1. Solve the diffusion equation with constant dissipation:

\[ u_t - ku_{xx} + bu = 0 \quad \text{for} \quad -\infty < x < \infty \quad u(x, 0) = \phi(x), \]

where \( b > 0 \) is a constant. [Hint: Make the change of variables \( u(x, t) = e^{-bt}v(x, t) \).]

**Answer**

Let \( u = e^{-bt}v \), then \( u_t = -be^{-bt}v + e^{-bt}v_t \) and \( u_{xx} = e^{-bt}v_{xx} \). Substituting these into the PDE for \( u \) gets

\[ v_t - kv_{xx} = 0. \]

The initial condition for \( v \) is \( v(x, 0) = e^0u(x, 0) = \phi(x) \).

The solution for \( u \) is

\[ u(x, t) = e^{-bt} \int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy. \]

2. Solve the diffusion equation with advection:

\[ u_t - ku_{xx} + Vu_x = 0 \quad \text{for} \quad -\infty < x < \infty \quad u(x, 0) = \phi(x), \]

where \( V \) is a constant. (Hint: Go to the moving frame of reference by substituting \( y = x - Vt \).)

**Answer**

Let \( \xi = x - Vt, \eta = t \). The inverse relations are \( x = \xi + V\eta, t = \eta \). The chain rules give \( \partial_x = \partial_\xi \) and \( \partial_t = -V\partial_\xi + \partial_\eta \). With the change of coordinates, the equation becomes

\[ w_\eta - kw_{\xi\xi} = 0 \]

where \( w(\xi, \eta) = u(x, t) \). The initial condition is \( w(\xi, 0) = u(\xi + 0 \cdot V, 0) = u(x, 0) = \phi(x) = \phi(\xi + 0 \cdot V) = \phi(\xi) \). The solution is

\[ w(\xi, \eta) = \int_{-\infty}^{\infty} S(\xi - y, \eta)\phi(y)dy. \]

Therefore,

\[ u(x, t) = \int_{-\infty}^{\infty} S(x - Vt - y, t)\phi(y)dy. \]
3. The purpose of this exercise is to show that the maximum principle is not true for the equation \( u_t = xu_{xx} \), which has a variable coefficient.

a. Verify that \( u = -2xt - x^2 \) is a solution. Find the location of its maximum in the rectangle \( \{-2 \leq x \leq 2, 0 \leq t \leq 1\} \)

b. Where precisely does the proof of the maximum principle break down for this equation?

**Answer**

a. \( u_t = -2x, \ u_x = -2t - 2x, \ u_{xx} = -2, \) so \( u_t - xu_{xx} = (-2x) - (-2x) = 0. \) If a maximum is at an interior point, then \( u_t = u_x = 0 \Rightarrow x = t = 0, \) which is in fact on the bottom boundary where \( u = 0 \ \forall x \in (-2, 2). \) There is no maximum in the interior.

On the left side, \( \{(x, t)| x = -2, 0 \leq t \leq 1\} \), \( u(-2, t) = 4(t - 1) \leq 0 \ \forall 0 \leq t \leq 1. \) On the right side, \( \{(x, t)| x = 2, 0 \leq t \leq 1\} \), \( u(2, t) = -4(t + 1) < 0 \ \forall 0 \leq t \leq 1 \) Therefore the maximum value of \( u \) on the bottom and sides of the rectangle is 0. On the top, \( \{(x, t)| -2 < x < 2, \ t = 1\} \), \( u(x, 1) = -2x - x^2. \) Note that the sole variable \( x \) is in an open interval \( (-2, 2) \). A maximum, if exists, will have a value of \( x \) in the interior of this interval with \( \frac{\partial}{\partial x} u(x, 1) = 0. \) Taking the derivative w.r.t. \( x \) produces \(-2 - 2x \) which is 0 at \( x = -1. \) Therefore, the maximum of \( u \) is at \((-1, 1)\) and the value is \( u(-1, 1) = 2 - 1 = 1. \)

b. At this maximum point \( u_t(-1, 1) = 2 > 0, \) and \( u_{xx}(-1, 1) = -2 < 0 \) but \( u \) still satisfies the PDE as the coefficient of \( u_{xx} \) is a variable.

4. Prove that if \( \phi(x) \) is any piecewise continuous function, then

\[
\lim_{t \to 0^+} \frac{1}{\sqrt{\pi}} \int_0^{\pm \infty} e^{-p^2} \phi(x + \sqrt{4kt} p) dp = \pm \frac{1}{2} \phi(x^\pm).
\]

**Answer**

Since \( \phi \) is bounded, \( \exists \) a number \( B \) so that \( |\phi(x)| < B \ \forall x. \)

Since \( \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-p^2} = 1/2, \) for any fixed \( x_0 \)

\[
\left[ \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-p^2} \phi(x_0 + \sqrt{4kt} p) dp \right] - \frac{1}{2} \phi(x_0^+) =
\]
\[
= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-p^2} [\phi(x_0 + \sqrt{4kt} p) - \phi(x_0^+)] dp.
\]

It is necessary to show that the last expression tends to 0 as \( t \to 0^+ \). Since the right hand limit of \( \phi \), as \( x \to x_0^+ \), is \( \phi(x_0^+) \), for any \( \epsilon > 0 \), \( \exists \delta > 0 \) so that

\[
|\phi(x_0 + \sqrt{4kt} p) - \phi(x_0^+)| < \epsilon \quad \text{for} \quad 0 < p < \delta/(\sqrt{4kt}).
\]

Now split the integral into two parts, one from 0 to \( \delta/(\sqrt{4kt}) \) (to be represented as \( \int_0^{\delta/(\sqrt{4kt})} \)) and the other from \( \delta/(\sqrt{4kt}) \) to \( \infty \) (to be represented as \( \int_{\delta/(\sqrt{4kt})}^{\infty} \)). The first part satisfies

\[
\left| \int_0^{\delta/(\sqrt{4kt})} \right| < \epsilon \cdot \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-p^2} dp < \frac{\epsilon}{2}.
\]

The second part satisfies

\[
\left| \int_{\delta/(\sqrt{4kt})}^{\infty} \right| < 2B \cdot \frac{1}{\sqrt{\pi}} \int_{\delta/(\sqrt{4kt})}^{+\infty} e^{-p^2} dp
\]

\[
< 2B \cdot \frac{1}{\sqrt{\pi}} \int_{\delta/(\sqrt{4kt})}^{+\infty} e^{-p^2} dp \quad \text{(as long as} \ t \ \text{is small enough to make} \ \delta/(\sqrt{4kt}) > 1)\]

\[
= 2B \cdot \frac{e^{-\delta/(\sqrt{4kt})}}{\sqrt{\pi}}
\]

For small enough \( t \), say \( t < \frac{1}{4k} \left( \frac{\delta}{\ln \left( \frac{4B}{\epsilon \sqrt{\pi}} \right)} \right)^2 \), it is less than \( \frac{\epsilon}{2} \). The sum of the two parts is therefore less than \( \epsilon \).

The \( \phi(x_0^-) \) case can be similarly proven.

5. Solve \( u_t = ku_{xx}; \ u(x, 0) = 0; \ u(0, t) = 1 \) on the half-line \( 0 < x < \infty \).

**Answer**

Let \( u \) be the solution of the problem and let \( v = u - 1 \), then \( v \) satisfies

\[
v_t = kv_{xx}, \ v(0, t) = u(0, t) - 1 = 0, \ v(x, 0) = u(x, 0) - 1 = -1.
\]

This a standard IBVP with the Dirichlet boundary condition. The solution is

\[
v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} [e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}](1) dy
\]

Let \((x - y)/\sqrt{4kt} = p\) and \((x + y)/\sqrt{4kt} = q\), then the result becomes

\[
eq \frac{1}{\sqrt{\pi}} \left[ \int_p^{-\infty} e^{-p^2} dp + \int_q^{\infty} e^{-q^2} dq \right]
\]
\[ u(x, t) = v + 1 = 1 - \text{erf}\left(\frac{x}{\sqrt{4kt}}\right). \]

6. Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

\[ u_t - ku_{xx} = f(x, t) \]

\[ u(0, t) = 0 \quad u(x, 0) = \phi(x) \quad 0 < x < \infty, \ 0 < t < \infty \]

using the method of reflection.

**Answer**

The problem can be split into two parts: (i) homogeneous diffusion with initial condition \( u(x, 0) = \phi(x) \), and (ii) the inhomogeneous problem \( w_t - kw_{xx} = f(x, t) \) with \( w(x, 0) = 0 \). Solution of the full problem is the sum of solutions to (i) and (ii). The solution of (i) is already given in the notes (wk9). We only need to consider (ii) here.

Use odd extension and define

\[ f_e(x) = \begin{cases} f(x) & \text{for } x > 0 \\ -f(-x) & \text{for } x < 0. \end{cases} \]

The extended solution on \(-\infty < x < \infty\) is

\[ w_o(x, t) = \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^\infty S(x - y, t - s) f_o(y, s) dy ds. \]

For \( x > 0 \),

\[ w(x, t) = \frac{1}{\sqrt{\pi}} \int_0^t \left[ -\int_{-\infty}^0 S(x - y, t - s) f(-y, s) dy + \int_0^\infty S(x - y, t - s) f(y, s) dy \right] ds \]

Let \( y' = -y \). The first inner integral becomes

\[ \int_{-\infty}^0 S(x - y, t - s) f(-y, s) dy = \int_0^\infty S(x + y', t) f(y', t - s) dy'. \]
Therefore, the solution is
\[
w(x, t) = \frac{1}{\sqrt{4\pi k}} \int_0^t \frac{1}{\sqrt{t-s}} \int_0^\infty \left[ e^{-(x-y)^2/4k(t-s)} - e^{-(x+y)^2/4k(t-s)} \right] f(y, s) dy ds.
\]

By setting \( x = 0 \) in the above equation, one can see that the boundary condition \( w(0, t) = 0 \) is satisfied. It is also obvious that \( w(x, 0) = 0 \).