<u>Solution to Set 1</u>

1. (a) $\vec{a} = -\frac{1}{2}\vec{b}$ therefore $\vec{a} \parallel \vec{b}$

   (b) Let $\vec{a} = c\vec{b}$, then $\vec{i} + \vec{j} = c\vec{j} + c\vec{k}$.
   There is no c that can create an $\vec{i}$ component. Therefore, $\vec{a}$ is not parallel to $\vec{b}$.

2. $\vec{a} \cdot \vec{b} = 2 - 3 - 4 = -5$ $\cos\theta = -\frac{5}{\sqrt{3}\sqrt{29}} = -0.536$

   $\theta = 2.137$ radian $= 122.4°$

3. Determine whether $\vec{PQ}$ and $\vec{PR}$ are perpendicular

   $\vec{PQ} = (2, 0, 1) - (-1, 3, 0) = (3, -3, 1)$;

   $\vec{PR} = (-1, 1, -6) - (-1, 3, 0) = (0, -2, -6)$

   $\vec{PQ} \cdot \vec{PR} = 6 - 6 = 0$ $\therefore \perp$

4. $\vec{a} \times \vec{b} = -\vec{i} + 2\vec{j} - \vec{k}$

   $\vec{c} \cdot (\vec{a} \times \vec{b}) = 1 + 2 + 1 = 4$

5. The diagonals are $\vec{u} + \vec{v}$ and $\pm(\vec{u} - \vec{v})$. If their lengths are equal, then $|\vec{u} + \vec{v}|^2 = |\vec{u} - \vec{v}|^2 = (\vec{u} \pm \vec{v}) \cdot (\vec{u} \pm \vec{v})$. Therefore, $2\vec{u} \cdot \vec{v} = -2\vec{u} \cdot \vec{v} \Rightarrow \vec{u} \cdot \vec{v} = 0 \Rightarrow \vec{u} \perp \vec{v} \Rightarrow$ the figure is a rectangle.

6. Let $\vec{P}$ and $\vec{A}$ be the position vectors of points P and A. The directed line segments $\vec{PO} = -\vec{P}$ and $\vec{PA} = \vec{A} - \vec{P}$ are perpendicular to each other. Therefore,

   $\vec{P} \cdot (\vec{P} - \vec{A}) = |\vec{P}|^2 - \cos\theta |\vec{P}||\vec{A}| = r^2 - ar\cos\theta = 0$

   where θ is the angle between $\vec{P}$ and the x-axis, $r = |\vec{P}|$, and $a = |\vec{A}|$. The above equation is equivalent to

   $r = a\cos\theta$,

   a standard polar equation for a circle with diameter a.

Solution to Set 2

1. Two of the sides of the triangle are represented by the line segments
$\vec{a} = (3,3,3) - (2,2,2) = (1,2,1)$ and $\vec{b} = (5,1,2) - (2,1,2) = (3,0,0)$.
The area of the triangle is given by $\frac{1}{2}|\vec{a} \times \vec{b}|$.
Since $\vec{a} \times \vec{b} = (\vec{i} + 2\vec{j} + \vec{k}) \times 3\vec{i} = 3\vec{j} - 6\vec{k}$,
the area is $\frac{1}{2}\sqrt{36+9} = \frac{3}{2}\sqrt{5}$.

2. vector form: $\vec{r} = -2\vec{i} + \vec{j} + t(3\vec{i} - 2\vec{j} + \vec{k}) \qquad -\infty < t < \infty$
parametric form: $\left\{ \begin{array}{lcl} x &=& -2+3t \\ y &=& 1-2t \\ z &=& t \end{array} \right.$

3. $\vec{r} - \vec{P}_0 = (x+1)\vec{i} + (y-1)\vec{j} + (z-3)\vec{k}$
$\vec{N} \cdot (\vec{r} - \vec{P}_0) = -2(x+1) + 15(y-1) - \frac{1}{2}(z-3) = 0$
equation: $\quad -4x + 30y - z - 31 = 0$

4. $(5,2,5) - (2,-1,4) = (3,3,1)$
$(2,1,3) - (2,-1,4) = (0,2,-1)$
Let $\vec{N} = (3\vec{i} + 3\vec{j} + \vec{k}) \times (2\vec{j} - \vec{k}) = -5\vec{i} + 3\vec{j} + 6\vec{k}$
equation: $\quad [(x-5)\vec{i} + (y-2)\vec{j} + (z-5)\vec{k}] \cdot (-5\vec{i} + 3\vec{j} + 6\vec{k}) = -5x + 3y + 6z - 11 = 0$

5. We are going to show that the vector $\vec{N} = a\vec{i} + b\vec{j} + c\vec{k}$ is perpendicular to any line lying on the plane (or any directed line segments on the plane).
Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be any two distinct points on a line that lies on the plane. The vector associated with the directed line segment connecting P to Q is $\vec{PQ} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$, and it is giving the direction of the line. Since P and Q are also on the plane, the equations $ax_1 + by_1 + cz_1 = d$ and $ax_2 + by_2 + cz_2 = d$ are both satisfied. Subtracting one from the other, one gets $a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$. This equation means that $\vec{N} \cdot \vec{PQ} = 0$. Therefore, $a\vec{i} + b\vec{j} + c\vec{k}$ is perpendicular to $\vec{PQ}$.

6. Choose the origin at one vertex of the triangle and let two of the sides be represented by the line segments (vectors) $\vec{a}$ and $\vec{b}$. The third side is then given by the line segment $\vec{b} - \vec{a}$. The midpoint of this side is at $\vec{a} + (\vec{b} - \vec{a})/2 = (\vec{a} + \vec{b})/2$.
The line connecting the origin to this point is parameterized by the equation $\vec{r} = t(\vec{a} + \vec{b})/2$. Similarly, the line connecting the end point of $\vec{a}$ (another vertex of the triangle) to the midpoint of the opposite line segment $\vec{b}$ is parameterized by the equation $\vec{r} = \vec{a} + s(\vec{b}/2 - \vec{a})$.
The two lines intersect when $t(\vec{a} + \vec{b})/2 = \vec{a} + s(\vec{b}/2 - \vec{a})$. In this equation, the coefficients of $\vec{a}$ and $\vec{b}$ can be equated respectively. This gives the solution $t = s = 2/3$. Therefore the two lines meet at the location $(\vec{a} + \vec{b})/3$.
Similarly, it can be shown that the line connecting the end point of $\vec{b}$ to the mid point of $\vec{a}$ intersects with these lines at the same point.

Solution to Set 3

1. (a) Component functions:
   $F_1(t) = \sqrt{t+1},\ F_2(t) = \sqrt{1-t},\ F_3(t) = 1$
   Domain:
   For F_1 to be defined (getting a real value), the following condition has to be satisfied
   $t + 1 \geq 0 \ \equiv\ t \geq -1$
   For F_2, the condition is
   $1 - t \geq 0 \ \equiv\ t \leq 1$
   No restrictions from F_3.
   Combining the conditions on all the components, the conditions for $\vec{F}$ are
   $-1 \leq t \leq 1$
   The domain is therefore $[-1, 1]$.

   (b) $\vec{F}(t) = \frac{1}{t}\vec{i} - \vec{j} + t\ln(t)\vec{k}$
   Component functions:
   $F_1(t) = \frac{1}{t},\ F_2(t) = -1,\ F_3(t) = t\ln(t)$
   Domain:
   For F_1 to be defined (getting a real value), the following condition has to be satisfied
   $t \neq 0$
   No conditions on F_2
   For F_3
   $t > 0$
   Combining the conditions on all the components, the condition for $\vec{F}$ is
   $t > 0$
   The domain is therefore $(0, \infty)$.

2. (a) A straight line.

   (b) A helix spiraling upward along the z-axis with projection on the xy-plane tracing a unit circle in the counterclockwise direction.

3. (a) $\lim\limits_{t\to 0}(\frac{\sin t}{t}\vec{i} + (t+\sqrt{2})\vec{j} + \frac{(e^{-t}-1)}{t}\vec{k}) = \vec{i} + \sqrt{2}\vec{j} - \vec{k}$

   (b) $\lim\limits_{t\to 0}\vec{F}(t) \ = \vec{0}$

4. (a) $\vec{F}'(t) = (2t\cos t - t^2\sin t)\vec{i} + (3t^2\sin t + t^3\cos t)\vec{j} + 4t^3\vec{k}$

   (b) $\vec{F} \times \vec{G} = \left(\frac{1+t^2}{2+\sin t} - \frac{e^t + e^{-t}}{\sqrt{t}+t}\right)\vec{k}$
   Therefore,
   $$(\vec{F} \times \vec{G})' = \left[\frac{(2+\sin t)2t - (1+t^2)\cos t}{(2+\sin t)^2} - \frac{(\sqrt{t}+t)(e^t - e^{-t}) - (e^t + e^{-t})(\frac{1}{2\sqrt{t}} + 1)}{(\sqrt{t}+t)^2}\right]\vec{k}$$

   (c) $(f(t)\vec{F}(t))' = 2t\ \vec{i} + \vec{j}$

   (d) $\vec{F} \circ f = \vec{i} + \frac{1}{t^2}\vec{j} + \frac{1}{t^4}\vec{k}$
   Therefore, $(\vec{F} \circ f)' = -\frac{2}{t^3}\vec{j} - \frac{4}{t^5}\vec{k}$

5. velocity: $\dot{\vec{r}}(t) = -\sin t\ \vec{i} + \cos t\ \vec{j} +\ \vec{k}$ speed: $\sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$
   acceleration: $\ddot{\vec{r}}(t) = -\cos t\ \vec{i} - \sin t\ \vec{j} = -\vec{r}$ $|\ddot{\vec{r}}(t)| = 1$

Solution to Set 4

1. (a) The square root is real only if the argument is non-negative, namely
   $1 - x^2 - y^2 \geq 0 \quad \Leftrightarrow \quad x^2 + y^2 \leq 1.$
   The domain is a closed disk (boundary included) centered at $(0, 0)$ with radius 1.

   (b) The logarithmic function is defined only for a positive argument, namely
   $x + y > 0.$
   The domain is an open half plane above the line $x + y = 0$ ($135°$ with the x-axis).

2. (a) $c = 0$ case: The level curve is defined by the equation
   $f(x, y) = \sqrt{1 - x^2 - y^2} = 0 \quad \Leftrightarrow \quad x^2 + y^2 = 1.$
   It is a circle centered at $(0, 0)$ with radius 1.
   $c = 1/\sqrt{2}$ case: The level curve satisfies
   $x^2 + y^2 = 1/2$
   and is a circle centered at $(0, 0)$ with radius $1/\sqrt{2}$.

   (b) $f(x, y) = x^2 - y^2; \quad c = -1, 0, 1$ $c = -1$ case: The level curve is the hyperbola $y^2 - x^2 = 1$.
   $c = 0$ case: The level curve is composed of the two lines $x = \pm y$.
   $c = 1$ case: The level curve is the hyperbola $x^2 - y^2 = 1$.

3. Sketch the graph of f

   (a) $f(x, y) = \sqrt{1 - x^2 - y^2}$ (an ellipsoid)

   (b) $f(x, y) = \sqrt{x^2 + y^2}$ (a cone)

   (c) $f(x, y) = x^2 + y^2 + 1$ (a paraboloid)

   (d) $f(x, y) = x^2 - y^2 + 1$ (a hyperbolic paraboloid)

4. (a) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x^2 - xy + 1}{x^2 + y^2} = 2$

   (b) $\displaystyle\lim_{(x,y)\to(ln2,0)} e^{2x+y^2} = 4$

5. Let $x = r\cos\theta$, $y = r\sin\theta$, then
   $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{r\to 0} \frac{r^2 \sin\theta\cos\theta}{r^2} = \sin\theta\cos\theta$$
   and is dependent on θ (not a unique constant). Therefore, the limit does not exist.

   Note: If one wants to think about taking limit along paths, one can get different paths by choosing different fixed values of θ and use the parameterization $\vec{r}_\theta(t) = t\,(\cos\theta, \sin\theta)$ for the paths.

6. Let $x = r\cos\theta$, $y = r\sin\theta$, then
   $$\lim_{(x,y)\to(0,0)} (x^2 + y^2)\sin\frac{1}{x^2 + y^2} = \lim_{r\to 0} r^2 \sin\frac{1}{r^2} = 0 = f(0, 0)$$
   Therefore, the function is continuous at $(0, 0)$.

Solution to Set 5

1. We proceed by showing that all points not in L are exterior points of L, so that L contains all the boundary points and is a close set.

   Let (x, y) be a point not in L, then by the definition of L, $x \neq 0$. Pick an ϵ which has the value $|x|$, then all the points inside the open disk $D_\epsilon(x, y)$ are not in L

   $((a, b) \in D_\epsilon(x, y) \quad \Rightarrow \quad |x - a| < \epsilon = |x| \quad \Rightarrow \quad a \neq 0)$.

   Thus (x, y) is an exterior point of L.

2. First, it is easy to show that $f_x(0, 0) = f_y(0, 0) = f(0, 0) = 0$. Then define $\epsilon_1(x, y) = y \sin(x/y)$ and $\epsilon_2(x, y) = 0$. In a small disk around $(0, 0)$, $\Delta x = x$, $\Delta y = y$, and $\Delta f = f(x, y) - f(0, 0)$ can be written in the form
$$\Delta f = f_x(0, 0)\Delta x + f_y(0, 0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$
   The limit $\lim\limits_{(x,y)\to(0,0)} \epsilon_1(x, y)$ is 0 as $\sin(x/y)$ is bounded by ± 1 and $\lim\limits_{(x,y)\to(0,0)} y = 0$. Therefore the definition of differentiability is satisfied at $(0, 0)$.

3. (a) $\dfrac{\partial}{\partial u} w = -\dfrac{1}{v} \sin \dfrac{u}{v} \qquad \dfrac{\partial}{\partial v} w = \dfrac{u}{v^2} \sin \dfrac{u}{v}$

   $\dfrac{\partial^2}{\partial u^2} w = -\dfrac{1}{v^2} \cos \dfrac{u}{v} \qquad \dfrac{\partial^2}{\partial v^2} w = -\dfrac{u}{v^3} \left(\dfrac{u}{v} \cos \dfrac{u}{v} + 2 \sin \dfrac{u}{v}\right)$

   $\dfrac{\partial^2}{\partial u \partial v} w = \dfrac{1}{v^2} \left(\dfrac{u}{v} \cos \dfrac{u}{v} + \sin \dfrac{u}{v}\right)$

   (b) $\dfrac{\partial}{\partial x} z = y x^{y-1} \qquad \dfrac{\partial}{\partial y} z = (\ln x) x^y$

   $\dfrac{\partial^2}{\partial x^2} z = y(y-1)x^{y-2} \qquad \dfrac{\partial^2}{\partial y^2} z = (\ln x)^2 x^y \qquad \dfrac{\partial^2}{\partial x \partial y} z = x^{y-1} + y(\ln x) x^{y-1}$

4. $f_x(x, y, z) = 4x^3 - 4xy\sqrt{z} \qquad f_y(x, y, z) = -2x^2\sqrt{z} + 3z^4 \qquad fxy = -4x\sqrt{z} = fyx$

5. $df(x, y) = f_x dx + f_y dy = [y^2 \ln(y/x) - y^2]dx + [2xyln(y/x) + xy]dy$

6. $f_x = \dfrac{2x}{x^2 + y^2} \qquad f_y = \dfrac{2y}{x^2 + y^2}$

   $f(0, 1) = \ln 1 = 0 \qquad f_x(0, 1) = 0 \quad f_y(0, 1) = 2$

   $f(-0.03, 0.98) \approx f(0, 1) + 0 \cdot (-0.03) + 2 \cdot (-0.02) = -0.04$

7.* Show that the expressions
$$\Delta f = f_x \Delta x + f_y \Delta y + \epsilon\sqrt{\Delta x^2 + \Delta y^2} \qquad \lim_{(x,y)\to(x_0,y_0)} \epsilon = 0 \tag{1}$$
   and
$$\Delta f = f_x \Delta x + f_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \qquad \lim_{(x,y)\to(x_0,y_0)} \epsilon_1 = \lim_{(x,y)\to(x_0,y_0)} \epsilon_2 = 0 \tag{2}$$
   can be equivalently used in the definition of *differentiability*.

   (1) $\Rightarrow$ (2)

   Let's use Δ to represent $\sqrt{\Delta x^2 + \Delta y^2}$. Then the last term of expression (1) can be rewritten as

$\epsilon \cdot \Delta = \epsilon \cdot \dfrac{\Delta}{|\Delta x| + |\Delta y|} \cdot (\beta(\Delta x)\Delta x + \beta(\Delta y)\Delta y)$

where the function β is defined as

$$\beta(\Delta x) = \begin{cases} \dfrac{|\Delta x|}{\Delta x} & \text{if} \quad \Delta x \neq 0 \\ 1 & \text{if} \quad \Delta x = 0 \end{cases} .$$

One can choose $\epsilon_1 = \epsilon \dfrac{\Delta}{|\Delta x| + |\Delta y|}\beta(\Delta x)$ and $\epsilon_2 = \epsilon \dfrac{\Delta}{|\Delta x| + |\Delta y|}\beta(\Delta y)$ to obtain expression (2).

As $|\epsilon_1| \leq |\epsilon|$ (with triangle inequality applied to the ratio $\dfrac{\Delta}{|\Delta x| + |\Delta y|}$), $\lim_{(\Delta x, \Delta y) \to (0,0)} \epsilon_1 = 0$ (The situation of ϵ_2 is similar).

<u>(2) $\Rightarrow$ (1)</u>

Given (2), ϵ can be idenified as $\epsilon = (\epsilon_1 \Delta x + \epsilon_2 \Delta y)/\Delta$. Clearly $\lim_{(\Delta x, \Delta y) \to (0,0)} (\epsilon_1 \Delta x + \epsilon_2 \Delta y)/\Delta = 0$ as $|\Delta x|$ and $|\Delta y|$ are both less than Δ.

Solution to Set 6

1. Find dz/dt

   (a) $\dfrac{\partial z}{\partial x} = 2x \qquad \dfrac{\partial z}{\partial y} = -2y$

   $\dfrac{dx}{dt} = \dfrac{1}{2}t^{-1/2} \qquad \dfrac{dy}{dt} = 2e^{2t}$

   $\dfrac{dz}{dt} = xt^{-1/2} - 4ye^{2t}$

   (b) $\dfrac{\partial z}{\partial x} = \cos x - y\sin xy \qquad \dfrac{\partial z}{\partial y} = -x\sin xy$

   $\dfrac{dx}{dt} = 2t \qquad \dfrac{dy}{dt} = 0$

   $\dfrac{dz}{dt} = (\cos x - y\sin xy)\, 2t$

2. (a) $\dfrac{\partial z}{\partial x} = \dfrac{1}{y^2} \qquad \dfrac{\partial z}{\partial y} = -\dfrac{2x}{y^3}$

   $\dfrac{\partial x}{\partial u} = 1 \qquad \dfrac{\partial x}{\partial v} = 1 \qquad \dfrac{\partial y}{\partial u} = 1 \qquad \dfrac{\partial y}{\partial v} = -1$

   $\dfrac{\partial z}{\partial u} = \dfrac{1}{y^2} - \dfrac{2x}{y^3} \qquad \dfrac{\partial z}{\partial v} = \dfrac{1}{y^2} + \dfrac{2x}{y^3}$

   (b) $\dfrac{\partial z}{\partial x} = 4xye^{x^2y} \qquad \dfrac{\partial z}{\partial y} = 2x^2e^{x^2y}$

   $\dfrac{\partial x}{\partial u} = \dfrac{1}{2}\sqrt{\dfrac{v}{u}} \qquad \dfrac{\partial x}{\partial v} = \dfrac{1}{2}\sqrt{\dfrac{u}{v}} \qquad \dfrac{\partial y}{\partial u} = -\dfrac{1}{u^2} \qquad \dfrac{\partial y}{\partial v} = 0$

   $\dfrac{\partial z}{\partial u} = 2xye^{x^2y}\sqrt{\dfrac{v}{u}} - 2x^2e^{x^2y}\dfrac{1}{u^2} \qquad \dfrac{\partial z}{\partial v} = 2xye^{x^2y}\sqrt{\dfrac{u}{v}}$

3. $f_x = y^2\cos xy^2 \qquad f_y = 2xy\cos xy^2$

   $f_x(\dfrac{1}{\pi}, \pi) = \pi^2\cos\pi = -\pi^2 \qquad f_y(\dfrac{1}{\pi}, \pi) = -2$

   unit vector along $\vec{u}$ is $\dfrac{1}{\sqrt{5}}(\vec{i} - 2\vec{j})$

   $D_a f(\dfrac{1}{\pi}, \pi) = \dfrac{1}{\sqrt{5}}(-\pi^2 + 4)$

4. The component of $\vec{\nabla} f^n$ along the $\vec{i}$ direction is $\partial f^n/\partial x = nf^{n-1}\partial f/\partial x$, and other components are similar (by replacing x with y or z). Therefore, $\vec{\nabla} f^n = nf^{n-1}(f_x\vec{i} + f_y\vec{j} + f_z\vec{k}) = nf^{n-1}\vec{\nabla} f$.

5. (a) Let $f(x, y) = \sin(\pi xy)$, the graph of the equation is the level curve $f(x, y) = \dfrac{\sqrt{3}}{2}$.

   $f_x = \pi y\cos(\pi xy) \qquad f_y = \pi x\cos(\pi xy)$

   $f_x(\dfrac{1}{6}, 2) = 2\pi\cos\dfrac{\pi}{3} = \pi \qquad f_y(\dfrac{1}{6}, 2) = \dfrac{\pi}{12}$

   A normal vector is $\pi\vec{i} + \dfrac{\pi}{12}\vec{j}$.

   (b) The graph is a level curve the function $f(x, y) = e^{x^2y}$.

   $f_x = 2xye^{x^2y} \qquad f_y = x^2e^{x^2y}$

   $f_x(1, \ln 3) = 6\ln 3 \qquad f_y(1, \ln 3) = 3$

   A normal vector is $6\ln 3\vec{i} + 3\vec{j}$.

6. (a) $f_x = y - 1 \qquad f_y = 1 + x$

$f_x(0,2) = 1 \qquad f_y(0,2) = 1$

A normal to the graph is $\vec{i} + \vec{j} - \vec{k}$.

An equation of the tangent plane is $(x-0) + (y-2) - (z-1) = 0$.

(b) $f_x = -\dfrac{x}{\sqrt{1 - x^2 - 2y^2}} \qquad f_y = -\dfrac{2y}{\sqrt{1 - x^2 - 2y^2}}$

$f_x(0,0) = 0 \qquad f_y(0,0) = 0$

A normal to the graph is $-\vec{k}$.

An equation of the tangent plane is $z - 1 = 0$.

Solution to Set 7

1. (a) $f_x = 2x - 6 \qquad f_y = 4y + 8$
$f_x = f_y = 0 \quad \Rightarrow \quad x = 3,\ y = -2 \qquad$ The ciritcal point is $(3, -2)$.
$f_{xx} = 2 \qquad f_{yy} = 4 \qquad f_{xy} = 0$
$D = f_{xx}f_{yy} - f_{xy}^2 = 8 > 0$ and $f_{xx} > 0 \qquad \therefore$ The point is a relative minimum.

(b) $g_x = ye^{xy} \qquad g_y = xe^{xy}$
$g_x = g_y = 0 \quad \Rightarrow \quad x = 0,\ y = 0 \qquad$ The ciritcal point is $(0, 0)$.
$g_{xx} = y^2e^{xy} \qquad g_{yy} = y^2e^{xy} \qquad g_{xy} = e^{xy} + xye^{xy}$
At $(0, 0),\ D = g_{xx}g_{yy} - g_{xy}^2 = -1 \qquad \therefore$ The point is a saddle point.

(c) $f_u = 3u^2 - 6v \qquad f_v = 3v^2 - 6u$
$f_u = f_v = 0 \quad \Rightarrow \quad v = \dfrac{u^2}{2}\ \&\ \dfrac{u^4}{4} - 2u = 0 \quad \Rightarrow \quad u(u^3 - 8) = 0 \quad \Rightarrow \quad u = 0$ or 2
The ciritcal points are $(0, 0)$ and $(2, 2)$.
$f_{uu} = 6u \qquad f_{vv} = 6v \qquad f_{uv} = -6$
$D = f_{uu}f_{vv} - f_{uv}^2 = 36(uv - 1)$
At $(0, 0),\ D = -36 < 0 \qquad \therefore$ The point is a saddle point.
At $(2, 2),\ D = 108 > 0\ \&\ f_{uu} = 12 > 0 \qquad \therefore$ The point is a relative minimum.

(d) $f_x = \dfrac{-x}{\sqrt{1 - x^2 - 2y^2}} \qquad f_y = f_x = \dfrac{-2y}{\sqrt{1 - x^2 - 2y^2}}$
$f_x = f_y = 0 \quad \Rightarrow \quad x = 0,\ y = 0 \qquad$ The ciritcal point is $(0, 0)$.
Note that $f_x,\ f_y$ are undefined at the boundary of the domain of f, but the boundary points need not be counted as critical points.
$f_{xx} = \dfrac{2y^2 - 1}{(1 - x^2 - 2y^2)^{3/2}} \qquad f_{yy} = \dfrac{2x^2 - 2}{(1 - x^2 - 2y^2)^{3/2}} \qquad f_{xy} = \dfrac{-2xy}{(1 - x^2 - 2y^2)^{3/2}}$
At $(0, 0),\ D = f_{xx}f_{yy} - f_{xy}^2 = 2 > 0\ \&\ f_{xx} = -1 < 0$
$\therefore$ The point is a relative maximum.

2. (a) The constraint is $g(x, y) = x^2 + y^2 = 4$.
$f_x = 1 \qquad f_y = 2y \qquad g_x = 2x \qquad g_y = 2y$
The Lagrange multiplier equations are:
$$\begin{cases} 1 &= \lambda 2x \\ 2y &= \lambda 2y \end{cases}$$
Consider the following cases:
(i) $\lambda = 0 \qquad \Rightarrow \quad 1 = 0$ impossible!
(ii) $\lambda \neq 0 \qquad$ then $\lambda 4xy = \lambda 2y$
$\Rightarrow \quad 2xy - y = (2x - 1)y = 0 \quad \Rightarrow \quad y = 0$ or $x = \dfrac{1}{2}$
$y = 0 \quad \Rightarrow \quad x = \pm 2$
$x = \dfrac{1}{2} \quad \Rightarrow \quad y = \pm\dfrac{\sqrt{15}}{2}$
Now compare values of all the solutions
$f(2, 0) = 2 \qquad f(-2, 0) = -2 \qquad f(\dfrac{1}{2}, \pm\dfrac{\sqrt{15}}{2}) = \dfrac{17}{4}$
$\therefore\ \dfrac{17}{4}$ is the maximum value, and -2 is the minimum value.

(b) The constraint is $g(x,y) = (x+1)^2 + y^2 = 1$.

$f_x = y \qquad f_y = x \qquad g_x = 2(x+1) \qquad g_y = 2y$

The Lagrange multiplier equations are:

$$\begin{cases} y &= \lambda 2(x+1) \\ x &= \lambda 2y \end{cases}$$

Consider the following cases:

(i) if $\lambda = 0$, then the two λ equations yield $x = y = 0$

(ii) if $\lambda \neq 0$, then $\lambda 2y^2 = \lambda 2x(x+1) \quad \Rightarrow \quad y^2 = x(x+1)$

Substituting that into the constraint equation $\quad \Rightarrow \quad x(2x+3) = 0 \quad \Rightarrow \quad x = 0$ or $x = -\dfrac{3}{2}$

$x = 0 \quad \Rightarrow \quad y = 0$ same solutions as case (i)

$x = -\dfrac{3}{2} \quad \Rightarrow \quad y = \pm\dfrac{\sqrt{3}}{2}$

Now compare values

$$f(0,0) = 0 \qquad f(-\frac{3}{2}, \frac{\sqrt{3}}{2}) = -\frac{3\sqrt{3}}{2} \qquad f(-\frac{3}{2}, -\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{2}$$

$\therefore \dfrac{3\sqrt{3}}{2}$ is the maximum value, and $-\dfrac{3\sqrt{3}}{2}$ is the minimum value.

(c) The constraint is $g(x,y,z) = x^2 + y^2 + z^2 = 1$.

$f_x = z^2 \qquad f_y = 3y^2 \qquad f_z = 2xz$

$g_x = 2x \qquad g_y = 2y \qquad g_z = 2z$

The Lagrange multiplier equations are:

$$\begin{cases} z^2 &= \lambda 2x \\ 3y^2 &= \lambda 2y \\ 2xz &= \lambda 2z \end{cases}$$

Consider the following cases:

(I) if $\lambda = 0$, then the first two λ equations yield $z = y = 0$

Substiting that into the constraint equation $\quad \Rightarrow \quad x = \pm 1$.

(II) if $\lambda \neq 0$, then the first and third λ equations yield $z^3 = 2x^2 z$.

(i) $x = 0 \quad \Rightarrow \quad z = 0$ & constraint $\quad \Rightarrow \quad y = \pm 1$

(ii) $y = 0$ & constraint $\quad \Rightarrow \quad x^2 = 1 - z^2$

$\Rightarrow \quad z(3z^2 - 2) = 0 \quad \Rightarrow \quad z = 0$ or $z = \pm\sqrt{\dfrac{2}{3}}$

$z = 0$ same solutions as case (I)

$z = \pm\sqrt{\dfrac{2}{3}} \quad \Rightarrow \quad x = \pm\sqrt{\dfrac{1}{3}}$ for either value of z.

(iii) $z = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad y = \pm 1$

(iv) $x, y, z \neq 0 \quad \Rightarrow \quad \lambda = \dfrac{z^2}{2x} = \dfrac{3y}{2} = x \quad \Rightarrow \quad z^2 = 2x^2$ & $y = \dfrac{2}{3}x$

$\Rightarrow \quad x^2 + \dfrac{4}{9}x^2 + 2x^2 = 1 \quad \Rightarrow \quad x = \pm\dfrac{3}{\sqrt{31}}$

$x = \dfrac{3}{\sqrt{31}} \quad \Rightarrow \quad y = \dfrac{2}{\sqrt{31}}$ and $z = \pm\dfrac{3\sqrt{2}}{\sqrt{31}}$

$x = -\dfrac{3}{\sqrt{31}} \quad \Rightarrow \quad y = -\dfrac{2}{\sqrt{31}}$ and $z = \pm\dfrac{3\sqrt{2}}{\sqrt{31}}$

Now compare values

$f(0,0,\pm 1) = 0 \qquad f(0,1,0) = 1 \qquad f(0,-1,0) = -1$

$f(\sqrt{\tfrac{1}{3}}, 0, \pm\sqrt{\tfrac{2}{3}}) = \tfrac{2}{3}\sqrt{\tfrac{2}{3}} \qquad f(-\sqrt{\tfrac{1}{3}}, 0, \pm\sqrt{\tfrac{2}{3}}) = -\tfrac{2}{3}\sqrt{\tfrac{2}{3}}$

$f(\pm\dfrac{3}{\sqrt{31}}, \pm\dfrac{2}{\sqrt{31}}, \pm\dfrac{3\sqrt{2}}{\sqrt{31}}) = \pm\dfrac{2}{\sqrt{31}}$

$\therefore$ 1 is the maximum value, and -1 is the minimum value.

(d) The constraint is $g(x,y,z) = x^2 + y^2 + 4z^2 = 6$.
$f_x = yz \qquad f_y = xz \qquad f_z = xy$
$g_x = 2x \qquad g_y = 2y \qquad g_z = 8z$
The Lagrange multiplier equations are:

$$\begin{cases} yz &= \lambda 2x \\ xz &= \lambda 2y \\ xy &= \lambda 8z \end{cases}$$

These equations $\quad \Rightarrow \quad f^2(x,y,z) = (xyz)^2 = \lambda^3 32xyz.$
Consider the following cases:
(i) if $\lambda = 0$ or any of $x,\ y,\ z,\ = 0,\qquad$ then $f(x,y,z) = 0$.
(ii) if all of $\lambda,\ x,\ y,\ z,\ \neq 0$, then $\dfrac{yz}{x} = \dfrac{xz}{y} = \dfrac{xy}{4z}$ and
$\Rightarrow \quad x^2 = y^2$ and $y^2 = 4z^2$
Substituting these into the constraint equation $\quad \Rightarrow \quad 3x^2 = 6 \quad \Rightarrow \quad x = \pm\sqrt{2}$
There are eight solutions: $(\pm\sqrt{2}, \pm\sqrt{2}, \pm\frac{1}{\sqrt{2}})$ which produce the values $\pm\sqrt{2}$.
Therefore, the maximum value is $\sqrt{2}$, and the minimum value is $-\sqrt{2}$.

Solution to Set 8

1. (a) $\displaystyle\int_0^1\int_0^1 e^{x+y}dxdy = \left(\int_0^1 e^x dx\right)\left(\int_0^1 e^y dy\right) = (e-1)^2$

   (b) $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}} xdxdy = \int_0^2 \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y^2}} dy = \frac{1}{2}\int_0^2 (4-y^2)dy = \frac{8}{3}$

2. The reversed interated integral is

   (a) $\displaystyle\int_0^1\int_0^x e^{x^2}dydx = \int_0^1 e^{x^2}xdx = \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}(e-1)$

   (b) $\displaystyle\int_0^{\pi^{2/3}}\int_0^{\sqrt{x}} \sin x^{3/2}dydx = \int_0^{\pi^{2/3}} (\sin x^{3/2})x^{1/2}dx = \frac{2}{3}\left[-\cos x^{3/2}\right]_0^{\pi^{2/3}} = \frac{4}{3}$

3. (a) $\displaystyle\iint_R (x+y)dA = \int_0^4\int_0^{y/2}(x+y)dxdy = \int_0^4\left[\frac{x^2}{2}+yx\right]_0^{y/2}dy = \int_0^4 \frac{5}{8}y^2dy = \frac{40}{3}$

   (b) $\displaystyle\iint_R xdA = \int_{-4}^0\int_0^{\sqrt{16-x^2}} xdydx = \int_{-4}^0 x\sqrt{16-x^2}dx = -\frac{64}{3}$

4. (a) $\displaystyle\iint_R xydA = \int_0^{2\pi}\int_0^5 r^2\cos\theta\sin\theta\, rdrd\theta = \left(\int_0^{2\pi}\cos\theta\sin\theta d\theta\right)\left(\int_0^5 r^3dr\right) = 0$

   (b) $\displaystyle\iint_R x^2dA = \int_0^{\pi}\int_0^{4\sin\theta} r^2\cos^2\theta\, rdrd\theta = 4^3\int_0^{\pi}\cos^2\theta\sin^4\theta d\theta = 4\pi$

5. (a) The projection R of the portion of the plane on the xy-plane is the triangle bounded by the two axes and the line $x = 8-2y$.

   A unit normal to the surface is $\quad \vec{n} = \dfrac{1}{\sqrt{14}}(\vec{i}+2\vec{j}+3\vec{k})$.

   $\vec{n}\cdot\vec{k} = \dfrac{3}{\sqrt{14}}$

   Surface area $\displaystyle = \iint_R \frac{dA}{|\vec{n}\cdot\vec{k}|} = \int_0^4\int_0^{8-2y}\frac{\sqrt{14}}{3}dxdy = \frac{16\sqrt{14}}{3}.$

   (b) The intersection of the paraboloid with the xy-plane is the circle $\quad x^2+y^2=9$.

   The projection R of the portion of paraboloid on the xy-plane is the disk with radius 3.

   A unit normal vector to the paraboloid is $\quad \vec{n} = \dfrac{1}{\sqrt{4x^2+4y^2+1}}(-2x\vec{i}-2y\vec{j}-\vec{k})$.

   Surface area $\displaystyle = \iint_R \frac{dA}{|\vec{n}\cdot\vec{k}|} = \iint \sqrt{4x^2+4y^2+1}\; dA.$

   In terms of polar coordinates, the integral becomes

   $\displaystyle\int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\; rdrd\theta = \frac{\pi}{6}(37^{3/2}-1) \approx 37.3\,\pi.$

6. (a) R is the annular region between $x^2+y^2=1$ and $x^2+y^2=4$;

$$\iint_{\sigma} z^2 dS = \iint_{R} (x^2+y^2)\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\, dA$$

$$= \sqrt{2}\iint_{R}(x^2+y^2)dA = \sqrt{2}\int_0^{2\pi}\int_1^2 r^3 dr d\theta = \frac{15}{2}\pi\sqrt{2}.$$

(b) The projection of the surface σ (parameterized by $z=+\sqrt{1-x^2}$) on the xy-plane is the region $R=[-1,1]\times[0,1]$. The suface integral becomes

$$\iint_{\sigma} x^2 y dS = \int_0^1\int_{-1}^1 x^2 y\sqrt{\left(\frac{-x}{\sqrt{1-x^2}}\right)^2+1}\, dx dy = \frac{1}{2}\int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}}dx.$$

With the substitution $u=\sin x$, the integral can be evaluated as $\displaystyle\int_{-\pi/2}^{\pi/2}\sin^2 u\ du = \pi/2$. Therefore, $\displaystyle\iint_{\sigma} x^2 y\, dS = \pi/4$.

Solution to Set 9

1. $\displaystyle\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-x^2}} x\cos z\,dy\,dx\,dz = \left(\int_0^{\pi/2}\cos z\,dz\right)\left(\int_0^1 x\sqrt{1-x^2}\,dx\right) = \frac{1}{3}$

2. (a) The parabolic sheet cuts the xy-plane at the lines $x = \pm 1$.

   $\displaystyle\therefore \iiint_D dv = \iint_R \int_0^{1-x^2} dz\,dA$ where R is the rectangular region $[-1,1]\times[-1,2]$.

   The volume is $\displaystyle\int_{-1}^{2}\int_{-1}^{1}(1-x^2)\,dx\,dy = 3\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 4.$

   (b) $\displaystyle\iiint_D dv = \iint_R \int_0^{x^2+y^2} dz\,dA$ where R is the triangular region with vertices $(0,0),\ (1,0),\ (0,1)$.

   The volume is $\displaystyle\int_0^1\int_0^{1-y}(x^2+y^2)\,dx\,dy = \int_0^1\left[\frac{1}{3}(1-y)^3 + y^2(1-y)\right]dy = \frac{1}{6}$

3. (a) $\displaystyle\iiint_D (x^2+y^2)\,dv = \int_0^{2\pi}\int_0^1\int_0^4 r^2\,dz\,r\,dr\,d\theta = 2\pi$

   (b) $\displaystyle\iiint_D z\,dv = \int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} z\,dz\,r\,dr\,d\theta = \frac{\pi}{16}$

4. (a) $\displaystyle\iiint_D x^2\,dv = \int_0^{2\pi}\int_0^{\pi}\int_2^3 \rho^2\sin^2\phi\cos^2\theta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

   $\displaystyle = \left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_2^3\rho^4\,d\rho\right) = \pi\times\frac{4}{3}\times\frac{1}{5}(3^5-2^5)\approx 56.3\,\pi$

   (b) The cone angle (from axis to edge) is $\displaystyle\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$

   $\displaystyle\iiint_D \frac{1}{x^2+y^2+z^2}\,dv = \int_0^{2\pi}\int_0^{\pi/6}\int_3^9 \frac{1}{\rho^2}\rho^2\,d\rho\sin\phi\,d\phi\,d\theta = 2\pi\times(1-\frac{\sqrt{3}}{2})\times 6\approx 1.6\,\pi$

5. (a) $\displaystyle x = \frac{1}{5}u + \frac{2}{5}v,\ y = -\frac{2}{5}u + \frac{1}{5}v,\ \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5};\qquad \frac{1}{5}\int_1^3\int_1^4 \frac{u}{v}\,du\,dv = \frac{3}{2}\ln 3.$

   (b) $x = u/3,\ y = v/2,\ z = w,\ \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 1/6$; S is the region in uvw-space enclosed by the sphere $u^2+v^2+w^2 = 36$. Therefore,

$$\iiint_D x^2\,dV_{xyz} = \iiint_S \frac{u^2}{9}\frac{1}{6}\,dV_{uvw} = \frac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^6(\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^6 \rho^4\sin^3\phi\cos^2\theta\,d\rho\,d\phi\,d\theta = \frac{192}{5}\pi$$

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Solution to Set 10

1. (a) $\vec{\nabla} \cdot \vec{F} = 2$
$\vec{\nabla} \times \vec{F} = (\partial_x y - \partial_y x)\vec{k} = \vec{0}$

(b) $\vec{\nabla} \cdot \vec{F} = \dfrac{y^2}{(x^2+y^2)^{3/2}} + \dfrac{x^2}{(x^2+y^2)^{3/2}} = \dfrac{1}{\sqrt{x^2+y^2}}$
$\vec{\nabla} \times \vec{F} = \left(\dfrac{-xy}{(x^2+y^2)^{3/2}} + \dfrac{xy}{(x^2+y^2)^{3/2}} \right) \vec{k} = \vec{0}$

(c) $\vec{\nabla} \cdot \vec{F} = 0$
$\vec{\nabla} \times \vec{F} = -2\vec{k}$

(d) $\vec{F}$ is not differentiable on the circle $x^2 + y^2 = 1$.
$$\vec{\nabla} \cdot \vec{F} = \begin{cases} 0 & \text{for} \quad x^2+y^2 < 1 \\ 0 & \text{for} \quad x^2+y^2 > 1 \end{cases}$$
$$\vec{\nabla} \times \vec{F} = \begin{cases} -2\vec{k} & \text{for} \quad x^2+y^2 < 1 \\ \left(\dfrac{x^2-y^2}{(x^2+y^2)^2} - \dfrac{x^2-y^2}{(x^2+y^2)^2} \right) \vec{k} = \vec{0} & \text{for} \quad x^2+y^2 > 1 \end{cases}$$

2. $\vec{\nabla} \cdot f\vec{F} = 3e^{x+y+z}$
$\vec{\nabla} \times f\vec{F} = \vec{0}$

3. Let $F_1,\ F_2,\ F_3$ be the components of $\vec{F}$. The x component of $\vec{\nabla} \times (\phi\vec{\nabla}F)$ is $\partial_y(\phi F_3) - \partial_z(\phi F_2) = \phi(\partial_y F_3 - \partial_z F_2) + (\phi_y F_3 - \phi_z F_2)$. It is the same as the x component of $\phi\vec{\nabla} \times \vec{F} + \vec{\nabla}\phi \times \vec{F}$. Similarly, the same can be shown for the y and z components.

4. First, note that $\dfrac{\partial f(r)}{\partial x} = f' \dfrac{\partial r}{\partial x}$. Therefore, $\vec{\nabla} f(r) = f' \vec{\nabla} r$.

Furthermore $\dfrac{\partial r}{\partial x} = \dfrac{\partial \sqrt{x^2+y^2+z^2}}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2+z^2}} = \dfrac{x}{r}$. Thus, $\vec{\nabla} r = \dfrac{\vec{r}}{r}$.

$\vec{\nabla} \cdot \vec{F} = \vec{\nabla} \cdot f\vec{r} = f\vec{\nabla} \cdot \vec{r} + (\vec{\nabla} f) \cdot \vec{r}$

Since $\vec{\nabla} \cdot \vec{r} = 3$ and $\vec{\nabla} f = f' \vec{\nabla} r = f' \dfrac{\vec{r}}{r}$,

$\vec{\nabla} \cdot \vec{F} = 3f + rf' = 0 \quad$ and $\quad \dfrac{f'}{f} = \dfrac{d}{dr} \ln|f| = -\dfrac{3}{r}$.

$\ln|f| = \displaystyle\int -\frac{3}{r} dr = -3\ln r + C \quad \Rightarrow \quad f = \frac{\pm e^C}{r^3}$.

Therefore,
$$\vec{F} = \frac{A}{r^2}\hat{r}$$
where A is an arbitrary constant and $\hat{r} = \vec{r}/r$ is the unit vector along the direction of $\vec{r}$.

Solution to Set 11

1. (a) $\dfrac{d\vec{r}}{dt} = \dfrac{3}{2}t^{1/2}(\vec{i}+\vec{j})$

   $\vec{F}\cdot\dfrac{d\vec{r}}{dt} = [x(t)+y(t)]\dfrac{3}{2}t^{1/2} = 3t^2$

   $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_1^2 3t^2 dt = 7$

   (b) $\dfrac{d\vec{r}}{dt} = 2(-\sin t\vec{i} + \cos t\vec{j})$

   $\vec{F}\cdot\dfrac{d\vec{r}}{dt} = -2x(t)\sin t + 2y(t)\cos t = 0$

   $\displaystyle\int_C \vec{F}\cdot d\vec{r} = 0$

   (c) $\dfrac{d\vec{r}}{dt} = \dfrac{3}{2}t^{1/2}(\vec{i}+\vec{j})$

   $\vec{F}\cdot\dfrac{d\vec{r}}{dt} = \dfrac{y(t)-x(t)}{x^2(t)+y^2(t)}\dfrac{3}{2}t^{1/2} = 0$

   $\displaystyle\int_C \vec{F}\cdot d\vec{r} = 0$

   (d) $\dfrac{d\vec{r}}{dt} = 2(-\sin t\vec{i} + \cos t\vec{j})$

   $\vec{F}\cdot\dfrac{d\vec{r}}{dt} = -\dfrac{2y(t)\sin t + 2x(t)\cos t}{x^2(t)+y^2(t)} = -1$

   $\displaystyle\int_C \vec{F}\cdot d\vec{r} = -2\pi$

   Note that $\vec{\nabla}\times\vec{F} = 0$ everywhere except at $(0,0)$ (that makes the line integral of the closed loop non-zero).

2. (a) Define the function f by performing a line integral along the line segments $(0,0)\to(x,0)$ and $(x,0)\to(x,y)$

   $f(x,y) = \displaystyle\int_0^x (e^t+0)dt + \int_0^y (x+2t)dt = e^x + xy + y^2$

   $\vec{\nabla}f = (e^x+y)\vec{i} + (x+2y)\vec{j} =$ the field

   $\therefore$ the field is conservative $\Rightarrow$ $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ is path independent.

   $\displaystyle\int_{(0,1)\to(2,3)} \vec{F}\cdot d\vec{r} = e^2 + 2\cdot 3 + 3^2 - (e^0+0+1) = e^2 + 13$

   (b) Define the function f by performing a line integral along the line segments $(0,0)\to(x,0)$ and $(x,0)\to(x,y)$

   $f(x,y) = \displaystyle\int_0^x (0+1)dt + \int_0^y 2x^2 t dt = x + x^2y^2$

   $\vec{\nabla}f = (1+2xy^2)\vec{i} + 2x^2y\vec{j} =$ the field

   $\therefore$ the field is conservative $\Rightarrow$ $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ is path independent.

   $\displaystyle\int_{(-1,2)\to(2,3)} \vec{F}\cdot d\vec{r} = 2 + 4\cdot 9 - (-1+4) = 35$

3. (a) Let R be the region enclosed by the closed curve C.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\partial_x 3x - \partial_y y) dA = \iint_R 2dA = 8\pi$$

(b) Let R be the region enclosed by the closed curve C.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\partial_x x^3 - \partial_y y^4) dA = 4\left[x^3\right]_{-2}^{2} - 4\left[y^4\right]_{-2}^{2} = 64$$

Solution to Set 12

1. (a) Σ is composed of two pieces of smooth surfaces, a hemisphere and a flat disk.
(i) The hemisphere is the graph of the function $f(x,y) = \sqrt{1-x^2-y^2}$.
$\partial_x f = -\dfrac{x}{\sqrt{1-x^2-y^2}} \qquad \partial_y f = -\dfrac{y}{\sqrt{1-x^2-y^2}}$
The projection of the sphere on the xy-plane is the unit disk D centered at the origin. The surface integral over the hemisphere is given by
$$\iint \vec{F}\cdot\vec{n}dS = \iint_D \left(-x\partial_x f - y\partial_y f + \sqrt{1-x^2-y^2}\right) dA = \iint_D \frac{1}{\sqrt{1-x^2-y^2}} dA$$
$$= \int_0^{2\pi}\int_0^1 \frac{rdrd\theta}{\sqrt{1-r^2}} = 2\pi$$
(ii) On the disk, the outward unit normal is $-\vec{k}$ and $\vec{F}\cdot\vec{n} = -z = 0$. Therefore, the flux integral over this piece is 0.
In total, $\displaystyle\iint_\Sigma \vec{F}\cdot\vec{n}dS = 2\pi$.

(b) Σ is composed of three smooth pieces, a cylindrical surface and two flat disks.
(i) The cylindrical surface
To evaluate the flux integral over the cylindrical surface, one may cut the cylinder into two pieces and project them onto the xz-plane (or the yz-plane).
The two pieces are graphs of the two functions $f_1(x,z) = \sqrt{4-x^2}$ and $f_2(x,z) = -\sqrt{4-x^2}$. The projection of the graphs on the xz-plane is the rectangular region $R = (-2,2)\times(0,2)$.
Note that for the graph of f_1 an outward-pointing normal is $-\partial_x f_1\vec{i} + \vec{j} - \partial_z f_1\vec{k}$ while that for the graph of f_2 is $\partial_x f_2\vec{i} - \vec{j} + \partial_z f_2\vec{k}$.
In terms of the components F_1, F_2, F_3 of the vector field (along the x, y, z directions respectively), the surface integral over the graph of f_1 is
$$\iint \vec{F}\cdot\vec{n}dS = \iint_R (-F_1\partial_x f_1 + F_2 - F_3\partial_z f_1)dA = \iint_R \left(\frac{x}{\sqrt{4-x^2}}\cdot xy + xy + 0\right) dA$$
$$= \int_0^2\int_{-2}^2 (x^2 + x\sqrt{4-x^2})dxdz = \frac{32}{3}.$$
The surface integral over the graph of f_2 is
$$= \iint_R (F_1\partial_x f_2 - F_2 + F_3\partial_z f_2)dA = \iint_R \left(\frac{x}{\sqrt{4-x^2}}\cdot xy - xy + 0\right) dA$$
$$= \int_0^2\int_{-2}^2 (-x^2 + x\sqrt{4-x^2})dxdz = -\frac{32}{3}.$$
The sum is therefore 0.

An alternative argument that can lead to this conclusion faster is to note that in 3D, the outward pointing unit normal to the cylindrical surface is $\dfrac{1}{\sqrt{x^2+y^2}}(x\vec{i}+y\vec{j})$.

Therefore $\vec{F}\cdot\vec{n} = \dfrac{x^2y+y^2x}{\sqrt{x^2+y^2}}$ and the surface integral can be found to be 0 by antisymmetry of the integrand or by writing it as
$$\int_0^2\int_0^{2\pi} \frac{8(\cos^2\theta\sin\theta + \sin^2\theta\cos\theta)}{2} 2d\theta dz = 0.$$
(ii) The flat disks
For the piece at $z=0$, $\vec{n} = -\vec{k}$, $\vec{F}\cdot\vec{n} = -z^2 = 0$.
For the piece at $z=2$, $\vec{n} = \vec{k}$, $\vec{F}\cdot\vec{n} = z^2 = 4$.

$$\therefore \iint \vec{F}\cdot\vec{n}dS = 16\pi$$

In total $\displaystyle\iint_{\Sigma} \vec{F}\cdot\vec{n}dS = 16\,\pi$

2. (a) $\vec{\nabla}\cdot\vec{F} = 3$

   $$\iiint \vec{\nabla}\cdot\vec{F}dV = 3\cdot\frac{1}{2}\cdot\frac{4\pi\,1^3}{3} = 2\pi$$

   (b) $\vec{\nabla}\cdot\vec{F} = y + x + 2z$

   $$\iiint \vec{\nabla}\cdot\vec{F}dV = \iiint (y+x)dV + 2\iiint zdV = 2\left(\int_0^2 zdz\right)(\pi 2^2) = 16\,\pi$$

   The integral with the integrand y (or x) is 0 due to antisymmetry.

3. (a) $\vec{\nabla}\times\vec{F} = \vec{i}+\vec{j}+\vec{k}$

   The surface Σ is the graph of the function $f(x,y) = 1 - x^2 - y^2$ over the unit disk D. For $\vec{n}$ directed downward

   $$\oint_C \vec{F}\cdot d\vec{r} = \iint_{\Sigma} \vec{\nabla}\times\vec{F}\cdot\vec{n}dS = \iint_D (-2x-2y-1)dA$$

   $$= -\int_0^1\int_0^{\pi/2}(2r\cos\theta + 2r\sin\theta + 1)d\theta rdr = -(\frac{4}{3}+\frac{\pi}{4}).$$

   (b) $\vec{\nabla}\times\vec{F} = 2z\vec{j} - y\vec{k}$

   The surface Σ is the graph of the function $f(x,y) = \sqrt{4-x^2-y^2}$ over the disk D centered at the origin with radius 2. For $\vec{n}$ directed upward

   $$\oint_C \vec{F}\cdot d\vec{r} = \iint_{\Sigma} \vec{\nabla}\times\vec{F}\cdot\vec{n}dS = \iint_D (2z\frac{y}{\sqrt{4-x^2-y^2}} - y)dA = \iint_D ydA = 0.$$