin t - \cos t$$

so we have $x + y + z = 3$ and $(x-1)^2 + (y-1)^2 = 1$

i.e. C lies on the surface $x + y + z = 3$ and $(x-1)^2 + (y-1)^2 = 1$.

In fact, C is the boundary of the elliptic disk in the plane $x+y+z = 3$ lying inside the cylinder $(x-1)^2 + (y-1)^2 = 1$.

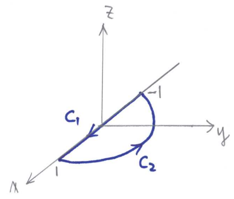
Let $f(x, y, z) = x + y + z = 3$, this is a level surface in 3D, so $\nabla f = (1, 1, 1) = \mathbf{n}$.

$$\text{Therefore } \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \widehat{\mathbf{n}}\, dS$$
$$= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA$$
$$= \iint_R \mathbf{k} \cdot (1, 1, 1)\, dA$$
$$= \iint_R dA$$
$$= \pi(1)^2 = \pi.$$

Alternatively, note that $\mathbf{F} = ye^x\,\mathbf{i} + (x + e^x)\,\mathbf{j} + z^2\,\mathbf{k} = x\,\mathbf{j} + \nabla\phi$, where $\phi = ye^x + \dfrac{z^3}{3}$. The curve C is a closed curve in a simply connected domain, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C x\,\mathbf{j} \cdot d\mathbf{r} + \oint_C \nabla\phi \cdot d\mathbf{r}$$
$$= \oint_C x\, dy$$
$$= \int_0^{2\pi} (1 + \cos t) \cos t\, dt$$
$$= \int_0^{2\pi} \cos^2 t\, dt = \pi.$$

Qu. 9 If S_1 and S_2 are two surfaces joining C_1 to C_2 each having upward normal, then the closed surface S_3 consisting of S_1 and $-S_2$ (that is , S_2 with downward normal) bound a region G in 3-space. Then



$$\iint_{S_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS - \iint_{S_2} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \iint_{S} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS + \iint_{-S_2} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$
$$= \oiint_{S_3} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$
$$= \pm \iiint_G \nabla \cdot \mathbf{F}\, dV$$
$$= 0$$

provided that $\nabla \cdot \mathbf{F} \equiv 0$. Since

$$\mathbf{F} = (\alpha x^2 - z)\,\mathbf{i} + (xy + y^3 + z)\,\mathbf{j} + \beta y^2 (z + 1)\,\mathbf{k}$$

we have

$$\nabla \cdot \mathbf{F} = 2\alpha x + x + 3y^2 + \beta y^2$$
$$= 0 \qquad \text{if} \quad \alpha = -\frac{1}{2} \quad \text{and} \quad \beta = -3.$$

In this case we can evaluate $\iint_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$ for any such surface S by evaluating the special case where S is the half-disk $H : \{x^2 + y^2 \leqslant 1,\ z = 0,\ y \geq 0\}$, with upward normal $\widehat{\mathbf{n}} = \mathbf{k}$. We have

$$\iint_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = -3 \iint_H y^2\, dA$$
$$= -3 \int_0^{\pi} \int_0^1 r^2 \sin\theta\, r\, dr\, d\theta$$
$$= -\frac{3\pi}{8}.$$