Exercise 14.1

Qu. 17 By symmetry

$$\iint\limits_{x^2+y^2\leqslant 1} (4x^2y^3 - x + 5)\, dA$$
$$= 0 + 0 + 5 \times (\text{area of disk with radius } 1) = 5\pi.$$

The first two terms of the integral equal to 0 because $4x^2y^3$ is odd function in y and x is odd function in x.

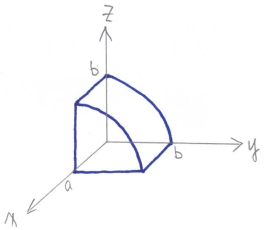
(see also page 15)

Qu. 22

$$\iint\limits_R \sqrt{b^2 - y^2}\, dA$$

$=$ volume of the quarter cylinder shown in the figure.

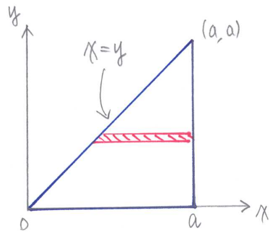
$$= \frac{1}{4}(\pi b^2) \cdot a$$
$$= \frac{1}{4}\pi a b^2.$$



Exercise 14.2

Qu. 12

$$\iint\limits_T \sqrt{a^2 - y^2}\, dA = \int_0^a \int_y^a \sqrt{a^2 - y^2}\, dx dy$$
$$= \int_0^a (a - y)\sqrt{a^2 - y^2}\, dy$$
$$= a \int_0^a (a^2 - y^2)\sqrt{a^2 - y^2}\, dy - \int_0^a y\sqrt{a^2 - y^2}\, dy$$
$$= a \cdot \frac{\pi a^2}{4} + \frac{1}{2}\int_0^a (a^2 - y^2)^{\frac{1}{2}} d(a^2 - y^2)$$
$$= \frac{\pi}{4}a^3 + \frac{1}{2}\frac{2}{3}(a^2 - y^2)^{\frac{3}{2}}\Big|_0^a$$
$$= \left(\frac{\pi}{4} - \frac{1}{3}\right) a^3.$$

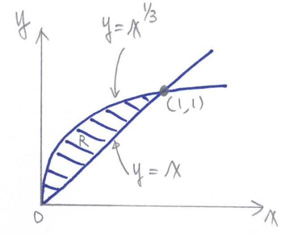


Qu. 18 The domain of integration:

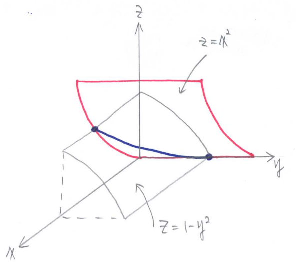
from $y = x$ to $y = x^{\frac{1}{3}}$

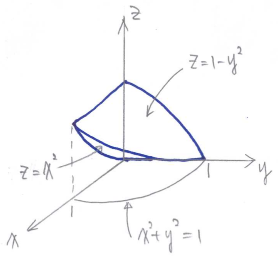
from $x = 0$ to $x = 1$

$$\int_0^1 \int_x^{x^{\frac{1}{3}}} \sqrt{1 - y^4}\, dy dx$$
$$= \iint\limits_R \sqrt{1 - y^4}\, dA \quad (\text{R as shown})$$
$$= \int_0^1 \int_y^{y^3} \sqrt{1 - y^4}\, dx dy$$
$$= \int_0^1 y\sqrt{1 - y^4}\, dy - \int_0^1 y^3\sqrt{1 - y^4}\, dy$$
$$= \frac{1}{2}\int_0^1 \sqrt{1 - u^2}\, du + \frac{1}{4}\int_0^1 (1 - y^4)^{\frac{1}{2}} d(-y^4), \quad \text{let } u = y^2, \text{ then } du = 2y dy$$
$$= \frac{1}{2}(\frac{\pi}{4} \times 1^2) + \frac{1}{4}\frac{2(1 - y^4)^{\frac{3}{2}}}{3}\Big|_0^1$$
$$= \frac{\pi}{8} - \frac{1}{6}.$$



Qu. 22 (see also page 15)





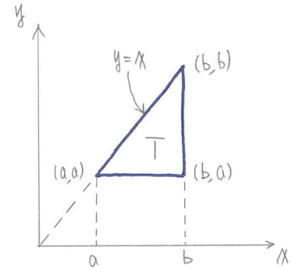
$z_1 = 1 - y^2$ and $z_2 = x^2$ intersect on the cylinder $x^2 + y^2 = 1$. The volume lying below $z = 1 - y^2$ and above $z = x^2$ is

$$\begin{aligned}
V &= \iint\limits_{x^2+y^2\leqslant 1} (z_1 - z_2)\, dA \\
&= \iint\limits_{x^2+y^2\leqslant 1} (1 - y^2 - x^2)\, dA \\
&= 4\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dydx \\
&= 4\int_0^1 \left[(1-x^2)y - \frac{y^3}{3}\right]\Bigg|_0^{\sqrt{1-x^2}} dx \\
&= \frac{8}{3}\int_0^1 (1-x^2)^{\frac{3}{2}}\, dx, \qquad \text{let } x = \sin u, \text{ then } dx = \cos u\, du \\
&= \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^4 u\, du \\
&= \frac{2}{3}\int_0^{\frac{\pi}{2}} (1 + \cos 2u)^2\, du \\
&= \frac{2}{3}\int_0^{\frac{\pi}{2}} (1 + 2\cos 2u + \frac{1+\cos 4u}{2})\, du \\
&= \frac{2}{3} \times \frac{3}{2} \times \frac{\pi}{2} \\
&= \frac{\pi}{2}.
\end{aligned}$$

Alternatively, using polar, we have

$$V = \int_0^{2\pi}\int_0^1 (1 - r^2) r\, drd\theta = 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]\Bigg|_0^1 = \frac{\pi}{2}.$$

Qu. 30 Since $F'(x) = f(x)$ and $G'(x) = g(x)$ on $a \leqslant x \leqslant b$, we have



$$\begin{aligned}
\mathrm{I}_1 &= \iint\limits_T f(x)g(y)\, dA \\
&= \int_a^b \int_a^x f(x)g(y)\, dydx \\
&= \int_a^b f(x)\left(\int_a^x G'(y)\, dy\right) dx \\
&= \int_a^b f(x)[G(x) - G(a)]\, dx \\
&= \int_a^b f(x)G(x)\, dx - G(a)\int_a^b f(x)\, dx \\
&= \int_a^b f(x)G(x)\, dx - G(a)F(b) + G(a)F(a).
\end{aligned}$$

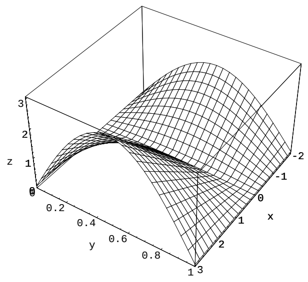
OR

$$\begin{aligned}
\mathrm{I}_2 &= \iint\limits_T f(x)g(y)\, dA \\
&= \int_a^b \int_y^b f(x)g(y)\, dxdy \\
&= \int_a^b g(y)\left(\int_y^b f(x)\, dx\right) dy \\
&= \int_a^b g(y)\, [F(b) - F(y)]\, dy \\
&= F(b)G(b) - F(b)G(a) - \int_a^b F(y)g(y)\, dx.
\end{aligned}$$

$\mathrm{I}_1 = \mathrm{I}_2$, thus

$$\int_a^b f(x)G(x)\, dx = F(b)G(b) - F(a)G(a) - \int_a^b F(y)g(y)\, dy.$$

Qu. $\int_{-2}^{3}\int_{0}^{1} |x| \sin \pi y \, dy dx.$



This is the volume of the region bounded by $z = |x| \sin \pi y$, the xy-plane, and the planes $x = -2$, $x = 3$, $y = 0$ and $y = 1$. The volume is

$$V = \int_{-2}^{3}\int_{0}^{1} |x| \sin \pi y \, dy dx$$
$$= \int_{-2}^{3} -\frac{|x|}{\pi} \cos \pi y \Big|_0^1 \, dx$$
$$= \int_{-2}^{3} \frac{2}{\pi} |x| \, dx.$$

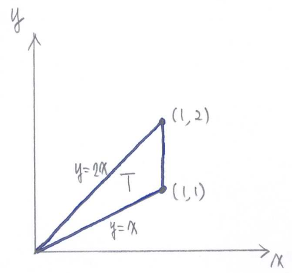
At this point we use the definition of absolute value to split this into two quantities

$$V = \int_{-2}^{0} -\frac{2}{\pi} x \, dx + \int_{0}^{3} \frac{2}{\pi} x \, dx$$
$$= \frac{4}{\pi} + \frac{9}{\pi}$$
$$= \frac{13}{\pi}.$$

Exercise 14.3

Qu. 4

$$\iint_T \frac{1}{x\sqrt{y}} \, dA = \int_0^1 \int_x^{2x} \frac{1}{x\sqrt{y}} \, dy$$
$$= 2\int_0^1 \frac{\sqrt{2x} - \sqrt{x}}{x} \, dx$$
$$= 2(\sqrt{2} - 1) \int_0^1 \frac{1}{\sqrt{x}} \, dx$$
$$= 4(\sqrt{2} - 1) \quad \text{(converges)} \quad \text{(why!!)}.$$



Qu. 5

$$\iint_Q \frac{x^2 + y^2}{(1 + x^2)(1 + y^2)} \, dA = 2 \iint_Q \frac{x^2}{(1 + x^2)(1 + y^2)} \, dA \quad \text{(by symmetry)}$$
$$= 2\int_0^\infty \frac{x^2}{1 + x^2} \, dx \times \int_0^\infty \frac{1}{1 + y^2} \, dy$$
$$= \pi \int_0^\infty \frac{x^2}{1 + x^2} \, dx$$

which diverges to infinity, since

$$\frac{x^2}{1 + x^2} \geqslant \frac{1}{2} \quad \text{on} \quad [1, \infty)$$

or

$$\frac{x^2}{1 + x^2} \to 1 \quad \text{as} \quad x \to \infty.$$

(see also page 15)

Qu. 21

$$\iint\limits_S \frac{x-y}{(x+y)^3}\,dA = \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dydx, \quad \text{let} \quad u = x+y, \quad \text{then} \quad du = dy$$

$$= \int_0^1\int_x^{x+1} \frac{2x-u}{u^3}\,dudx$$

$$= \int_0^1 \left(\frac{1}{u} - \frac{x}{u^2}\right)\Bigg|_x^{x+1} dx$$

$$= \int_0^1 \frac{1}{(1+x)^2}\,dx = \frac{1}{2}.$$

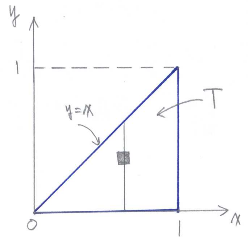
other iteration:

$$\iint\limits_S \frac{x-y}{(x+y)^3}\,dA = \int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dxdy, \quad \text{let} \quad u = x+y, \quad \text{then} \quad du = dy$$

$$= \int_0^1\int_y^{y+1} \frac{u-2y}{u^3}\,dudy$$

$$= \int_0^1 \left(\frac{y}{u^2} - \frac{1}{u}\right)\Bigg|_y^{y+1} dy$$

$$= -\int_0^1 \frac{1}{(1+y)^2}\,dy = -\frac{1}{2}.$$



These seemingly contradictory results are explained by the fact that the given double integral is improper and does not, in fact, exist, that is, it does not converge. To see this, we calculate the integral over a certain subset of the square S, namely the triangle T defined by $0 < x < 1$, $0 < y < x$

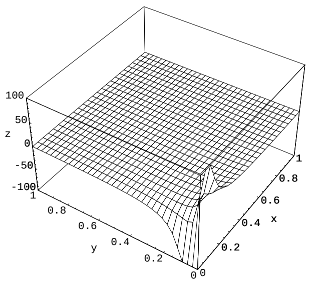
$$\iint\limits_T \frac{x-y}{(x+y)^3}\,dA = \int_0^1\int_0^x \frac{x-y}{(x+y)^3}\,dydx, \quad \text{let } u = x+y, \quad \text{then} \quad du = dy$$

$$= \int_0^1\int_x^{2x} \frac{2x-u}{u^3}\,dudx$$

$$= \int_0^1 \left(\frac{1}{u} - \frac{x}{u^2}\right)\Bigg|_x^{2x} dx$$

$$= \frac{1}{4}\int_0^1 \frac{dx}{x}$$

which diverges to infinity.



Qu. 30 If $R = \{(x,y)\,\big|\, a \leqslant x \leqslant a+h,\ b \leqslant y \leqslant b+k\}$

$$\iint\limits_R f_{xy}(x,y)\,dA = \int_0^{a+h}\int_b^{b+k} f_{xy}(x,y)\,dydx$$

$$= \int_a^{a+h} [f_x(x,b+k) - f_x(x,b)]\,dx$$

$$= f(a+h,b+k) - f(a,b+k) - f(a+h,b) + f(a,b).$$

Similarly,

$$\iint\limits_R f_{yx}(x,y)\,dA = \int_b^{b+k}\int_a^{a+h} f_{yx}(x,y)\,dxdy$$

$$= f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b).$$

Thus

$$\iint\limits_R f_{xy}(x,y)\,dA = \iint\limits_R f_{yx}(x,y)\,dA.$$

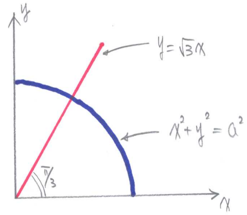
Divide both sides of this identity by hk and let $(h,k) \to (0,0)$ to obtain, using the mean-value theorem,

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Exercise 14.4

Qu. 11

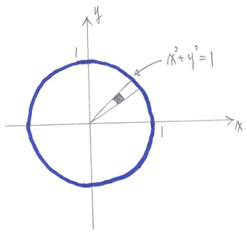
$$\iint_S (x+y)\,dA = \int_0^{\frac{\pi}{3}}\int_0^a (r\cos\theta + r\sin\theta) r\,dr d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_0^a d\theta$$
$$= \frac{a^3}{3}[\sin\theta - \cos\theta]\Big|_0^{\frac{\pi}{3}}$$
$$= \frac{(\sqrt{3}+1)}{6}a^3.$$



(see also page 15)

Qu. 14

$$\iint_{x^2+y^2\leqslant 1} \ln(x^2+y^2)\,dA = \int_0^{2\pi}\int_{0+}^1 (\ln r^2)\cdot r\,dr d\theta$$
$$= 4\pi\int_{0+}^1 r\ln r\,dr$$
$$= 4\pi\left[\frac{r^2}{2}\ln r\Big|_{0+}^1 - \int_0^1 \frac{r^2}{2}\frac{1}{r}\,dr\right]$$
$$= 4\pi\left[0 - 0 - \frac{1}{4}\right]$$
$$= -\pi.$$



Note that the integral is improper, but converges since

$$\lim_{r\to 0+} r^2\ln r = 0.$$

Qu. 22

$$x^2+y^2+z^2 = a^2$$

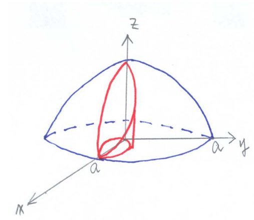
This is a sphere centre at $(0,0,0)$ with radius a. In cylind. coord. $r^2+z^2=a^2$

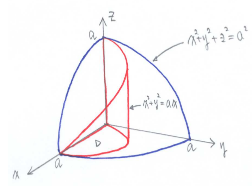
$$x^2+y^2 = ax \quad \Rightarrow \quad \left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2.$$

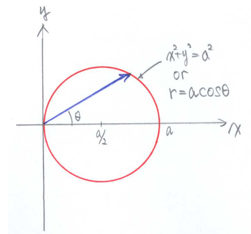
This is a cylinder centre at $\left(\frac{a}{2}, 0\right)$ with radius $\frac{a}{2}$.

In polar coord. $r = a\cos\theta$.

By symmetry, one quarter of the required volume lies in the first octant.







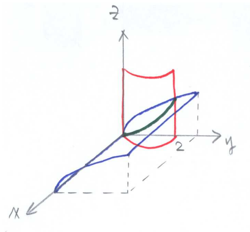
$$V = 4\iint_D \sqrt{a^2-x^2-y^2}\,dA$$
$$= 4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\cdot r\,dr$$
$$= -\frac{4}{2}\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta} (a^2-r^2)^{\frac{1}{2}}\,d(a^2-r^2)$$
$$= -2\int_0^{\frac{\pi}{2}} \frac{2}{3}(a^2-r^2)^{3/2}\Big|_0^{a\cos\theta} d\theta$$
$$= -\frac{4}{3}a^3\int_0^{\frac{\pi}{2}} \left(|\sin\theta|^3 - 1\right) d\theta$$
$$= \frac{4}{3}a^3\int_0^{\frac{\pi}{2}} (1-\sin^3\theta)\,d\theta \quad \text{since} \quad \sin\theta > 0 \quad \text{for} \quad 0\leqslant\theta\leqslant\frac{\pi}{2}$$
$$= \frac{4}{3}a^3\frac{\pi}{2} + \frac{4}{3}a^3\int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\,d(\cos\theta)$$
$$= \frac{2}{3}\pi a^3 + \frac{4}{3}a^3\left[\cos\theta - \frac{\cos^3\theta}{3}\right]\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{2}{3}\pi a^3 + \frac{4}{3}a^3\left[-(1-\frac{1}{3})\right]$$
$$= \frac{2}{9}a^3(3\pi - 4).$$

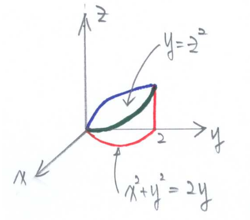
Qu. 26 One quarter of the required volume V is shown in the figure.

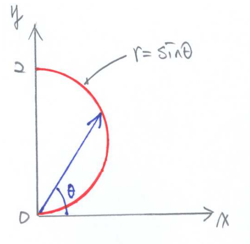
$$x^2 + y^2 = 2y$$
$$x^2 + (y-1)^2 = 1^2$$

This is a circle centre at $(0,1)$ with radius 1.

In polar coord. $r = 2\sin\theta$.







$$
\begin{aligned}
\therefore V &= 4\iint_D \sqrt{y}\, dA \\
&= 4\int_0^{\frac{\pi}{2}}\int_0^{2\sin\theta} \sqrt{r\sin\theta}\, r dr d\theta \\
&= 4\int_0^{\frac{\pi}{2}} \sin\theta\, \frac{2}{5}\, r^{5/2}\Big|_0^{2\sin\theta} d\theta \\
&= \frac{32\sqrt{2}}{5}\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta \\
&= \frac{64\sqrt{2}}{15}.
\end{aligned}
$$

Qu. 37

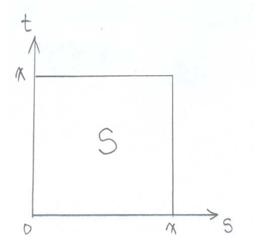
$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\, ds.$$

Thus

$$[\mathrm{Erf}(x)]^2 = \frac{4}{\pi}\iint_S e^{-(s^2+t^2)}\, dsdt,$$

where S is the square

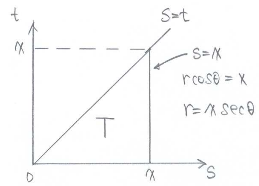
$$S = \{(s,t)\,\big|\, 0 \leqslant s \leqslant x,\ 0 \leqslant t \leqslant x\}.$$



By symmetry,

$$[\mathrm{Erf}(x)]^2 = \frac{8}{\pi}\iint_T e^{-(s^2+t^2)}\, dsdt,$$

where $T = \{(s,t)\,\big|\, 0 \leqslant s \leqslant x,\ 0 \leqslant t \leqslant s\}.$



In polar, we have

$$
\begin{aligned}
[\mathrm{Erf}(x)]^2 &= \frac{8}{\pi}\int_0^{\frac{\pi}{4}}\int_0^{x\sec\theta} e^{-r^2}\, r\, dr d\theta \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} (-e^{-r^2})\Big|_0^{x\sec\theta} d\theta \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \left(1 - e^{-x^2/\cos^2\theta}\right) d\theta.
\end{aligned}
$$

Since $\cos^2\theta \leqslant 1$, we have $e^{-x^2/\cos^2\theta} \leqslant e^{-x^2}$, so

$$
\begin{aligned}
[\mathrm{Erf}(x)]^2 &\geqslant \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \left(1 - e^{-x^2}\right) d\theta \\
&= 1 - e^{-x^2}
\end{aligned}
$$

$$\therefore\ \mathrm{Erf}(x) \geqslant \sqrt{1 - e^{-x^2}}.$$

Qu. The cone $z^2 = x^2 + y^2$ and the sphere $x^2 + (y-a)^2 + z^2 = a^2$ intersect where

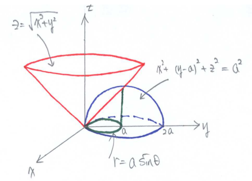
$$x^2 + y^2 = a^2 - x^2 - y^2 + 2ay - a^2$$

i.e. on the cylinder

$$x^2 + y^2 = ay$$
$$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2.$$

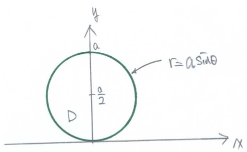
In polar

$$r = a\sin\theta.$$



The volume V lying outside the cone and inside the sphere lies on four octants; one quarter of it is in the first octant. To calculate V, we first calculate the volume V_1 under the cone and inside the cylinder, i.e.

$$\begin{aligned} V_1 &= \iint_D \sqrt{x^2+y^2}\, dA \\ &= \int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} r \cdot r\, dr d\theta \\ &= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta \\ &= \frac{2}{9} a^3. \end{aligned}$$



Then calculate

$$\begin{aligned} V_2 &= \text{the volume inside the sphere and inside the cylinder} \\ &= \frac{2}{9} a^3 (3\pi - 4) \quad \text{(see Ex. 14.4 Qu. 22)} \end{aligned}$$

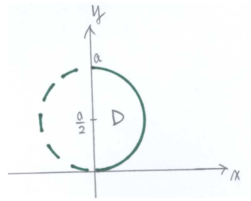
$\therefore$ The volume inside the sphere and outside the cylinder

$$V_3 = \frac{4}{3}\pi a^3 - V_2.$$

$\therefore$ The required volume V

$$\begin{aligned} V &= 4V_1 + V_3 \\ &= \frac{8}{9}a^3 + \frac{4}{3}\pi a^3 - \frac{2}{3}\pi a^3 - \frac{8}{9}a^3 \\ &= \frac{16}{9}a^3 + \frac{2\pi}{3}a^3. \end{aligned}$$

Alternatively, let V_4 be the volume inside the cone and outside the sphere, then

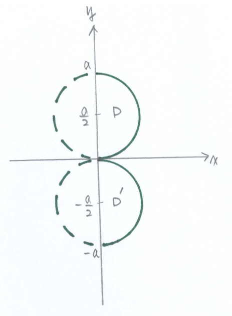


$$\begin{aligned} \frac{1}{4}V_4 &= \iint_D \int_{\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-(y-a)^2}} dz dA \\ &= \iint_D \left[\sqrt{a^2-x^2-(y-a)^2} - \sqrt{x^2+y^2}\right] dA \\ &= \iint_D \sqrt{a^2-x^2-(y-a)^2}\, dA - \iint_D \sqrt{x^2+y^2}\, dA \\ &= V_1 - V_2, \end{aligned}$$

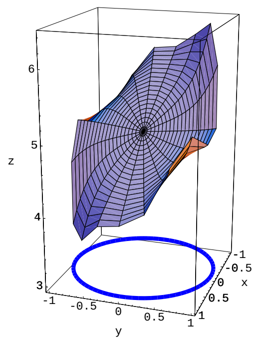
where

$$\begin{aligned} V_1 &= \iint_D \sqrt{a^2-x^2-(y-a)^2}\, dA, \quad \text{let} \quad y' = y - a \\ &= \iint_{D'} \sqrt{a^2-x^2-(y')^2}\, dA', \quad \text{where} \quad dA' = dxdy' \\ &= \frac{1}{4}\frac{2}{9}a^3(3\pi - 4) \quad \text{(from Ex. 14.4 Qu. 22)} \end{aligned}$$

$$\begin{aligned} V_2 &= \iint_D \sqrt{x^2+y^2}\, dA \\ &= \frac{2}{9}a^3 \quad \text{(from above).} \end{aligned}$$

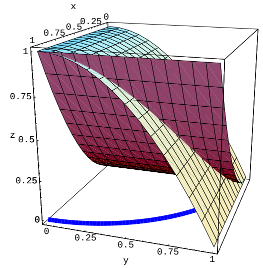


$$\begin{aligned} \therefore \quad V &= \text{(volume of the entire sphere)} - V_4 \\ &= \frac{16}{9}a^3 + \frac{2\pi}{3}a^3. \end{aligned}$$

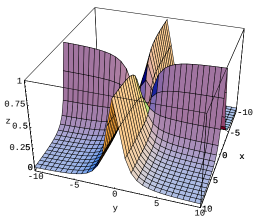


Ex. 14.1, Qu. 17

$f(x, y) = 4x^2y^3 - x + 5$

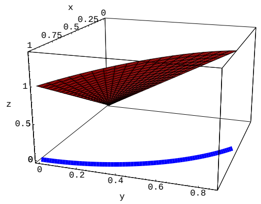


Ex. 14.2, Qu. 22



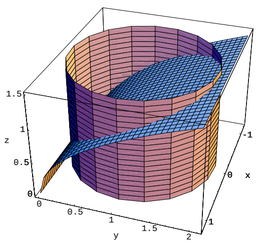
Ex. 14.3, Qu. 5

$$f(x, y) = \frac{x^2 + y^2}{(1 + x^2)(1 + y^2)}$$

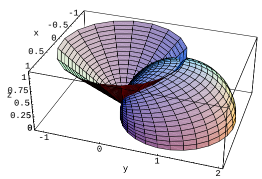


Ex. 14.4, Qu. 11

$f(x, y) = x + y$



Ex. 14.4, Qu. 26



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