Exercise 15.1

Qu. 7

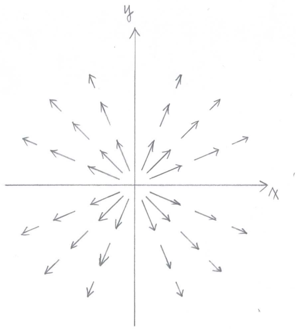
$$
\begin{aligned}
\mathbf{F} &= \nabla \ln(x^2+y^2) \\
&= \frac{1}{x^2+y^2}(2x\,\mathbf{i} + 2y\,\mathbf{j})
\end{aligned}
$$

Note that $\|\mathbf{F}\| = 2\Big/\sqrt{x^2+y^2}$, i.e. the length of the vector decreases like $\dfrac{1}{r}$.

The field lines satisfy

$$
\begin{aligned}
\frac{dx}{x} &= \frac{dy}{y} \\
\ln x &= \ln y + c_1
\end{aligned}
$$

Thus they are radial lines $y = cx$ and $x = 0$.



Qu. 9

$$\mathbf{v}(x,y,z) = y\,\mathbf{i} - y\,\mathbf{j} - y\,\mathbf{k}.$$

The streamlines satisfy

$$dx = -dy = -dz.$$

Thus

$$y + x = c_1 \quad \text{and} \quad z + x = c_2.$$

Let $x = t$, then $y = c_1 - t$, $z = c_2 - t$, then

$$
\begin{aligned}
\mathbf{r}(t) &= t\,\mathbf{i} + (c_1 - t)\,\mathbf{j} + (c_2 - t)\,\mathbf{k} \\
&= (0, c_1, c_2) + t(\mathbf{i} - \mathbf{j} - \mathbf{k})
\end{aligned}
$$

i.e. the streamlines are straight lines parallel to $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

Qu. 16

$$\mathbf{v}(x,y) = x\,\mathbf{i} + (x+y)\,\mathbf{j}$$

The field lines satisfy

$$
\begin{aligned}
\frac{dx}{x} &= \frac{dy}{x+y} \\
\frac{dy}{dx} &= \frac{x+y}{x} \\
\text{Let} \qquad y &= xv(x) \\
\frac{dy}{dx} &= v + x\frac{dv}{dx} \\
v + x\frac{dv}{dx} &= \frac{x(1+v)}{x} = 1 + v \\
\therefore\ \frac{dv}{dx} &= \frac{1}{x} \\
v(x) &= \ln(x) + c
\end{aligned}
$$

$\therefore$ The field lines have equations $y = x\,\ln|x| + cx$.

Exercise15.3

Qu. 2

$$
\begin{aligned}
C: x = t\cos t,\ y = t\sin t,\ z = t, \qquad 0 \leqslant t \leqslant 2\pi \\
ds &= [(x'(t))^2 + (y'(t))^2 + (z'(t))^2]^{1/2} \\
&= [(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1]^{1/2}\, dt \\
&= \sqrt{2+t^2}\, dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_C z\, dS &= \int_0^{2\pi} t\sqrt{2+t^2}\, dt \\
&= \frac{1}{2}\frac{2}{3}(2+t^2)^{3/2}\Big|_0^{2\pi} \\
&= \frac{1}{3}[(2+4\pi^2)^{3/2} - 2^{3/2}].
\end{aligned}
$$

Qu. 8 The curve C of intersection of $x^2+z^2=1$ and $y=x^2$ is:
let $x=\cos t$, $z=\sin t$ and $y=\cos^2 t$, i.e. in parametrized form

$$
\mathbf{r}(t) = \cos t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \sin t\,\mathbf{k}, \qquad 0 \leqslant t \leqslant 2\pi
$$

Thus

$$
\begin{aligned}
ds &= [\sin^2 t + 4\sin^2 t\cos^2 t + \cos^2 t]^{\frac{1}{2}}\, dt \\
&= \sqrt{1+\sin^2 2t}\, dt \\
\therefore \int_C \sqrt{1+4x^2z^2}\, ds &= \int_0^{2\pi} \sqrt{1+4\cos^2 t\cdot\sin^2 t}\ \sqrt{1+\sin^2 2t}\, dt \\
&= \int_0^{2\pi} (1+\sin^2 2t)\, dt \\
&= \int_0^{2\pi} (1+\frac{1-\cos 4t}{2})\, dt \\
&= \frac{3}{2}\cdot 2\pi \\
&= 3\pi.
\end{aligned}
$$

Qu. 15 The parabola $z^2 = x^2+y^2$, $x+z=1$, can be parametrized in terms of $y=t$ since

$$
\begin{aligned}
(1-x)^2 = z^2 = x^2+y^2 = x^2+t^2 \\
\Rightarrow \quad 1-2x = t^2 \quad \Rightarrow \quad x = \frac{(1-t^2)}{2}
\end{aligned}
$$

and

$$
z = 1-x = \frac{(1-t^2)}{2}.
$$

Thus

$$
\begin{aligned}
ds &= \sqrt{t^2+1+t^2}\, dt \\
&= \sqrt{1+2t^2}\, dt
\end{aligned}
$$

and

$$
\begin{aligned}
\int_C \frac{1}{(2y^2+1)^{3/2}}\, ds &= \int_{-\infty}^{\infty} \frac{\sqrt{1+2t^2}}{(2t^2+1)^{3/2}}\, dt \\
&= 2\int_0^{\infty} \frac{1}{1+2t^2}\, dt \\
&= \sqrt{2}\tan^{-1}(\sqrt{2}t)\Big|_0^{\infty} \\
&= \sqrt{2}\ \frac{\pi}{2} \\
&= \frac{\sqrt{2}}{2}\pi.
\end{aligned}
$$

Exercise 15.4

Qu. 3

$$\mathbf{F}(\mathbf{r}) = z\,\mathbf{i} - y\,\mathbf{j} + 2x\,\mathbf{k}$$

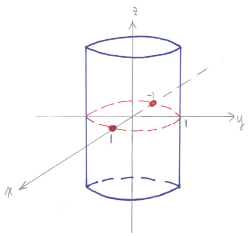
The curve C:

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\mathbf{r_0} + \mathbf{t}\mathbf{r_1} \qquad \mathbf{0 \leqslant t \leqslant 1}\\ &= t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}\end{aligned}$$

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t + t - t)\,dt = \frac{t^2}{2}\bigg|_0^1 = \frac{1}{2}.$$

Qu. 5 The curve of intersection can be parametrized as

$$x = \cos t$$
$$y = \sin t$$
$$z = \sin t.$$



At the points:

$$(-1,0,0), \quad \text{i.e.} \quad x = -1, \quad t = -\pi \text{ or } \pi$$
$$(1,0,0), \quad \text{i.e.} \quad x = 1, \quad t = 0,\ 2\pi$$

$$\begin{aligned}\therefore \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{-\pi}^0 (\sin^2 t\,\mathbf{i} + \cos t \sin t\,\mathbf{j} + \cos t \sin t\,\mathbf{k})\cdot(-\sin t, \cos t, \cos t)\,dt\\ &= \int_{-\pi}^0 (-\sin^3 t + \cos^2 t \sin t + \cos^2 t \sin t)\,dt\\ &= \int_{-\pi}^0 (3\cos^2 t - 1)\,d(\cos t)\\ &= [\cos^3 t - \cos t]\Big|_{-\pi}^0\\ &= [-1-(-1)] - [1-1] = 0.\end{aligned}$$

Alternatively, note that $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k} = \nabla(xyz)$ i.e. $\mathbf{F}$ is a conservative vector field in an open simply-connected domain.

$\therefore C$: a curve from $(-1,0,0)$ to $(1,0,0)$.

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = xyz\Big|_{(-1,0,0)}^{(1,0,0)} = 0 - 0 = 0.$$

Since $\mathbf{F}$ is conservative, it does not matter what curve.

Qu. 11

$$\mathbf{F} = Ax\,\ln z\,\mathbf{i} + By^2 z\,\mathbf{j} + (\frac{x^2}{z} + y^3)\,\mathbf{k} \text{ is conservative iff}$$

$$\mathbf{F} = \nabla\phi \quad \text{or} \quad \nabla\times\mathbf{F} = \mathbf{0}.$$

$$\text{i.e.} \quad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ Ax\,\ln z & By^2 z & \dfrac{x^2}{2} + y^3\end{vmatrix}$$

$$= (3y^2 - By^2)\,\mathbf{i} - (\frac{2x}{z} - \frac{Ax}{z})\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}.$$

i.e. $B = 3$ and $A = 2$. $\therefore \mathbf{F}(\mathbf{r}) = 2x\,\ln z\,\mathbf{i} + 3y^2 z\,\mathbf{j} + (\frac{x^2}{z} + y^3)\,\mathbf{k}$. Hence

$$\frac{\partial\phi}{\partial x} = 2x\,\ln z \tag{1}$$

$$\frac{\partial\phi}{\partial y} = 3y^2 z \tag{2}$$

$$\frac{\partial\phi}{\partial z} = \frac{x^2}{z} + y^3. \tag{3}$$

From (1), we have

$$\phi = x^2\,\ln z + g(y,z)$$
$$\phi_y = g_y(y,z).$$

From (2), we have

$$g_y = 3y^2 z \quad \Rightarrow \quad g = y^3 z + f(z)$$
$$\phi = x^2\,\ln z + y^3 z + f(z)$$
$$\phi_z = \frac{x^2}{z} + y^3 + f'(z).$$

From (3), we have

$$f'(z) = 0 \quad \Rightarrow \quad f(z) = \text{const}$$
$$\therefore \phi(x,y,z) = x^2\,\ln z + y^3 z + \text{const}.$$

If C is the straight line from (1,1,1) to (2,1,2), then

$$\mathbf{r}(t) = (1-t)(1,1,1) + t(2,1,2) = (t+1, 1, t+1), \qquad 0 \leqslant t \leqslant 1$$

$$\begin{aligned}\therefore \int_C 2x\,\ln z\,dx + 2y^2 z\,dy + y^3\,dz &= \int_C \nabla\phi\cdot\, d\mathbf{r} - \int_C (y^2 z\,dy + \frac{x^2}{z}\,dz)\\ &= (x^2\,\ln z + y^3 z)\Big|_{(1,1,1)}^{(2,1,2)} - \int_0^1 [(t+1)(0) + (t+1)]\,dt\\ &= 4\ln 2 + 2 - 1 + (\frac{t^2}{2} + t)\Big|_0^1\\ &= 4\ln 2 - \frac{1}{2}.\end{aligned}$$

Qu. 13

$$
\begin{aligned}
\mathrm{I} &= \int_C (2x\sin(\pi y) - e^z)\,dx + (\pi x^2\cos(\pi y) - 3e^z)\,dy - xe^z\,dz \\
&= \int_C \mathbf{F}\cdot d\mathbf{r},
\end{aligned}
$$

where

$$\mathbf{F} = (2x\sin(\pi y) - e^z, -\pi x^2\cos(\pi y) - 3e^z, -xe^z).$$

By observation,

$$
\begin{aligned}
\nabla\phi &= \nabla(x^2\sin(\pi y) - xe^z) \\
&= (2x\sin(\pi y) - e^z, \pi x^2\cos(\pi y), -xe^z) \\
&= \mathbf{F} + (0, 3e^z, 0) \\
\therefore\ \mathrm{I} &= \int_C \nabla\phi\cdot d\mathbf{r} - 3\int_C e^z\,dy \\
&= [x^2\sin(\pi y) - xe^z]\Big|_{(0,0,0)}^{(1,1,\ln 2)} - 3\int_0^1 e^{\ln(1+x)}\,dx \\
&= -\frac{13}{2}.
\end{aligned}
$$

Qu. 14 (a) $S = \{(x,y)\,|\,x>0,\ y\geqslant 0\}$ is a simply connected domain.

(b) $S = \{(x,y)\,|\,x=0,\ y\geqslant 0\}$ is not a domain. It has empty interior.

(c) $S = \{(x,y)\,|\,x\neq 0,\ y>0\}$ is a domain but is not connected. There is no path in S from $(-1,1)$ to $(1,1)$.

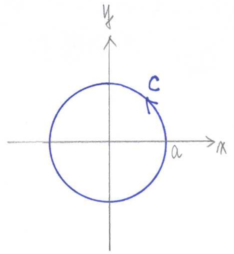
(d) $S = \{(x,y,z)\,|\,x^2>1\}$ is a domain but is not connected. There is no path in S from $(-2,0,0)$ to $(2,0,0)$.

(e) $S = \{(x,y,z)\,|\,x^2+y^2>1\}$ is a connected domain but is not simply connected. The circle $x^2+y^2=2,\ z=0$ lies in S, but cannot be shrunk through S to a point since it surrounds the cylinder $x^2+y^2\leqslant 1$ which is outside S.

(f) $S = \{(x,y,z)\,|\,x^2+y^2+z^2>1\}$ is a simply connected domain even though it has a ball-shaped "hole" in it.

Qu. 22 (a) $C: x = a\cos t,\ y = a\sin t, \quad 0\leqslant t\leqslant 2\pi$

$$\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2\cos^2 t + a^2\sin^2 t}{a^2\cos^2 t + a^2\sin^2 t}\,dt = 1.$$



(b) $C = C_1 + C_2 + C_3 + C_4$

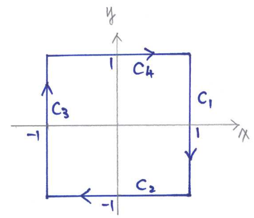
On $C_1: x = 1,\ dx = 0, \quad -1\leqslant y\leqslant 1$

On $C_2: y = -1,\ dy = 0, \quad -1\leqslant x\leqslant 1$

On $C_3: x = -1,\ dx = 0, \quad -1\leqslant y\leqslant 1$

On $C_4: y = 1,\ dy = 0, \quad -1\leqslant x\leqslant 1$.

$$
\begin{aligned}
\therefore\ \frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} &= \frac{1}{2\pi}\left[\int_{-1}^1\frac{dy}{1+y^2} + \int_{-1}^1\frac{dx}{x^2+1} + \int_{-1}^1\frac{-dy}{1+y^2} + \int_{-1}^1\frac{-dx}{x^2+1}\right] \\
&= -\frac{2}{\pi}\int_{-1}^1\frac{1}{1+t^2}\,dt \\
&= -\frac{2}{\pi}\tan^{-1}(t)\Big|_{-1}^{1} \\
&= -\frac{2}{\pi}(\frac{\pi}{4}+\frac{\pi}{4}) = 1.
\end{aligned}
$$



(c) $C = C_1 + C_2 + C_3 + C_4$

On $C_1: y = 0,\ dy = 0, \quad 1\leqslant x\leqslant 2$

On $C_2: x = 2\cos t,\ y = 2\sin t, \quad 0\leqslant t\leqslant \pi$

On $C_3: y = 0,\ dy = 0, \quad -2\leqslant x\leqslant -1$

On $C_4: x = \cos t,\ y = \sin t, \quad t$ goes from π to 0

$$
\begin{aligned}
\therefore\ \frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} &= \frac{1}{2\pi}\left[0 + \int_0^\pi \frac{4\cos^2 t + 4\sin^2 t}{4\cos^2 t + 4\sin^2 t}\,dt + 0 + \int_\pi^0\frac{\cos^2 t + \sin^2 t}{\cos^2 t + \sin^2 t}\,dt\right] \\
&= \frac{1}{2\pi}(\pi - \pi) = 0.
\end{aligned}
$$