Exercise 15.2

Qu. 5

$$\begin{aligned}
\mathbf{F}(\mathbf{r}) &= (2xy - z^2)\,\mathbf{i} + (2yz + x^2)\,\mathbf{j} - (2zx - y^2)\,\mathbf{k} \\
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy - z^2 & 2yz + x^2 & y^2 - 2xz \end{vmatrix} \\
&= (2x - 2x)\,\mathbf{i} - (-2z + 2z)\,\mathbf{j} + (2x - 2x)\,\mathbf{k} \\
&= \mathbf{0}.
\end{aligned}$$

$\therefore \mathbf{F}$ is a conservative vector field in an open simply-connected region ($\mathbb{R}^3$), i.e. $\mathbf{F} = \nabla\phi$.

Therefore

$$\frac{\partial \phi}{\partial x} = 2xy - z^2 \quad (1)$$

$$\frac{\partial \phi}{\partial y} = 2yz + x^2 \quad (2)$$

$$\frac{\partial \phi}{\partial z} = y^2 - 2xz. \quad (3)$$

From (1), we have

$$\begin{aligned}
\phi &= x^2y - xz^2 + g(y, z) \\
\phi_y &= x^2 + g_y.
\end{aligned}$$

From (2), we have

$$\begin{aligned}
g_y &= 2yz \qquad \Rightarrow \qquad g(x, y) = y^2z + f(z) \\
\therefore \phi &= x^2y - xz^2 + y^2z + f(z) \\
\phi_z &= -2xz + y^2 + f'(z).
\end{aligned}$$

From (3), we have

$$f'(z) = 0 \qquad \Rightarrow \qquad f(z) = c. \qquad \text{(constant)}$$

$\therefore \phi(x, y, z) = x^2y - xz^2 + y^2z + c$ is a scalar potential for $\mathbf{F}$, $\mathbf{F}$ is conservative on $\mathbb{R}^3$.

Qu. 9

$$\begin{aligned}
\mathbf{F} &= \frac{2x}{z}\,\mathbf{i} + \frac{2y}{z}\,\mathbf{j} - \frac{x^2 + y^2}{z^2}\,\mathbf{k} \\
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{2x}{z} & \dfrac{2y}{z} & -\dfrac{x^2 + y^2}{z^2} \end{vmatrix} \\
&= \left(-\frac{2y}{z^2} + \frac{2y}{z^2}\right)\mathbf{i} - \left(-\frac{2x}{z^2} + \frac{2x}{z^2}\right)\mathbf{j} + 0\,\mathbf{k} \\
&= \mathbf{0}.
\end{aligned}$$

Therefore, $\mathbf{F}$ may be conservative in $\mathbb{R}^3$ except on the plane $z = 0$ where it is not defined. If $\mathbf{F} = \nabla\phi$, then

$$\frac{\partial \phi}{\partial x} = \frac{2x}{z} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \frac{2y}{z} \quad (2)$$

$$\frac{\partial \phi}{\partial z} = -\frac{x^2 + y^2}{z^2}. \quad (3)$$

From (1), we have

$$\begin{aligned}
\phi &= \frac{x^2}{z} + g(y, z) \\
\phi_y &= g_y.
\end{aligned}$$

From (2), we have

$$\begin{aligned}
g_y &= \frac{2y}{z} \qquad \Rightarrow \qquad g(x, y) = \frac{y^2}{z} + f(z) \\
\therefore\ \phi &= \frac{x^2 + y^2}{z} + f(z) \\
\phi_z &= -\frac{x^2 + y^2}{z^2} + f'(z).
\end{aligned}$$

From (3), we have

$$f'(z) = 0 \qquad \Rightarrow \qquad f(z) = c.$$

$$\therefore \phi(x, y, z) = -\frac{x^2 + y^2}{z} + c,$$

is a potential for $\mathbf{F}$, and $\mathbf{F}$ is conservative in $\mathbb{R}^3$ except on the plane $z = 0$.

The equipotential surfaces have equations $\phi = c$, i.e.

$$\frac{x^2+y^2}{z} = c \qquad \text{or} \qquad cz = x^2+y^2.$$

Thus the equipotential surfaces are circular paraboloids.

The field lines of $\mathbf{F}$ satisfy

$$\frac{dx}{2x/z} = \frac{dy}{2y/z} = \frac{dz}{-(x^2+y^2)/z^2}$$

From the first equation, $\dfrac{dx}{x} = \dfrac{dy}{y}$, so $y = Ax$ for an arbitrary constant A. Therefore

$$\frac{dx}{2x} = \frac{z\,dz}{-(x^2+y^2)} = \frac{z\,dz}{-x^2(1+A^2)}$$

so $-(1+A^2)x\,dx = 2z\,dz$, hence

$$\frac{1+A^2}{2}x^2 + z^2 = \frac{B}{2} \qquad \text{or} \qquad x^2+y^2+2z^2 = B, \text{ where } B \text{ is a second arbitrary constant.}$$

The field lines of $\mathbf{F}$ are ellipses in which the vectical planes containing the z-axis intersect the ellipsoids $x^2+y^2+2z^2 = B$. These ellipses are orthogonal to all the equipotential surfaces of $\mathbf{F}$.

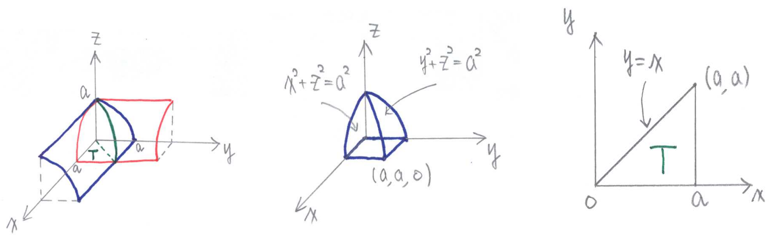
Qu. 7

$$\begin{aligned}
\phi(\mathbf{r}) &= \frac{1}{\|\mathbf{r}-\mathbf{r_0}\|^2} \quad \text{if} \quad \mathbf{F} = \nabla\phi, \quad \text{then} \\
[\nabla\phi]_i &= \partial_i[(r_j - r_{0j})\cdot(r_j - r_{0j})]^{-1} \\
&= -1[(r_j - r_{0j})(r_j - r_{0j})]^{-2}\,\partial_i[(r_j - r_{0j})(r_j - r_{0j})] \\
&= -\frac{1}{\|\mathbf{r}-\mathbf{r_0}\|^4}[(r_j - r_{0j})\delta_{ij} + (r_j - r_{0j})\delta_{ij}] \\
&= -\frac{2(r_i - r_{0i})}{\|\mathbf{r}-\mathbf{r_0}\|^4} \\
\therefore \mathbf{F} = \nabla\phi &= -2\frac{\mathbf{r}-\mathbf{r_0}}{\|\mathbf{r}-\mathbf{r_0}\|^4}.
\end{aligned}$$

Exercise 15.5

Qu. 10 One-eighth of the required area in the first octant above the triangle T with vertices (0,0,0), (a,0,0) and (a,a,0). And

$$\begin{aligned}
z &= \sqrt{a^2 - x^2} \\
z_y &= 0, \quad z_x = \frac{x}{\sqrt{a^2-x^2}} \\
\therefore S &= 8\iint_S dS \\
&= 8\iint_T \sqrt{1+(z_x)^2+(z_y)^2}\,dA \\
&= 8\iint_T \frac{a}{\sqrt{a^2-x^2}}\,dA \\
&= 8a\int_0^a\int_0^x \frac{1}{\sqrt{a^2-x^2}}\,dy\,dx \\
&= 8a\int_0^a \frac{x}{\sqrt{a^2-x^2}}\,dx \\
&= -8a\sqrt{a^2-x^2}\Big|_0^a \\
&= 8a^2.
\end{aligned}$$



Qu. 14 The intersection of the plane $z = 1 + y$ and the cone $z = \sqrt{2(x^2+y^2)}$ has projection onto the xy-plane the elliptic disk E bounded by

$$(1+y)^2 = 2(x^2+y^2)$$
$$1 + 2y + y^2 = 2x^2 + 2y^2$$

$$E : x^2 + \frac{(y-1)^2}{2} = 1$$

Note that E has area $A = \pi(1)(\sqrt{2})$ and centroid (0,1) and $z_x = \dfrac{\sqrt{2}x}{\sqrt{x^2+y^2}}$, $z_y = \dfrac{\sqrt{2}y}{\sqrt{x^2+y^2}}$.

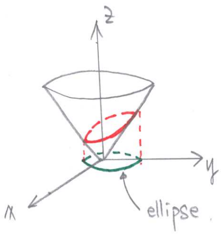
$$\begin{aligned} dS &= \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA \\ &= \sqrt{1 + \frac{2(x^2+y^2)}{x^2+y^2}}\, dA \\ &= \sqrt{3}\, dA \end{aligned}$$

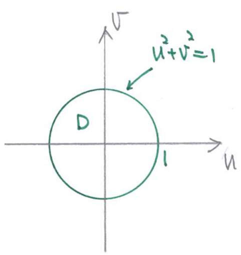
$$\therefore \iint_S y\, dS = \sqrt{3} \iint_E y\, dA.$$

Let $x = u$, $y = 1 + \sqrt{2}v$, then

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{vmatrix} = \sqrt{2}.$$

$$\begin{aligned} \therefore\ \sqrt{3}\iint_E y\, dA &= \sqrt{3}\iint_D (1+\sqrt{2}v)\sqrt{2}\, dA \\ &= \sqrt{3}\iint_D \sqrt{2}\, dA + 2\sqrt{3}\iint_D v\, dA \\ &= \sqrt{6}\pi + 2\sqrt{3}\int_0^{2\pi}\int_0^1 r\sin\theta\, r\, dr\, d\theta \\ &= \sqrt{6}\pi. \end{aligned}$$





Exercise 15.6

Qu. 1 $\mathbf{F} = x\,\mathbf{i} + z\,\mathbf{j}$

The surface S of the tetrahedron has four faces :

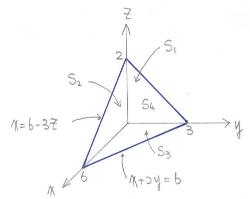
On $S_1 : x = 0,\ \widehat{\mathbf{n}} = -\mathbf{i},\ \mathbf{F}\cdot\widehat{\mathbf{n}} = 0$

On $S_2 : y = 0,\ \widehat{\mathbf{n}} = -\mathbf{j},\ \mathbf{F}\cdot\widehat{\mathbf{n}} = -z,\ dS = dx\,dz$

On $S_3 : z = 0,\ \widehat{\mathbf{n}} = -\mathbf{k},\ \mathbf{F}\cdot\widehat{\mathbf{n}} = 0$

On $S_4 :\ x + 2y + 3z = 6,\ \widehat{\mathbf{n}} = \dfrac{(\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k})}{\sqrt{14}},\ \mathbf{F}\cdot\widehat{\mathbf{n}} = \dfrac{(x+2z)}{\sqrt{14}}$

and $z_x = -\dfrac{1}{3},\ z_y = -\dfrac{2}{3},\ dS = \dfrac{\sqrt{14}}{3}\, dA_{xy}$



we have

$$\iint_{S_1} \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS = 0$$

$$\iint_{S_2} \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS = \int_0^2\int_0^{6-3z} z\, dx\, dz = -\int_0^2 (6z - 3z^2)\, dz = -4$$

$$\iint_{S_3} \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS = 0$$

$$\begin{aligned} \iint_{S_4} \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS &= \frac{1}{\sqrt{14}}\iint_{S_4} (x+2z)\, dS \\ &= \frac{1}{\sqrt{14}}\frac{\sqrt{14}}{3}\iint_{S_3}\left[x + 2(\frac{6-x-2y}{3})\right] dA_{xy} \\ &= \frac{1}{3}\iint_{S_3} (4 + \frac{1}{3}x - \frac{4}{3}y)\, dA_{xy} \\ &= \frac{1}{3}\int_0^3\int_0^{6-2y} (4 + \frac{1}{3}x - \frac{4}{3}y)\, dx\, dy \\ &= \frac{1}{3}\int_0^3 [4x + \frac{1}{6}x^2 - \frac{4}{3}xy]\Big|_0^{6-2y} dy \\ &= \frac{1}{3}\int_0^3 [30 - 20y + \frac{10}{3}y^2]\, dy \\ &= \frac{1}{3}[30y - 10y^2 + \frac{10}{3}y^3]\Big|_0^3 \\ &= 10. \end{aligned}$$

$\therefore$ The flux of $\mathbf{F}$ out of the tetrahedron is

$$\iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\, dS = 0 - 4 + 0 + 10 = 6.$$

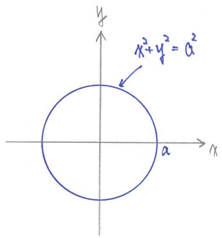
Qu. 6 For $z = x^2 - y^2,\ z_x = 2x,\ z_y = -2y$

$$\therefore dS = \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA_{xy}$$
$$= \sqrt{1 + 4x^2 + 4y^2}\, dA_{xy}.$$

Also, let $g(x, y, z) = z - x^2 + y^2 = 0$ this is a level surface in 3D.

$$\therefore \widehat{\mathbf{n}} = \nabla g = (-2x, 2y, 1) \quad \text{which is upward and}$$
$$\widehat{\mathbf{n}} = \frac{(2x, -2y, -1)}{\sqrt{1 + 4x^2 + 4y^2}}$$

$$\therefore \iint_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \iint_R \mathbf{F} \cdot \widehat{\mathbf{n}}\, dA_{xy}$$
$$= \iint_R (-2x^2 + 2xy + 1)\, dA_{xy}$$
$$= \int_0^{2\pi} \int_0^a (-2r^2 \cos^2\theta + 2r^2 \cos\theta \sin\theta + 1) r\, dr\, d\theta$$
$$= \pi a^2 - 2\pi \frac{a^4}{4}$$
$$= \frac{\pi}{2} a^2 (2 - a^2).$$



Qu. 10

$$S : \mathbf{r} = u^2 v\, \mathbf{i} + u v^2\, \mathbf{j} + v^3\, \mathbf{k} \qquad 0 \leqslant u \leqslant 1, \qquad 0 \leqslant v \leqslant 1$$
$$\mathbf{r}_u = 2uv\, \mathbf{i} + v^2\, \mathbf{j}, \qquad \mathbf{r}_v = u^2\, \mathbf{i} + 2uv\, \mathbf{j} + 3v^2\, \mathbf{k}$$
$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2uv & v^2 & 0 \\ u^2 & 2uv & 3v^2 \end{vmatrix} = 3v^4\, \mathbf{i} - 6uv^3\, \mathbf{j} + 3u^2 v^2\, \mathbf{k}$$

On S, we have $\mathbf{F} = 2u^2 v\, \mathbf{i} + u v^2\, \mathbf{j} + v^3\, \mathbf{k}$

$$\therefore S = \iint_S \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$
$$= \iint_S (6u^2 v^5 - 6u^2 v^5 + 3u^2 v^5)\, dS$$
$$= \int_0^1 \int_0^1 (3u^2 v^5)\, dv\, du$$
$$= \frac{1}{2} \int_0^1 u^2\, du = \frac{1}{6}.$$

Qu. 15 The flux of the plane vector field $\mathbf{F}$ across the piecewise smooth curve C, in the direction of the unit normal $\widehat{\mathbf{n}}$ to the curve, is

$$\int_C \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds$$

The flux of $\mathbf{F} = x\, \mathbf{i} + y\, \mathbf{j}$ outward across.

(a) The circle $x^2 + y^2 = a^2$

Let $g(x, y) = x^2 + y^2 = a^2$, this is a level curve in 2D.

$$\therefore\ \nabla g = 2(x, y), \quad \text{i.e. } \widehat{\mathbf{n}} = \frac{(x, y)}{a}$$

$$\therefore \quad \oint_C \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds = \oint_C \frac{x^2 + y^2}{a}\, ds$$
$$= a \oint_C ds$$
$$= 2\pi a^2.$$

(b)

$$C = C_1 + C_2 + C_3 + C_4$$

C_1 : $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 = (1 - t)(1, -1) + t(1, 1) = (1, 2t - 1), \qquad 0 \leqslant t \leqslant 1$

$\mathbf{r}'(t) = (0, 2), \quad ds = \|\mathbf{r}'(t)\|\, dt = 2\, dt$

$$\therefore \int_{C_1} \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds = \int_0^1 (1, 2t - 1) \cdot \mathbf{i} \times 2\, dt = \int_0^1 2\, dt = 2.$$

C_2 : $\mathbf{r}(t) = (1 - t)(1, 1) + t(-1, 1) = (1 - 2t, 1), \qquad 0 \leqslant t \leqslant 1$

$\mathbf{r}'(t) = (-2, 0), \ ds = \|\mathbf{r}'(t)\|\, dt = 2\, dt$

$$\therefore \int_{C_2} \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds = \int_0^1 (1 - 2t, 1) \cdot \mathbf{j} \times 2\, dt = \int_0^1 2\, dt = 2.$$

C_3 : $\mathbf{r}(t) = (1 - t)(1, 1) + t(-1, 1) = (-1, 1 - 2t), \qquad 0 \leqslant t \leqslant 1$

$\mathbf{r}'(t) = (0, -2), \ ds = \|\mathbf{r}'(t)\|\, dt = 2\, dt$

$$\therefore \int_{C_3} \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds = \int_0^1 (-1, 1 - 2t) \cdot (-\mathbf{j}) 2\, dt = \int_0^1 2\, dt = 2.$$

C_4 : $\mathbf{r}(t) = (1 - t)(-1, 1) + t(1, -1) = (-1, 1 - 2t), \qquad 0 \leqslant t \leqslant 1$

$\mathbf{r}'(t) = (0, -2), \ ds = \|\mathbf{r}'(t)\|\, dt = 2\, dt$

$$\therefore \int_{C_4} \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds = \int_0^1 (-1, 1 - 2t) \cdot (-\mathbf{i}) 2\, dt = \int_0^1 2\, dt = 2.$$

$$\therefore \int_C \mathbf{F} \cdot \widehat{\mathbf{n}}\, ds = 8.$$