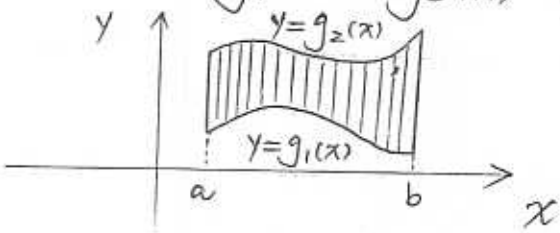
# 15.2 Double Integral over Non-Rectangular Regions.

Type I region: Bounded on the left and right by vertical lines $x=a$ and $x=b$, and bounded below and above by continuous curves $y=g_1(x)$ and $y=g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$.




$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b A(x)\,dx$$

$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$, evaluated by holding $x$ fixed.

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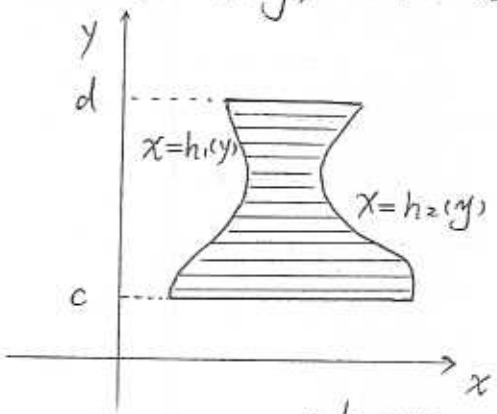
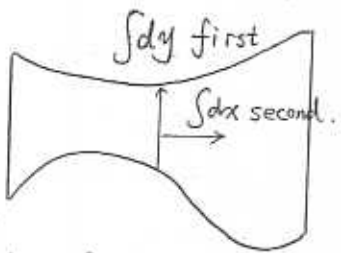
Type II region: Bounded below and above by horizontal lines $y=c$ and $y=d$, and bounded left and right by continuous curves $x=h_1(y)$ and $x=h_2(y)$, where $h_1(y) \le h_2(y)$ for $c \le y \le d$.




$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d A(y)\,dy$$

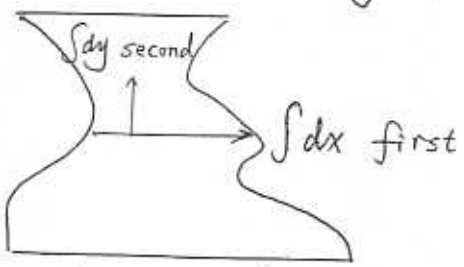
$A(y) = \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx$, evaluated by holding $y$ fixed.

Type I.




$\int dy$ first $\rightarrow A(x)$

Type II.



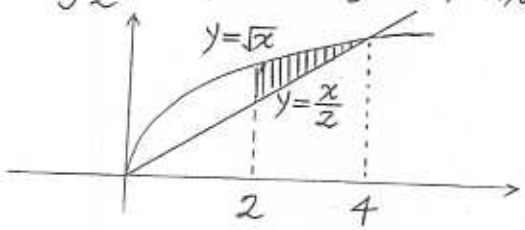

$\int dx$ first $\rightarrow A(y)$

Ex. Evaluate $\iint_R xy\,dA$ over the region $R$ enclosed by $y=\frac{1}{2}x$, $y=\sqrt{x}$, $x=2$, and $x=4$.

Region $R$ is type I. $y=\sqrt{x}$ is a curve above $y=\frac{x}{2}$ because $\sqrt{x} \geq x/2$ for $2 \leq x \leq 4$.

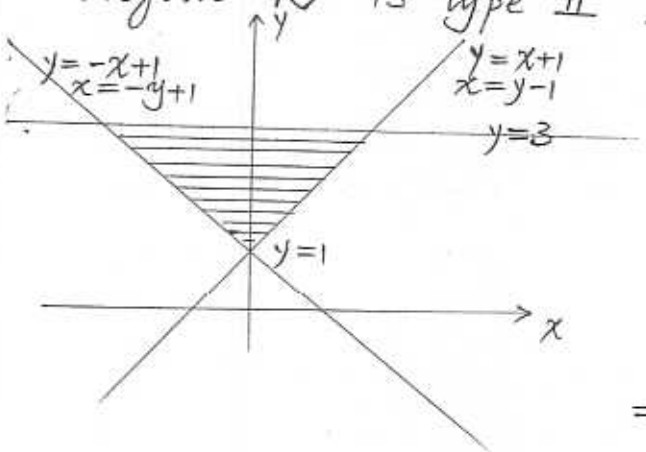
$$\iint_R xy\,dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\,dy\,dx = \int_2^4 \frac{1}{2}xy^2 \Big|_{y=\frac{x}{2}}^{\sqrt{x}} dx$$

$$= \int_2^4 \left(\frac{1}{2}x^2 - \frac{1}{8}x^3\right) dx = \left(\frac{1}{6}x^3 - \frac{1}{32}x^4\right)\Big|_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6}.$$




Ex. Evaluate $\iint_R (2x-y^2)\,dA$ over the triangular region enclosed by $y=-x+1$, $y=x+1$, and $y=3$.

Region $R$ is type II, bounded by $y=1$ and $y=3$ below and upper, and bounded by $x=-y+1$ and $x=y-1$ left and right.




$$\iint_R (2x-y^2)\,dA = \int_1^3 \int_{-y+1}^{y-1} (2x-y^2)\,dx\,dy$$

$$= \int_1^3 (x^2 - y^2x)\Big|_{x=-y+1}^{y-1} dy$$

$$= \int_1^3 \left[\left((y-1)^2 - y^2(y-1)\right) - \left((-y+1)^2 - y^2(-y+1)\right)\right] dy$$

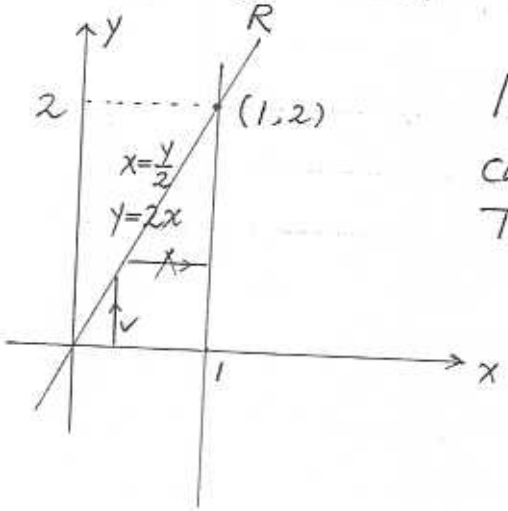
$$= \int_1^3 2(y^2-y^3)\,dy = \left(\frac{2}{3}y^3 - \frac{1}{2}y^4\right)\Big|_1^3$$

$$= \left(\frac{2}{3}\times 27 - \frac{1}{2}\times 81\right) - \left(\frac{2}{3} - \frac{1}{2}\right)$$

$$= -\frac{68}{3}$$

## Reversing the order of integration

Ex. $\iint_R e^{x^2}\,dA = \int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$




If $R$ is treated as type II, then $\int_{\frac{y}{2}}^1 e^{x^2}dx$ cannot be carried out.

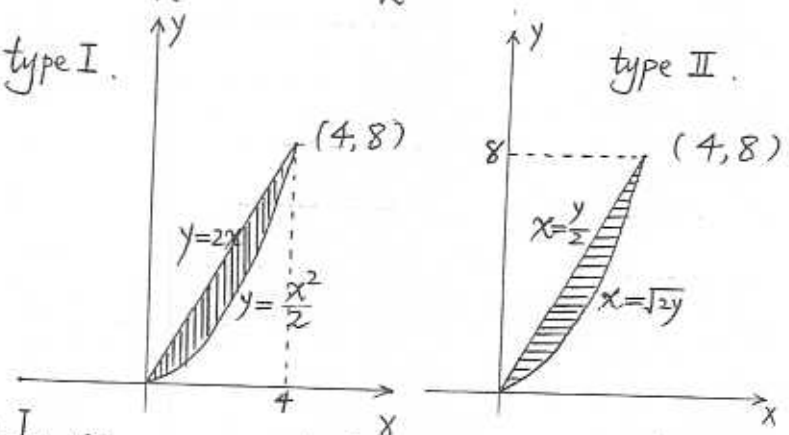
Treating $R$ as type I, $\iint_R e^{x^2}dA$ becomes

$$\int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx$$
$$= \int_0^1 ye^{x^2}\Big|_{y=0}^{2x}\,dx = \int_0^1 2xe^{x^2}dx$$
$$= \int_0^1 e^{x^2}dx^2 = e^{x^2}\Big]_0^1 = e-1.$$

## Area calculated as a double integral

When $f(x,y)=1$, Volume $V = 1 \times$ (area of $R$)

$$\iint_R 1\cdot dA = \iint_R dA = \text{area of } R.$$




Ex. Area enclosed by $y=2x$ and $y=\frac{x^2}{2}$. also by $x=\frac{y}{2}$ and $x=\sqrt{2y}$.

type I. $$\iint_R dA = \int_0^4 \int_{\frac{x^2}{2}}^{2x} dy\,dx = \int_0^4 (2x-\frac{x^2}{2})\,dx = (x^2-\frac{x^3}{6})\Big|_0^4 = 16-\frac{64}{6} = \frac{16}{3}$$

type II $$\iint_R dA = \int_0^8 \int_{\frac{y}{2}}^{\sqrt{2y}} dx\,dy = \int_0^8 (\sqrt{2y}-\frac{y}{2})\,dy = (\frac{2\sqrt{2}}{3}y^{\frac{3}{2}} - y^2/4)\Big]_0^8$$
$$= \frac{2\sqrt{2}}{3}\cdot 16\sqrt{2} - \frac{64}{4} = \frac{16}{3}$$

equal!