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FOREWORD

The Conference on Scientific Computation '94 was held on March 17–19, 1994 at the Chinese University of Hong Kong. It is the third of its kind in Hong Kong. The last two were held in 1990 and 1991. There has been a tremendous growth in the area of scientific computing in Hong Kong in the past few years—as is evidenced by the acquistion of three supercomputers by the local universities. The aim of this series of conferences is to promote the research interest in scientific computation for local mathematicians and engineers and to foster contacts and exchanges with experts from other parts of the world.

There were twelve talks in the 1994 conference giving by invited speakers from the US, mainland China and local institutions. The keynote address was given by Professor Gene Golub of Standford University. The title of his talk is *"Matrices, Moments and Quadrature"*.Besides the invited talks there were also twenty-two contributed talks by local researchers in the field. More than 80 people attended the conference.

The proceedings of the Conference is published in this issue of the SEAMS Bulletin of Mathematics. It contains twenty-one papers from the Conference and is in four major areas: numerical linear algebra, algorithms, wavelets and differential equations. The editors of the Proceedings are Wei-Min Xue (Chief Editor), Raymond Chan, Daniel Ho and Yue-Kuen Kwok.

On behalf of the organizers of the Conference, we would like to thank the following sponsors for their gracious supports: the Hong Kong Mathematical Society, the Institute of Mathematical Sciences, the Department of Computer Science and the Department of Mathematics at the Chinese University of Hong Kong, and the Departments of Mathematics of the City University of Hong Kong, Hong Kong Baptist University and Hong Kong University of Science and Technology.

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THE 'STRATIFIED' APPROXIMATION FOR COMPUTING GEOPHYSICAL FLOWS

Kwing L. Chan
Department of Mathematics
Hong Kong University of Science and Technology
Clear Water Bay, Hong Kong

Abstract In an earlier paper, Chan et al. [1] introduced the 'stratified' approximation to simplify the Navier Stokes equations for numerical solution with the spectral method. The resulting equations were written in spherical geometry. The errors associated with this approximation are now analysed in greater detail to include cases where the horizontal variations of the thermodynamic variables are larger than M^2 where $M(< 1)$ is the Mach number. We also write down the stratified form of the hydrodynamics equations in Cartesian geometry.

1. THE STRATIFIED APPROXIMATION

In geophysical fluid dynamics, the principal equations to be solved are the Navier Stokes equations for a rotating, compressible fluid under gravity:

$$\partial_t \rho = -\nabla \cdot M \tag{1}$$

$$\partial_t M = -\nabla \cdot (MM/\rho) + \nabla \cdot \sigma - \nabla p + \rho g - 2\Omega \times M \tag{2}$$

$$\partial_t p = -\nabla \cdot (pM/\rho) - (\Gamma - 1)p\nabla \cdot (M/\rho) - \nabla_{ad}\Gamma(\nabla \cdot f - \varepsilon) \tag{3}$$

where ∂_t is the time derivative; ρ is the density; $M(= \rho V)$ is the mass flux; V is the velocity; σ is the viscous stress tensor; p is the pressure; Ω is the angular velocity of the rotating frame; g is the gravitational acceleration; r is the position vector; Γ is the adiabatic exponent defined by $[\partial ln(p)/\partial ln(\rho)]_{ad}$; $\nabla_{ad} = [\partial ln(T)/\partial ln(p)]_{ad}$ is the adiabatic temperature gradient; T is the temperature; f is the radiative or conductive energy flux; and ε is a heating/cooling rate per unit volume (including the viscous dissipation of kinetic energy).

There are many advantages in handling geophysical flow problems with the spectral approach [2, 3]. However, the application of the spectral approach to the compressible Navier Stokes equations (1)–(3) is troubled by the occurrence of the $1/\rho$ factor in the nonlinear terms. Its presence generates high-order coupling in the spectral expansions (in contrast to only second-order coupling in the incompressible case) and thus makes de-aliasing of the transforms much more difficult. To circumvent this problem, Chan et al. [1] introduced the so called 'stratified' approximation which reduces the order of the nonlinearity to two while preserving the compressibility and some conservation properties of

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the fluid equations. The idea was based on the observation that the horizontal variations of ρ, p, and T are generally much smaller than their respective mean values.

In other words, let q be one of the above thermodynamic variables; q can be decomposed into two parts of widely different magnitudes:

$$q = q_0 + q_1 \tag{4}$$

where q_0 is the horizontal mean and is only a function of the vertical coordinate (z); q_1 represents the horizontal variation which can be expressed in terms of a series of horizontal harmonic functions whose coefficients are functions of z. After substituting such expressions into Eqs. (1)–(3), the following procedures can be performed:

(i) The horizontal variation of density is ignored in the nonlinear advection terms of the momentum equation.

(ii) Terms containing products of two or more horizontal variations of the thermodynamic variables are ignored in the advection terms of the energy equation.

The assumptions made by this 'stratified' approximation are a subset of those of the popular anelastic approximation [4–5]. In Ref. [1], discussion of the validity of the stratified approximation was very brief and was hinged on the validity of the anelastic approximation. The justification of the anelastic approximation [5], however, assumes that the relative horizontal fluctuations of the thermodynamic variables (i.e. q_1/q_0) are on the order of M^2 where M is the Mach number. In some geophysical situations, it is quite possible that this assumption does not hold (see later discussion), and is therefore necessary to reconsider the error estimates.

2. ERROR ESTIMATES

For small-scale or nonrotating flows, a balance is approximately maintained between the pressure gradient and the nonlinear advection terms of the momentum equation, and it is quite reasonable to assume that the relative fluctuations of the thermodynamic quantities are on the order of M^2 [5, 6]. For large-scale geophysical flows, however, the Coriolis term of Eq. (2) often changes the situation. A prominent example is the prevalence of 'geostrophic balance' in planetary atmospheres where for latitude $> 10°$, the horizontal pressure gradient is primarily balanced by the Coriolis force, so that the horizontal wind blows perpendicularly to the pressure gradient. In such case, the magnitudes of p_1 and V should be related primarily as $p_1/l \sim \rho v \Omega$ where l and v are certain characteristic horizontal length and velocity, respectively. Therefore

$$\frac{p_1}{p_0} \sim \left(\frac{a\Omega}{c}\right)\left(\frac{v}{c}\right) \tag{5}$$

in which the radius of the planet a is used for l, and the square of the sound speed c approximates the ratio p/ρ. The rightmost factor in the above equation is the Mach number M. The factor in front of it is the rotational speed of the planet divided by the sound speed; for Earth and Jupiter, it is on the order of 1 and 10, respectively. Therefore, the effect of rotation on the horizontal pressure variation is very significant, and the relative fluctuations of the thermodynamic quantities are first-order in M.

What is then the size of errors introduced to Eq. (2) by the stratified approximation? According to Procedure (i) specified in Section 1, the density is allowed to move freely in and out of a horizontal derivative in a nonlinear term; that is equivalent to ignoring $v\partial_i\rho$ in comparison with $\rho\partial_i v$. The relative error is therefore

$$\frac{v(\rho_1/l)}{\rho(v/l_v)} \sim \left(\frac{l_v}{l}\right)\left(\frac{\rho_1}{\rho_0}\right). \tag{6}$$

When the rotational effect is important as discussed above, this ratio is on the order of M. However, in such case, the nonlinear term itself is small compared to the pressure term or the Coriolis term. Relative to the pressure term which is significant in all situations, the net relative error is

$$\left(\frac{l_v}{l}\right)\left(\frac{\rho_1}{\rho_0}\right)\frac{(\rho v^2/l_v)}{(p_1/l)} \sim M^2 \tag{7}$$

where we have used $\rho_1/\rho_0 \sim p_1/p_0$. Therefore, the relative error introduced to Eq. (2) is always second-order in the Mach number, even when rotational effects are significant. Note that this is facilitated by the avoidance of any approximation in the pressure gradient and coriolis terms.

The generation of significant variations of the thermodynamic quantities in the global scale, estimated by Eq. (5), is usually induced by the horizontal variation of an external energy source. For example, solar heating of the Earth's atmosphere depends on latitude; the differential heating generates pressure variations between the equator and the poles. Such process is described by the last two terms of Eq. (3) which, under such circumstance, are usually the dominant terms. The energy equation is therefore mainly controlled by a balance between the local heating (h) and cooling rates:

$$h \sim p_1/\tau. \tag{8}$$

The cooling is estimated here as p_1/τ where τ is a time scale describing radiative loss or heat conduction. In case that the time variation of the pressure is significant, τ can also be interpreted as the time scale of temporal change (for example, $\tau = 1day$ for diurnal tides).

According to Procedure (ii) of the stratified approximation, the relative errors introduced within the advection terms of Eq. (3) (first two terms on the right hand side) is on the order of ρ_1/ρ_0. For small scales or when external heating is insignificant, this fraction is second order in M, but as discussed earlier, it may become first order in the global scale. In such case, however, the errors should be compared with the magnitude of the dominant energy terms and the ratio becomes:

$$\left(\frac{v(p_1/a)}{p_1/\tau}\right)\left(\frac{\rho_1}{\rho_0}\right) \sim \left(\frac{\tau c}{a}\right)\left(\frac{v}{c}\right)\left(\frac{\rho_1}{\rho_0}\right) \tag{9}$$

(Note that $\Gamma\nabla_{ad}$ is on the order of 1.) Therefore, the relative error is formally second order in M, even when a large-scale differential heating is present. Care, however, has to be taken because of the presence of the $(c\tau/a)$ factor. In the case of the Earth, τ is on the order of 1–10 days so that this factor is on the order of 1–10. To ensure good accuracy, it may be necessary to include higher order nonlinearities of the density or pressure variations in the energy equation. For the computation of the zonally averaged circulation in the Earth's lower and middle atmosphere (up to 100 km in height), Procedure (ii) turns out to be quite accurate [7] since the v that appears in Eq. (9) is only about 3% of c. (The magnitude of the meridional velocity is used as the horizontal pressure differential is mainly between the equator and the poles.)

Another point of caution concerns the computation of the energy source/sink term. This term usually depends on the temperature. While computing the temperature from the pressure and density (prognostic variables), it is necessary to make sure that the approximations made are sufficiently accurate, and therefore second or higher order products of the variations of the thermodynamic variables may need to be included. For this reason, Procedure (ii) put down in Section 1 specifies that second or higher order products of horizontal variations of the thermodynamic variables are *only* ignored in the *advection terms,* not necessarily in all the terms. The wording is a little different and more accurate compared to that of Ref. [1].

3. STRATIFIED EQUATIONS IN CARTESIAN COORDINATES

To present the application of the stratified approximation concretely, we write down here the approximate equations in Cartesian coordinates, and in a curl-divergence formulation for easy comparison with the spherical case of Ref. [1]. Let U represent the column vector containing the five diagnostic variables $\rho, M_z, \delta(\equiv \partial_x M_x + \partial_y M_y), \zeta(\equiv \partial_x M_y - \partial_y M_x)$, and p. The fluid equations can be written as:

$$\partial_t U = W + C + D + N + S, \tag{10}$$

where W, C, D, N, and S represent the 'wave' (acoustic and gravity waves), Coriolis, diffusive, nonlinear, and source/sink terms, respectively. The expressions for them are:

$$W = \begin{pmatrix} -\partial_z M_z - \delta \\ -\partial_z p + \rho g \\ -\nabla_H^2 p \\ 0 \\ -\partial_z(p_0 M_r/\rho_0) - (\Gamma - 1)p_0\partial_z(M_r/\rho_0) - \Gamma p_0\delta/\rho_0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 \\ 2(\Omega_y M_x - \Omega_x M_y) \\ 2\Omega_z\zeta + 2(\Omega_x\partial_y - \Omega_y\partial_x)M_z \\ -2\Omega_z\delta + 2(\Omega_x\partial_x + \Omega_y\partial_y)M_z \\ 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 \\ 2\partial_z(\mu\partial_z V_z) + \mu\left[\nabla_H^2 V_z + \partial_z\delta_V\right] + \partial_z[\lambda(\nabla\cdot V)] \\ \partial_z\left[\mu\left(\nabla_H^2 V_z + \partial_z\delta_V\right)\right] + 2\mu\nabla_H^2\delta_V + \lambda\left(\partial_z\nabla_H^2 V_z + \nabla_H^2\delta_V\right) \\ \partial_z(\mu\partial_z\zeta_V) + \mu\nabla_H^2\zeta_V \\ \nabla_{ad}\Gamma\left[\partial_z(\kappa_p\partial_z p + \kappa_\rho\partial_z\rho) + \kappa_p\nabla_H^2 p + \kappa_\rho\nabla_H^2\rho\right] \end{pmatrix},$$

$$N = \begin{pmatrix} 0 \\ -\partial_z(M_z V_z) - \nabla_H\cdot(M_H V_H) \\ -\partial_z\nabla_H\cdot(V_z M_H) + \nabla_H\times(\zeta V_H) - \nabla_H\cdot(\delta V_H) - \nabla_H^2(M_x V_x + M_y V_y)/2 \\ -\partial_z\nabla_H\times(V_z M_H) - \nabla_H\cdot(\zeta V_H) - \nabla_H\times(\delta V_H) \\ -\nabla\cdot[(p_1 - p_0\rho_1/\rho_0)V] - (\Gamma-1)[p_1\nabla\cdot V - p_0\nabla\cdot(\rho_1 V/\rho_0)] \end{pmatrix},$$

and S contains only the term $\nabla_{ad}\Gamma\varepsilon$ for $\partial_t p$. The subscript H denotes the horizontal component of a certain quantity; therefore ∇_H is the horizontal gradient operator; similarly ∇_H^2 is defined to be $\partial_x^2 + \partial_y^2$; $\delta_V \equiv (\partial_x V_x + \partial_y V_y)$; $\zeta_V \equiv (\partial_x V_y - \partial_y V_x)$. In the derivation of D, we assume that the viscosity and diffusivity coefficients μ, λ, κ_p, and κ_ρ are functions of z only; if necessary, their horizontal variations can be included in N which is to be computed by the transform method [2].

Eq. 10 is suitable for applying the spectral approach. Suppose that the domain of calculation is a rectangular box with widths 2π and periodic horizontal boundary conditions, an arbitrary variables q can be represented by a finite Fourier sum:

$$q = \sum q_{nm} e^{i(nx+my)}. \tag{11}$$

The diagnostic variables M_x and M_y can be calculated from the prognostic variables δ and ζ componentwise:

$$(M_x)_{nm} = -i(n\delta_{nm} - m\zeta_{nm})/(n^2+m^2) \tag{12}$$
$$(M_y)_{nm} = -i(m\delta_{nm} + n\zeta_{nm})/(n^2+m^2) \tag{13}$$

for $n^2 + m^2 \neq 0$.

Notice that the only horizontal differential operator that appears in W is ∇_H^2. Functions of the form $e^{i(mx+ny)}$ are eigenfunctions of the operator. (For different boundary conditions, other types of eigenfunctions of the Laplacian operator can be chosen for the spectral expansions.) This property makes the time-implicit treatment of W rather simple. Only a block-tri-diagonal matrix generated by the operator ∂_z needs to be solved for each time step. The CFL conditions associated with the acoustic and gravity waves originated form W can be suppressed. This is of great advantage for computing subsonic geophysical flows.

In the present case, $e^{i(mx+ny)}$ is also an eigenfunction of the horizontal operators of W and C, and S contains no spatial derivatives. Therefore, W, C and the linearized part of S can all be included in the time-implicit treatment.

Eq. 10 and the expressions for the various terms are very similar to those of the spherical case. There is, however, one significant difference. The Coriolis terms here do not couple the different harmonic functions. This makes the implicit handling of C much simpler, and is a result of Ω being independent of position. In a β-plane situation where Ω is a linear function of a horizontal coordinate, say y, coupling between different eigenfunctions of ∇_H^2 will occur.

Proving conservation of total horizontal momentum and conservation of total mass (when there is no boundary exchange) is straightforward. Conservation of horizontal momentum results from the fact that the stratified approximation does not need to pull ρ outside the divergence operator of the advection term.

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