MATH 2023 Multivariable Calculus Problem Sets

Note: Problems with asterisks represent supplementary informations. You may want to read their solutions if you like, but you don't need to work on them.

Set 1

1. Check if $\vec{a} \parallel \vec{b}$

   (a) $\vec{a} = \vec{i} + 2\vec{j} - \vec{k}, \quad \vec{b} = -2\vec{i} - 4\vec{j} + 2\vec{k}$

   (b) $\vec{a} = \vec{i} + \vec{j}, \quad \vec{b} = \vec{j} + \vec{k}$

2. Find the dot product and the angle between the two vectors
   $\vec{a} = \vec{i} + \vec{j} - \vec{k}, \quad \vec{b} = 2\vec{i} - 3\vec{j} + 4\vec{k}$

3. Determine whether $\vec{PQ}$ and $\vec{PR}$ are perpendicular
   $P = (-1, 3, 0), \quad Q = (2, 0, 1), \quad R = (-1, 1, -6)$.

4. Find $\vec{a} \times \vec{b}$ and $\vec{c} \cdot (\vec{a} \times \vec{b})$
   $\vec{a} = \vec{i} + \vec{j} + \vec{k}, \quad \vec{b} = \vec{i} - \vec{k}, \quad \vec{c} = -\vec{i} + \vec{j} - \vec{k}$.

5. Let $\vec{u}$ and $\vec{v}$ be adjacent sides of a paralelogram. Use vectors to show that the parallegram is a rectangle if the diagonals are equal in length.

6. Consider the two-dimensional xy-plane. Let O be the origin and A be a point lying on the x-axis. Let P be an arbitrary point of a curve. If the angle $\angle OPA$ (P being the vertex) is always a right angle, what is the geometical object traced by P (the curve)?

Set 2

1. What is the area of the triangle which has vertices at $(2,1,2)$, $(3,3,3)$, $(5,1,2)$?

2. Find a vector equation, and the parametric equations for the line that contains the point $(-2,1,0)$, and is parallel to the vector $3\vec{\imath} - 2\vec{\jmath} + \vec{k}$.

3. Find an equation of the plane that contains the point $(-1,1,3)$ and has normal vector $-2\vec{\imath} + 15\vec{\jmath} - \frac{1}{2}\vec{k}$.

4. Find an equation of the plane that contains the points $(2,-1,4)$, $(5,2,5)$, and $(2,1,3)$.

5. Show that the vector $a\vec{\imath} + b\vec{\jmath} + c\vec{k}$ is a normal to the plane $ax + by + cz = d$ where $a,\ b,\ c,\ d$ are constants.

6. Use vectors to show that for any triangle the three lines drawn from each vertex to the midpoint of the opposite side all pass through the same point.

Set 3

1. Determine the component functions and domain of the given function

   (a) $\vec{F} = \sqrt{t+1}\vec{i} + \sqrt{1-t}\vec{j} + \vec{k}$

   (b) $\vec{F}(t) = (t\vec{i} + \vec{j}) \times (ln(t)\vec{j} + \frac{1}{t}\vec{k})$

2. Sketch the curve traced out by the vector-valued function. Indicate the direction in which the curve is traced out.

   (a) $\vec{F}(t) = t\vec{i} + t\vec{j} + t\vec{k}$

   (b) $\vec{F}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$

3. Compute the limits or explain why it does not exist

   (a) $\lim\limits_{t\to 0}(\frac{\sin t}{t}\vec{i} + (t+\sqrt{2})\vec{j} + \frac{(e^{-t}-1)}{t}\vec{k})$

   (b) $\lim\limits_{t\to 0}\vec{F}(t)$ where

$$\vec{F}(t) = \begin{cases} t\vec{i} + e^{-1/t^2}\vec{j} + t^2\vec{k} & \text{for} \quad t \neq 0 \\ \frac{1}{2}\vec{j} & \text{for} \quad t = 0 \end{cases}$$

4. Find the derivatives of the function

   (a) $\vec{F}(t) = t^2\cos t\vec{i} + t^3\sin t\vec{j} + t^4\vec{k}$

   (b) $\vec{F} \times \vec{G}$ where

$$\vec{F}(t) = (1+t^2)\vec{i} + (e^t + e^{-t})\vec{j},$$
$$\vec{G}(t) = \frac{1}{\sqrt{t}+t}\vec{i} + \frac{1}{2+\sin t}\vec{j}$$

   (c) $f(t)\vec{F}(t)$ where $f = t^2, \quad \vec{F} = \vec{i} + \frac{1}{t}\vec{j} + \frac{1}{t^2}\vec{k}$

   (d) $\vec{F} \circ f$ where $f = t^2, \quad \vec{F} = \vec{i} + \frac{1}{t}\vec{j} + \frac{1}{t^2}\vec{k}$

5. Find the velocity, speed, acceleration, and magnitude of acceleration of an object having the given position function

   $\vec{r}(t) = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$

Set 4

1. Give the domain of definition for which the function is defined and real, and indicate this domain graphically

   (a) $f(x,y) = \sqrt{1 - x^2 - y^2}$

   (b) $f(x,y) = ln(x+y)$

2. Sketch the level curve $f(x,y) = c$ for

   (a) $f(x,y) = \sqrt{1 - x^2 - y^2}; \quad c = 0, 1/\sqrt{2}$

   (b) $f(x,y) = x^2 - y^2; \quad c = -1, 0, 1$

3. Sketch the graph of f

   (a) $f(x,y) = \sqrt{1 - x^2 - y^2}$ (an ellipsoid)

   (b) $f(x,y) = \sqrt{x^2 + y^2}$ (a cone)

   (c) $f(x,y) = x^2 + y^2 + 1$ (a paraboloid)

   (d) $f(x,y) = x^2 - y^2 + 1$ (a hyperbolic paraboloid)

4. Evaluate the limit

   (a) $\lim\limits_{(x,y)\to(1,0)} \dfrac{x^2 - xy + 1}{x^2 + y^2}$

   (b) $\lim\limits_{(x,y)\to(ln2,0)} e^{2x+y^2}$

5. Determine whether the limit exists

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

6. Investigate the continuity of the following function at $(0,0)$

$$f(x,y) = \begin{cases} (x^2 + y^2)\sin\dfrac{1}{x^2 + y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

Set 5

1. Show that the line $L = \{(x, y)|x = 0\}$ is a close set in $\mathcal{R}^2$.

2. Let $f(x, y) = \begin{cases} xy\sin(x/y) & y \neq 0 \\ 0 & y = 0 \end{cases},$

   show whether $f(x, y)$ is differentiable at $(0, 0)$.

3. Find the first and second partial derivatives of the function

   (a) $w = \cos\frac{u}{v}$

   (b) $z = x^y$

4. Find f_{xy} and f_{yx} and check whether they are equal

   $f(x, y, z) = x^4 - 2x^2y\sqrt{z} + 3yz^4 + 2$

5. Find the differential of f for

   $f(x, y) = xy^2ln(y/x)$

6. Find the approximate value of f at given point
   $f(x, y) = ln(x^2 + y^2); \quad (-0.03,\ 0.98)$

7.* Show that the expressions

$$\Delta f = f_x\Delta x + f_y\Delta y + \epsilon\sqrt{\Delta x^2 + \Delta y^2} \qquad \lim_{(x,y)\to(x_0,y_0)} \epsilon = 0 \tag{1}$$

and

$$\Delta f = f_x\Delta x + f_y\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \qquad \lim_{(x,y)\to(x_0,y_0)} \epsilon_1 = \lim_{(x,y)\to(x_0,y_0)} \epsilon_2 = 0 \tag{2}$$

can be equivalently used in the definition of *differentiability*.

Set 6

1. Find dz/dt

   (a) $z = x^2 - y^2 + 1; \quad x = \sqrt{t}, y = e^{2t}$

   (b) $z = \sin x + \cos xy; \quad x = t^2, y = 1$

2. Compute $\partial z/\partial u$ and $\partial z/\partial v$

   (a) $z = \frac{x}{y^2}; \quad x = u + v - 1, \quad y = u - v - 1$

   (b) $z = 2e^{x^2 y}; \quad x = \sqrt{uv}, \quad y = 1/u$

3. Find the directional derivatives of f at the point P in the direction of $\overrightarrow{a}$

   $f(x, y) = \sin xy^2; \quad P = (\frac{1}{\pi}, \pi); \quad \overrightarrow{a} = \overrightarrow{i} - 2\overrightarrow{j}$

4. If f is differentiable, show that $\vec{\nabla}(f^n) = nf^{n-1}\vec{\nabla}f$.

5. Find a vector that is normal to the graph of the equation on the given point

   (a) $\sin \pi xy = \sqrt{3}/2; \quad (\frac{1}{6}, 2)$

   (b) $e^{x^2 y} = 3; \quad (1, ln3)$

6. Find an equation of the plane tangent to the graph of the given function at the indicated point

   (a) $f(x, y) = xy - x + y - 1; \quad (0, 2, 1)$

   (b) $f(x, y) = \sqrt{1 - x^2 - 2y^2}; \quad (0, 0, 1)$

Set 7

1. Find all critical points. Determine whether each critical point yields a relative maximum value, a relative minimum value, or a saddle point

   (a) $f(x, y) = x^2 + 2y^2 - 6x + 8y + 3$

   (b) $g(x, y) = e^{xy}$

   (c) $f(u, v) = u^3 + v^3 - 6uv$

   (d) $f(x, y) = \sqrt{1 - x^2 - 2y^2}$

2. Find the extreme values of f subject to the given constraint. In each case assume that the extreme values exist

   (a) $f(x, y) = x + y^2; \quad x^2 + y^2 = 4$

   (b) $f(x, y) = xy; \quad (x + 1)^2 + y^2 = 1$

   (c) $f(x, y, z) = y^3 + xz^2; \quad x^2 + y^2 + z^2 = 1$

   (d) $f(x, y, z) = xyz; \quad x^2 + y^2 + 4z^2 = 6$

Set 8

1. Evaluate the iterated integral

   (a) $\int_0^1 \int_0^1 e^{x+y} dxdy$

   (b) $\int_0^2 \int_0^{\sqrt{4-y^2}} xdxdy$

2. Reverse the order of integration and evaluate the resulting integral

   (a) $\int_0^1 \int_y^1 e^{x^2} dxdy$

   (b) $\int_0^{\pi^{1/3}} \int_{y^2}^{\pi^{2/3}} \sin x^{3/2} dxdy$

3. Express the double integral as an iterated integral and evaluate it

   (a) $\iint\limits_R (x+y)dA$; R is the triangular region bounded by the lines $y = 2x, x = 0,$ and $y = 4$

   (b) $\iint\limits_R xdA$; R is the portion of the disk $x^2 + y^2 \leq 16$ in the second quadrant

4. Express the integral as an iterated integral in polar coordinates, and then evaluate it

   (a) $\iint\limits_R xydA$, where R is the region bounded by the circle $r = 5$

   (b) $\iint\limits_R x^2dA$, where R is the region bounded by the circle $r = 4\sin\theta$

5. Find the surface area of the given surface

   (a) The portion of the plane $x + 2y + 3z = 8$ in the first octant

   (b) The portion of the paraboloid $z = 9 - x^2 - y^2$ above the xy plane

6. Evaluate the surface integral $\iint\limits_\sigma f(x, y, z)dS$

   (a) $f(x, y, z) = z^2$; σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ between planes $z = 1$ and $z = 2$.

   (b) $f(x, y, z) = x^2y$; σ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$, $y = 1$, and above the xy-plane.

Set 9

1. Sketch the solid region described by the limits of the iterated integral and evaluate the integral
$$\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-x^2}} x\cos z\,dy\,dx\,dz.$$

2. Find the volume V of the solid region

   (a) The solid region bounded above by the parabolic sheet $z = 1 - x^2$, below by the xy plane, and on the sides by the planes $y = -1$ and $y = 2$.

   (b) The solid region in the first octant bounded by the paraboloid $z = x^2 + y^2$, the plane $x + y = 1$, and the coordinate planes.

3. Express the triple integral as an iterated integral in cylindrical coordinates, then evaluate it

   (a) $\iiint\limits_D (x^2 + y^2)dv$, where D is the solid region bounded by the cylinder $x^2+y^2 = 1$ and the planes $z = 0$ and $z = 4$.

   (b) $\iiint\limits_D zdv$, where D is the portion of the ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

4. Express the integral as an iterated integral in spherical coordinates, then evaluate it

   (a) $\iiint\limits_D x^2dv$ where D is the solid region between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

   (b) $\iiint\limits_D \frac{1}{x^2 + y^2 + z^2}dv$ where D is the solid region above the xy plane bounded by the cone $z = \sqrt{3x^2 + 3y^2}$ and the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 81$.

5. (a) Use the transformation $u = x - 2y,\ v = 2x + y$ to find
$$\iint\limits_R \frac{x - 2y}{2x + y}dA$$
   where R is the rectangular region enclosed by the lines $x - 2y = 1,\ x - 2y = 4,\ 2x + y = 1,\ 2x + y = 3$.

   (b) Evaluate $\iiint\limits_D x^2dV$, where D is the region enclosed by the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.

Set 10

1. Represent the vector field $\vec{F}(x,y)$ graphically. Then find the divergence and curl of the field.

   (a) $\vec{F} = x\vec{\imath} + y\vec{\jmath}$

   (b) $\vec{F} = \dfrac{x}{\sqrt{x^2+y^2}}\vec{\imath} + \dfrac{y}{\sqrt{x^2+y^2}}\vec{\jmath}$

   (c) $\vec{F} = y\vec{\imath} - x\vec{\jmath}$

   (d) $\vec{F} = \begin{cases} y\vec{\imath} - x\vec{\jmath} & \text{for} \quad x^2+y^2 \leq 1 \\ \dfrac{y}{x^2+y^2}\vec{\imath} - \dfrac{x}{x^2+y^2}\vec{\jmath} & \text{for} \quad x^2+y^2 > 1 \end{cases}$

2. Find the divergence and curl of $f\vec{F}$ where $f = e^{x+y+z}$ and $\vec{F} = \vec{\imath} + \vec{\jmath} + \vec{k}$.

3. Let $\vec{F}(x,y,z)$ be a vector field and $\phi(x,y,z)$ be a scalar function. Assuming that all the derivatives involved exists and are continuous, show that
   $\vec{\nabla} \times \phi\vec{F} = \phi\vec{\nabla} \times \vec{F} + \vec{\nabla}\phi \times \vec{F}$.

4. If $\vec{F}$ is a divergence-free vector field of the form $\vec{F} = f(r)\vec{r}$ where $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ is the radius vector and $r = |\vec{r}|$ is its magnitude, show that $\vec{F}$ is an inverse-square field.

Set 11

1. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where C is parameterized by $\vec{r}(t)$. Also, show $\vec{F}$ and C graphically

   (a) $\vec{F}(x,y) = x\vec{i} + y\vec{j} \quad \vec{r}(t) = t^{3/2}\vec{i} + t^{3/2}\vec{j} \quad 1 \le t \le 2$

   (b) $\vec{F}$ same as case (a) $\vec{r}(t) = 2\cos t\vec{i} + 2\sin t\vec{j} \quad 0 \le t \le 2\pi$

   (c) $\vec{F} = \dfrac{y}{x^2+y^2}\vec{i} - \dfrac{x}{x^2+y^2}\vec{j}; \quad \vec{r}(t) = t^{3/2}\vec{i} + t^{3/2}\vec{j} \quad 1 \le t \le 2$

   (d) $\vec{F}$ same as case (c); $r(t) = 2\cos t\vec{i} + 2\sin t\vec{j} \quad 0 < t \le 2\pi$

2. Show that the line integral is independent of path, and evaluate the integral

   (a) $\int_C (e^x + y)dx + (x+2y)dy;$ C is any piecewise smooth curve in the xy plane from $(0,1)$ to $(2,3)$

   (b) $\int_C (2xy^2+1)dx + 2x^2y dy;$ C is any piecewise smooth curve in the xy-plane from $(-1,2)$ to $(2,3)$

3. Use Green's Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is oriented counter-clockwise

   (a) $\vec{F}(x,y) = y\vec{i} + 3x\vec{j},$ C is the circle $x^2+y^2=4$

   (b) $\vec{F}(x,y) = y^4\vec{i} + x^3\vec{j},$ C is the square with vertices $(-2,-2), (-2,2), (2,-2)$ and $(2,2)$

<u>Set 12</u>

1. Evaluate $\iint\limits_{\Sigma} \vec{F} \cdot \vec{n} ds$

   (a) $F(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$; Σ is composed of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk in the xy plane bounded by the circle $x^2 + y^2 = 1$

   (b) $\vec{F}(x, y, z) = xy\vec{i} + xy\vec{j} + z^2\vec{k}$; Σ is the boundary of the solid region inside the cylinder $x^2 + y^2 = 4$ and between the planes $z = 0$ and $z = 2$

2. Use the Divergence Theorem to evaluate $\iint\limits_{\Sigma} \vec{F} \cdot \vec{n} ds$ for the cases in Problem 1.

3. Use the Stokes Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the curve that bounds Σ and that has the induced orientation from Σ

   (a) $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$; Σ is the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant; $\vec{n}$ is directed downward

   (b) $\vec{F} = y^2\vec{i} + xy\vec{j} - 2xz\vec{k}$; Σ is the hemisphere $z = \sqrt{4 - x^2 - y^2}$; $\vec{n}$ is directed upward.