Exercise 14.5

Qu. 4

$$\iiint_R x\,dV = \int_0^a\int_0^{b(1-\frac{x}{a})}\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} x\,dz$$

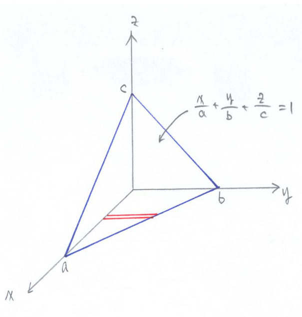
$$= c\int_0^a\int_0^{b(1-\frac{x}{a})} x(1-\frac{x}{a}-\frac{y}{b})\,dy$$

$$= c\int_0^a x\left[b(1-\frac{x}{a})^2-\frac{b^2}{2b}(1-\frac{x}{a})^2\right]dx$$

$$= \frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2 x\,dx$$

$$= \frac{bc}{2}\int_0^a\left(1-2\frac{x}{a}+\frac{x^2}{a^2}\right)x\,dx$$

$$= \frac{1}{24}a^2bc.$$

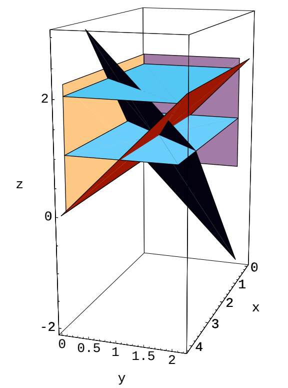


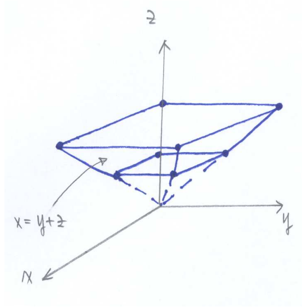
Qu. 11 Note that the region is

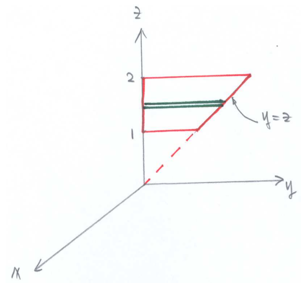
from $x = 0$ to $x = y + z$,

from $y = 0$ to $y = z$,

from $z = 1$ to $z = 2$.







Note also that for the plane $z = x - y$, when

$x = y = 0,\quad z = 0$

$y = 0,\quad z = x$

and $x = 0,\quad z = -y$

$$\therefore \iiint_R \frac{1}{(x+y+z)^3}\,dV = \int_1^2\int_0^z\int_0^{y+z}\frac{1}{(x+y+z)^3}\,dx\,dy\,dz$$

$$= \int_1^2\int_0^z \left.\frac{-1}{2(x+y+z)^2}\right|_0^{y+z} dy\,dz$$

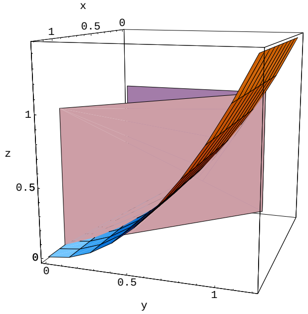
$$= \frac{3}{8}\int_1^2\int_0^z \frac{1}{(y+z)^2}\,dy\,dz$$

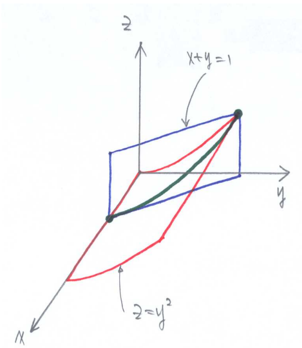
$$= \frac{3}{8}\int_1^2 \left.\frac{-1}{(y+z)}\right|_0^z dz$$

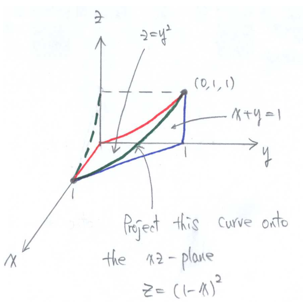
$$= \frac{3}{16}\int_1^2 \frac{1}{z}\,dz$$

$$= \frac{3}{16}\ln 2.$$

Qu. 16







$$\iiint_R f(x,y,z)\,dV = \int_0^1\int_0^{1-x}\int_0^{y^2} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1\int_{\sqrt{z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{(1-x)^2}\int_{\sqrt{z}}^{1-x} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{1-\sqrt{z}}\int_{\sqrt{z}}^{1-x} f(x,y,z)\,dy\,dx\,dz.$$

Qu. 19 Note that the integration region is

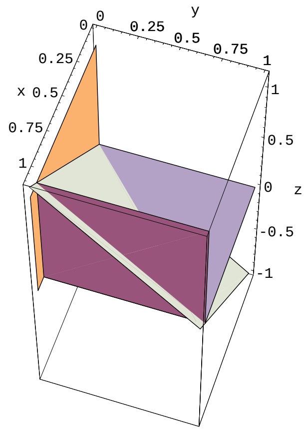
from $y=0$ to $y=x-z$,

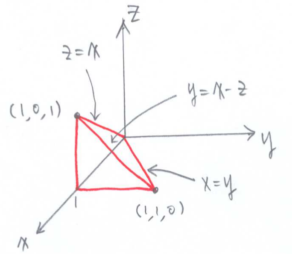
from $x=z$ to $x=1$,

from $z=0$ to $z=1$.

Note also that for the plane $y=x-z$, we have

$$x=z=0, \quad y=0$$
$$x=0, \quad y=z$$
$$y=0, \quad z=x$$
$$z=0, \quad y=x$$





$$\therefore \int_0^1\int_z^1\int_0^{x-z} f(x,y,z)\,dy\,dx\,dz = \iiint_R f(x,y,z)\,dV \quad (R \text{ is the tetrahedron in the figure})$$

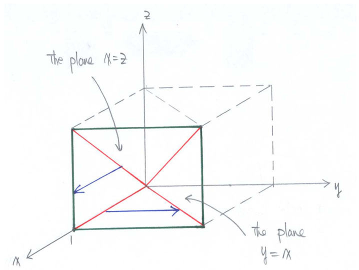
$$= \int_0^1\int_0^{x}\int_0^{x-y} f(x,y,z)\,dz\,dy\,dx.$$

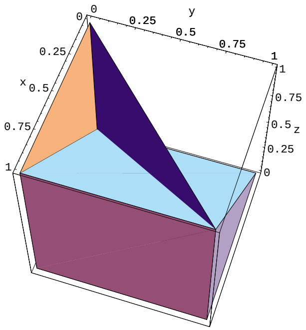
Qu. 27 Note that the integration region is

from $y = 0$ to $y = x$,

from $x = z$ to $x = 1$,

from $z = 0$ to $z = 1$.





$$\int_0^1 \int_z^1 \int_0^x e^{x^3}\, dy\, dx\, dz$$

$$= \iiint_R e^{x^3}\, dV \quad (R \text{ is the pyramid in the figure with vertex (0,0,0), rectangle base } x = 1)$$

$$= \int_0^1 \int_0^x \int_0^x e^{x^3}\, dz\, dy\, dx$$

$$= \int_0^1 x^2 e^{x^3}\, dx$$

$$= \frac{1}{3}(e-1).$$

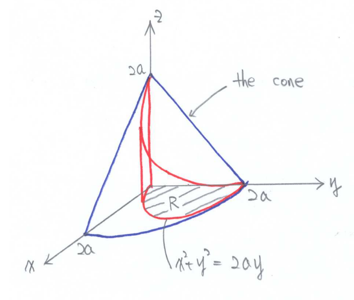
Exercise 14.6

Qu. 19 One half of the required volume V lies in the first octant, inside the cylinder

$$x^2 + y^2 = 2ay \quad \Rightarrow \quad x^2 + (y-a)^2 = a^2$$

In polar,

$$r = 2a\ \sin\theta$$



$$V = 2 \iint_R \int_0^{2a-\sqrt{x^2+y^2}} dz\, dA$$

$$= 2 \iint_R (2a - \sqrt{x^2+y^2})\, dA$$

$$= 2 \int_0^{\pi/2} \int_0^{2a\sin\theta} (2a - r) r\, dr\, d\theta \qquad \text{(in polar)}$$

$$= 2a \int_0^{\pi/2} 4a^2 \sin^2\theta\, d\theta - \frac{2}{3} \int_0^{\pi/2} 8a^3 \sin^3\theta\, d\theta$$

$$= 4a^3 \int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta + \frac{16}{3} a^3 \int_0^{\pi/2} (1 - \cos^2\theta)\, d(\cos\theta)$$

$$= 2\pi a^3 - \frac{32}{9} a^3.$$

Qu. 25

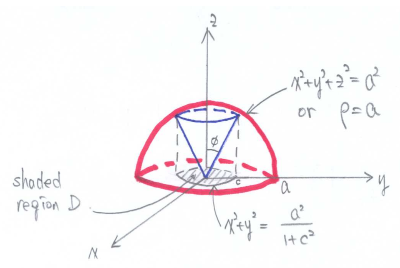
$$\iiint_B (x^2 + y^2)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^a (\rho^2 \sin^2\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi} \sin^3\phi\, d\phi \times \int_0^a \rho^4\, d\rho$$

$$= 2\pi \left(\frac{4}{3}\right) \frac{a^5}{5}$$

$$= \frac{8}{15} \pi a^5.$$

Qu. 30 In spherical coord



$$\iiint_R (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\tan^{-1}(\frac{1}{c})}\int_0^a (\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{2\pi a^5}{5}\int_0^{\tan^{-1}(\frac{1}{c})} \sin\phi(1-\cos^2\phi)\,d\phi$$

$$= \frac{2\pi a^5}{5}\int_{\frac{c}{\sqrt{c^2+1}}}^{1} (1-u^2)\,du, \qquad \text{let} \quad u=\cos\phi, \quad \text{then} \quad du = -\sin\phi\,d\phi$$

$$= \frac{2\pi a^5}{5}\left(u-\frac{u^3}{3}\right)\Bigg|_{\frac{c}{\sqrt{c^2+1}}}^{1}$$

$$= \frac{2\pi a^5}{5}\left[\frac{2}{3} - \frac{c}{\sqrt{c^2+1}} + \frac{c^3}{3(c^2+1)^{3/2}}\right].$$

In cylindrical coord:

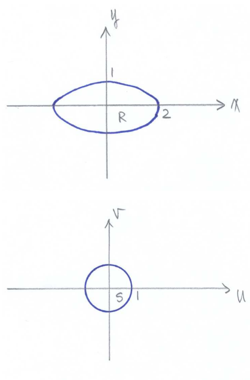
$$\iiint_R (x^2+y^2)\,dV = \iint_R\int_{c\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} dz\,dA$$

$$= \iint_R \left(\sqrt{a^2-x^2-y^2} - c\sqrt{x^2+y^2}\right)dA$$

$$= \int_0^{2\pi}\int_0^{\frac{a}{\sqrt{1+c^2}}} \left[\sqrt{a^2-r^2} - cr\right] r\,dr\,d\theta = \text{Ans.}$$

Exercise 14.7

Qu. 2



$$z = \frac{3x-4y}{5}, \quad \frac{\partial z}{\partial x} = \frac{3}{5}, \quad \frac{\partial z}{\partial y} = \frac{4}{5}$$

$$dS = \sqrt{1+(z_x)^2+(z_y)^2}\,dA$$

$$= \sqrt{1+\frac{3^2+4^2}{5^2}}\,dA = \sqrt{2}\,dA.$$

$$S = \iint_R \sqrt{2}\,dA$$

Let $x = 2u, \quad y = v, \quad$ then $\quad u^2+v^2=1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2.$$

$$\therefore \quad \sqrt{2}\iint_R dA = \sqrt{2}\iint_S 2\,dA, \quad \text{where } S \text{ is a circle with radius 1 in } uv\text{-plane}$$

$$= 2\sqrt{2}\pi.$$

Qu. 6

$$z = 1-x^2-y^2, \quad \frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y$$

$$dS = \sqrt{1+(z_x)^2+(z_y)^2}\,dA = \sqrt{1+4x^2+4y^2}\,dA$$

$$\therefore S = \iint_R \sqrt{1+4(x^2+y^2)}\,dA$$

$$= \int_0^{\pi/2}\int_0^1 \sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$= \frac{\pi}{2}\frac{1}{8}\frac{2(1+4r^2)^{3/2}}{3}\Bigg|_0^1$$

$$= \frac{\pi(5\sqrt{5}-1)}{24}.$$

Qu. 10 The area elements on $z = 2xy$ and $z = x^2 + y^2$ are

$$dS_1 = \sqrt{1 + (2y)^2 + (2x)^2}\, dA = \sqrt{1 + 4x^2 + 4y^2}\, dA$$
$$dS_2 = \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \sqrt{1 + 4x^2 + 4y^2}\, dA$$

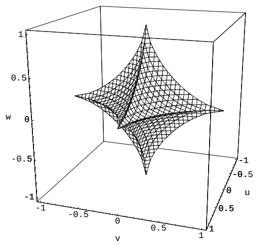
Since these elements are equal, the area of the parts of both surfaces defined over any region of the xy-plane will be equal.

Qu. Let $x = u^3$, $y = v^3$, $z = w^3$, then the region R bounded by the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ gets mapped to the ball B bounded by $u^2 + v^2 + w^2 = a^{2/3}$. Assume that $a > 0$. Since

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 27u^2v^2w^2,$$

the volume of R is

$$V = 27 \iiint_B u^2v^2w^2\, du\, dv\, dw.$$



In spherical coord,

$$dudvdw = (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi),$$

we have

$$\begin{aligned}
V &= 27 \int_0^{2\pi}\int_0^{\pi}\int_0^{a^{1/3}} \cos^2\theta \sin^2\theta \sin^5\phi \cos^2\phi\, \rho^8\, d\rho\, d\phi\, d\theta \\
&= 3a^3 \int_0^{2\pi} \frac{\sin^2(2\theta)}{4} \int_0^{\pi} (1 - \cos^2\phi)^2 \cos^2\phi \sin\phi\, d\phi\, d\theta \\
&= 3a^3 \int_0^{2\pi} \frac{1 - \cos(4\theta)}{8}\, d\theta \cdot \int_{-1}^{1} (1 - t^2)^2 t^2\, dt, \qquad \text{let} \quad t = \cos\phi \\
&= \frac{3a^3}{8}(2\pi)2 \int_0^1 (t^2 - 2t^4 + t^6)\, dt \\
&= \frac{4}{35}\pi a^3.
\end{aligned}$$