A new gas-kinetic scheme based on analytical solutions of the BGK equation

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A B S T R A C T

Up to an arbitrary order of the Chapman–Enskog expansion of the kinetic Bhatnagar–Gross–Krook (BGK) equation, a corresponding analytic solution can be obtained. Based on such a compact exact solution, a new gas-kinetic scheme is constructed for the compressible Navier–Stokes equations. Instead of using a discontinuous initial condition in the gas-kinetic BGK–NS method Xu [K. Xu, A gas-kinetic BGK scheme for the Navier–Stokes equations, and its connection with artificial dissipation and Godunov method, J. Comput. Phys. 171 (2001) 289–335.], the new scheme starts with a continuous initial flow distribution at a cell interface which is obtained through an upwind-biased WENO reconstruction, and uses the time accurate solution for the flux evaluation. The new kinetic scheme not only preserves favorable properties of the existing BGK–NS method, such as stability, high resolution, and good performance in capturing discontinuity, but also achieves a very high efficiency, which is even more efficient than the same order well-defined classical finite difference scheme based on the macroscopic governing equations directly. The stability, accuracy, and efficiency of the new scheme are evaluated quantitatively through numerical tests. The new scheme captures sharp discontinuity without post-shock oscillation, and has high accuracy in resolving viscous solution. Due to the use of time-accurate analytical solution, the overall performance of the new scheme is superior in comparison with the finite difference or finite volume schemes which start from the same initial reconstruction and use the numerical Runge–Kutta methods for time accuracy.

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1. Introduction

In the past years, the gas-kinetic scheme (GKS) has been well developed for compressible flow simulation [15,16,10,5]. The GKS uses the kinetic equation to model the physical process around a cell interface. With an initial discontinuity, the kinetic gas evolution covers the process from the kinetic scale free particle transport to the hydrodynamic scale Navier–Stokes solution.

In this paper, based on the Nth order Chapman–Enskog expansion of the kinetic Bhatnagar–Gross–Krook (BGK) equation [1], a corresponding compact analytic solution is obtained. Based on such an analytical solution, a new gas-kinetic scheme with high order accuracy can be constructed. A reconstructed continuous initial condition is used in the new scheme for the gas evolution and flux evaluation. Here, the well-defined Weighted Essential Non-Oscillatory (WENO) procedure is implemented for the initial data reconstruction [6,4,12]. In comparison with the BGK–NS method [16], the new kinetic scheme has the following distinguishable features. First, the new scheme is based on an analytical solution of the BGK equation,
which avoids the modelling for the temporal derivative in the previous method. Second, the new kinetic scheme uses a continuous initial reconstruction, which significantly reduces the computational cost associated with the time evolution from a discontinuous initial data. The current scheme is much more efficient than the BGK–NS method [16], and is about 1.5 times faster than the traditional finite difference schemes based on the macroscopic governing equations. Besides inheriting the favourable properties of the BGK–NS method, such as stability, high resolution, and robustness in capturing discontinuities, the present new third-order kinetic scheme is probably the fastest one among all third-order shock capturing schemes starting from the same initial reconstruction. Detailed comparison of the time efficiency is presented through many numerical examples.

In Section 2, a compact exact solution up to an arbitrary order of the Chapman–Enskog expansion of the BGK equation is introduced. Section 3 presents the corresponding numerical scheme and the third-order WENO reconstruction. Many benchmark numerical tests are included in Section 4, where the performance of the new scheme is validated in one and two dimensional flow problems for both the Euler and Navier–Stokes solutions. The last section is the conclusion.

2. Analytical solution for the truncated Chapman–Enskog expansion of BGK equation

To simplify the deduction, we start from one-dimensional BGK equation,

\[ f_t + u f_x = \frac{g - f}{\tau}, \quad (1) \]

where \((x, t)\) are space and time variables, \(\xi\) is internal variable, \(f(x, t, u, \xi)\) is the distribution function, \(g(x, t, u, \xi)\) is the equilibrium state approached by \(f\), and \(\tau\) is collision time. The \(N\)th order Chapman–Enskog expansion of the BGK model is

\[ f = \sum_{n=0}^{N} (-\tau)^n D^n g, \quad (2) \]

where \(D = \partial_t + u \partial_x\) and \(\partial_x \equiv \frac{\partial}{\partial x}\).

To expand the distribution function \(f(x, t, u, \xi)\) in the form

\[ f = \sum_{n=-2}^{N-2} t^{n+2} H_n(x - ut, u) \quad (3) \]

and substitute it into BGK equation, we can get

\[ g = \sum_{n=-2}^{N-2} t^{n+1} (t + (n + 2)\tau) H_n(x - ut, u). \quad (4) \]

It can be verified that the above \(f\) and \(g\) exactly satisfy \(N\)th order Chapman–Enskog expansion. To substitute the initial conditions

\[ f(x, u, t = 0, \xi) = f_0(x, u), \quad g(x, u, t = 0, \xi) = g_0(x, u), \quad (5) \]

into (3) and (4) to determine \(H_{-2}\) and \(H_{-1}\), we can obtain the solutions of the distribution functions \(f(x, t, u, \xi)\) and \(g(x, t, u, \xi)\) in the following forms,

\[ f = (1 - \frac{1}{\tau}) f_0(x - ut, u) + \frac{1}{\tau} g_0(x - ut, u) + t^2 \sum_{n=0}^{N-2} t^n H_n(x - ut, u), \]

\[ g = -\frac{1}{\tau} f_0(x - ut, u) + (1 + \frac{\xi}{\tau}) g_0(x - ut, u) + t^2 \sum_{n=0}^{N-2} [t + (n + 2)\tau] t^{n+1} H_n(x - ut, u), \quad (6) \]

where \(H_n(x, u)\) \((n = 0, 1, 2, \ldots)\) are functions to be determined later. Hence

\[ f - g = f_0(x - ut, u) - g_0(x - ut, u) - \tau \sum_{n=0}^{N-2} (2 + n) t^{n+1} H_n(x - ut, u), \]

\[ D(f - g) = -\tau \sum_{n=0}^{N-2} (n + 2)(n + 1) t^n H_n(x - ut, u), \quad (7) \]

\[ D^2(f - g) = -\tau \sum_{n=0}^{N-2} (n + 2)(n + 1) n t^{n-1} H_n(x - ut, u), \]

\[ \ldots \]

Based on the compatibility condition of the BGK equation

\[ \int (f - g) \psi_d d\xi = 0, \quad (8) \]
where \(d\Xi = du\xi\) and \(\psi = [1, u, (u^2 + \xi^2)/2]\), \(\alpha = 1, 2, 3\), the solution for the conservative macroscopic variables \(W(x, t)\) is

\[
W_x = \int \left[ \left( 1 - \frac{N-1}{Nt} \right) f_0(x - ut, u) + \frac{N-1}{Nt} g_0(x - ut, u) \right] \psi_x d\Xi, \tag{9}
\]

from which the equilibrium state \(g(x, t, u, \zeta)\) can be uniquely determined. Therefore, the real non-equilibrium distribution function \(f(x, t, u, \zeta)\) is obtained as

\[
f = \frac{Nt - (N - 1)t}{Nt + t} f_0(x - ut, u) + \frac{(N - 1)t}{Nt + t} g_0(x - ut, u) + \frac{t}{Nt + t} g(x, u, t) + \frac{t^2}{Nt + t} \sum_{n=0}^{N-3} (N - 2 - n)t^n H_n(x - ut, u). \tag{10}\]

Eqs. (9) and (10) are the analytical solutions of the BGK equation up to the \(N\)th-order Chapman–Enskog approximation and also the analytical solutions of the BGK equation up to the \(N\)th-order time accuracy, which full satisfy the compatibility condition (8).

To show the procedures to find the solutions and the function \(H_n(x, u)\), now let’s get the specific solution in different orders, which start from \(N = 2\).

2.1. \(N = 2\), the solution of the Navier–Stokes equations

When \(N = 2\), the analytic solution recovers the Navier–Stokes equations. From Eq. (7) and (6), we have

\[
\begin{align*}
    f &= (1 - \frac{t}{2t}) f_0(x - ut, u) + \frac{t}{2t} g_0(x - ut, u), \\
    g &= -\frac{t}{2t} f_0(x - ut, u) + \frac{1 + \frac{t}{2t}}{2t} g_0(x - ut, u) + (t^2 + 2t^2) H_0(x - ut, u),
\end{align*} \tag{11a}\]

\[
\begin{align*}
    f - g &= f_0(x - ut, u) - g_0(x - ut, u) - 2\frac{t}{2t} H_0(x - ut, u).
\end{align*} \tag{11b}\]

Then, the compatibility condition (8) leads (11b) to

\[
\int H_0(x - ut, u) \psi_x d\Xi = \frac{1}{2t} \int \left[ f_0(x - ut, u) - g_0(x - ut, u) \right] \psi_x d\Xi. \tag{12}\]

Combining the above equation with Eq. (11a), we have the solutions for the macroscopic variables

\[
W_x = \int f \psi_x d\Xi = \int \left[ \left( 1 - \frac{t}{2t} \right) f_0(x - ut, u) + \frac{t}{2t} g_0(x - ut, u) \right] \psi_x d\Xi, \tag{13}\]

and the non-equilibrium distribution function

\[
f = \frac{2t - t}{2t + t} f_0(x - ut, u) + \frac{t}{2t + t} \left[ g_0(x - ut, u) + g(x, u, t) \right], \tag{14}\]

where \(g(x, u, t)\) is the equilibrium state determined by \(W_x\).

In order to recover the Euler solution, we can set \(\tau = 0\) and \(f_0(x, u) = g_0(x, u)\), such that

\[
\begin{align*}
    W_x &= \int f f_0(x - ut, u) \psi_x d\Xi, \\
    f &= g(x, u, t).
\end{align*} \tag{15}\]

2.2. \(N = 3\), the solution of BGK–Burnett equation

When \(N = 3\), the BGK model gives the BGK–Burnett equations. From Eq. (7) and (6), we have

\[
\begin{align*}
    f &= (1 - \frac{t}{3t}) f_0(x - ut, u) + \frac{t}{3t} g_0(x - ut, u) + t^2 H_1(x - ut, u), \\
    g &= -\frac{t}{3t} f_0(x - ut, u) + \frac{1 + \frac{t}{3t}}{3t} g_0(x - ut, u) + (t^2 + 2t^2) H_0(x - ut, u) + (t^3 + 3t^2) H_1(x - ut, u),
\end{align*} \tag{16a}\]

\[
\begin{align*}
    f - g &= f_0(x - ut, u) - g_0(x - ut, u) - 2\frac{t}{3t} H_0(x - ut, u) - 6t H_1(x - ut, u),
\end{align*} \tag{16b}\]

\[
D(f - g) = -\tau(2H_0(x - ut, u) + 6t H_1(x - ut, u)). \tag{16c}\]

Substituting \(t = 0\) into (16c) leads to

\[
H_0(x, u) = -\frac{1}{2t} D(f - g)_{t=0} = \frac{1}{2} (D^2 g - \tau D^3 g)_{t=0}. \tag{17}\]

where \(D^2 g_{t=0}\) and \(D^3 g_{t=0}\) can be derived from initial conditions and Chapman–Enskog expansion [9,7].

Apply the compatibility condition to Eq. (16b), we have
Combining it with (16a) leads to

\[
W_x = \int \left[ \left( 1 - \frac{2t}{3\tau} \right) f_0(x - ut, u) + \frac{2}{3\tau} g_0(x - ut, u) + \frac{t^2}{3} H_0(x - ut, u) \right] \psi_x d\Xi,
\]  

and

\[
f = \frac{3\tau - 2t}{3\tau + t} f_0(x - ut, u) + \frac{2t}{3\tau + t} g_0(x - ut, u) + \frac{t}{3\tau + t} g(x, u, t) + \frac{t^2}{3\tau + t} \delta_0(x - ut, u).
\]  

Similar procedures can be done to derive the solutions of Nth order Chapman–Enskog expansion, which are shown as (9) and (10). In this paper we focus on the solution of Navier–Stokes equations and construct the corresponding gas-kinetic scheme. In two-dimensional case, the BGK equation is

\[
f_t + uf_x + uf_y = \frac{g - f}{\tau},
\]  

and the corresponding analytic solutions to the Nth order Chapman–Enskog expansion are

\[
W_x = \int \left[ \left( 1 - \frac{\left(-\frac{1}{N}\right) t}{N\tau + t} \right) f_0(x - ut, y - vt, u, v) + \frac{\left(-\frac{1}{N}\right) t}{N\tau + t} g_0(x - ut, y - vt, u, v) + \frac{1}{N\tau + t} \sum_{n=0}^{N-1} (N - 2 - n) t^n H_n(x - ut, y - vt, u, v) \right] \psi_x d\Xi,
\]  

\[
f = \frac{N\tau - (N - 1) t}{N\tau + t} f_0(x - ut, y - vt, u, v) + \frac{(N - 1) t}{N\tau + t} g_0(x - ut, y - vt, u, v) + \frac{t^2}{N\tau + t} \sum_{n=0}^{N-1} (N - 2 - n) t^n H_n(x - ut, y - vt, u, v).
\]

When taking \( N = 2 \), the above formulation presents solution for the two-dimensional Navier–Stokes equations,

\[
W_x = \int \left[ \left( 1 - \frac{1}{2\tau} \right) f_0(x - ut, y - vt, u, v) + \frac{1}{2\tau} g_0(x - ut, y - vt, u, v) \right] \psi_x d\Xi,
\]  

\[
f = \frac{1}{2\tau + t} f_0(x - ut, y - vt, u, v) + \frac{1}{2\tau + t} g_0(x - ut, y - vt, u, v) + g(x, y, u, v, t),
\]

where \( g(x, y, u, v, t) \) is a Maxwellian distribution determined by \( W_x \).

### 3. A new gas-kinetic scheme for the Euler and Navier–Stokes equations

Now we construct a numerical scheme for the Navier–Stokes equations based on the previous analytic solution (24). The following scheme as a BGK solver has 2nd-order time accuracy.

#### 3.1. The construction of the scheme

At each cell interface, a central goal for any finite volume scheme is to evaluate the numerical fluxes. In the gas kinetic scheme, this flux is obtained through the integration of a gas distribution function. The analytic solution of the gas distribution function presented in the previous section depends on the initial condition \( f_0 \) and the equilibrium state \( g_0 \). Similar to the approach in [16], continuous initial distribution functions can be defined as

\[
g_0(x, u) = g^e[1 + ax + by],
\]  

\[
f_0(x, u) = g^e[1 + ax + by + \tau(au + bv + A)],
\]

where \( g^e \) is the Maxwellian distribution at the cell interface, \( a, b, \) and \( A \) are spatial and temporal derivatives of the equilibrium state, which are fully determined in the same way as BGK–NS method [16]. Substitute the above initial conditions into (24), with the definition of moments of an equilibrium state

\[
M(\cdot) = \int (\cdot) g^e d\Xi,
\]

we can get the time dependent macroscopic variables and the corresponding fluxes at the cell interface,

\[
W_x = W_x^e - pM_1 - qM_2,
\]  

\[
F_x(x, t) = \frac{2\tau + 1}{2\tau + t} F_x + \frac{1}{2\tau + t} (F_{gx} + F_{wx}).
\]

where \( W_x^e = \int \psi_x^e g^e d\Xi \).
3.2. The 3rd-Order WENO reconstruction

Once the fluxes at the cell interface are determined, we can update the cell-averaged conservative variables. In two-dimensional space, for a rectangular mesh with cell boundaries bounded by the straight lines, \( x = x_{i-1/2} \), \( x = x_{i+1/2} \), \( y = y_{j-1/2} \) and \( y = y_{j+1/2} \), the update of the cell-averaged conservative variables \( W_{ij} \) in a control volume \( \Delta x \Delta y = (x_{i+1/2} - x_{i-1/2}) \times (y_{j+1/2} - y_{j-1/2}) \) from time step \( t_n \) to \( t_{n+1} \) becomes

\[
W_{ij}^{n+1} = W_{ij}^n + \frac{1}{\Delta x \Delta y} \int_{t_n}^{t_{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} G(t,x,y_{j-1/2}) - G(t,x,y_{j+1/2}) \, dx dt + \frac{1}{\Delta x \Delta y} \int_{t_n}^{t_{n+1}} \int_{y_{j-1/2}}^{y_{j+1/2}} [F(t,x_{i-1/2},y) - F(t,x_{i+1/2},y)] \, dy dt,
\]

where the time integration is calculated by Gauss quadrature. In this paper we will use two-points 3rd-order Gauss–Legendre quadrature for time integration.

In order to get the correct heat flux and Pr number, the same technique by modifying the energy flux in [16] is used here as well.

### 3.2. The 3rd-Order WENO reconstruction

In this paper, the macroscopic flow variables at the cell interface are interpolated by the 3rd-order WENO reconstruction with some modification.

In one-dimensional case, a 3rd-order WENO reconstruction has the following procedures:

1. With the cell averaged conservative variables \( \overline{W}_i \), at the cell interface \( i + 1/2 \), the averaged value is \( \overline{W}_{i+1/2} = (\overline{W}_i + \overline{W}_{i+1})/2 \), which is used to calculate the left and right eigenvector matrix \( L_{i+1/2} \) and \( R_{i+1/2} \) and the characteristic value \( c_{1,2,3}(x = 1, 2, 3) \).

2. Change the conservative variables \( W_{i+1/2}(l = -1, \ldots, 2) \) to characteristic ones \( \overline{W}_{i+1/2}(l = -1, \ldots, 2) \) by

\[
\overline{W}_{i+1/2} = L_{i+1/2} W_{i+1/2} \quad (l = -1, \ldots, 2).
\]

3. For \( x \)-th component, consider the following interpolation

\[
W_{x,i+1/2} = \frac{1}{2} \overline{W}_{x,i} - \frac{1}{2} \overline{W}_{x,i+1}, \quad W_{x,i+1/2} = \frac{1}{2} (\overline{W}_{x,i} + \overline{W}_{x,i+1}), \quad W_{x,i+1/2} = \frac{1}{2} \overline{W}_{x,i+1} - \frac{1}{2} \overline{W}_{x,i},
\]

\[
\beta_1 = \overline{W}_{x,i} - \overline{W}_{x,i-1}, \quad \beta_2 = \overline{W}_{x,i+1} - \overline{W}_{x,i}, \quad \beta_3 = \overline{W}_{x,i+2} - \overline{W}_{x,i+1}, \quad d_1 = 1/3, \quad d_2 = 2/3, \quad d_3 = 1/3, \quad \epsilon = 1e - 6;
\]

\[
w_s = \beta_s / \sum_{j=1}^{2} \beta_j (s = 1, 2), \quad \overline{W}_{x,i+1/2} = \sum_{j=1}^{2} w_s \overline{W}_{x,i+1/2,j},
\]

\[
w_s = \beta_s / \sum_{j=3}^{2} \beta_j (s = 2, 3), \quad \overline{W}_{x,i+1/2} = \sum_{j=1}^{2} w_s \overline{W}_{x,i+1/2,j}.
\]
When needed, the computed \( \tilde{W}_{i+1/2} \) and calculate \( W_{i+1/2} = R_{i+1/2} \tilde{W}_{i+1/2} \). If \( c < -\epsilon_0 \), we just interpolate \( W_{x,i+1/2} \) to calculate \( W_{i+1/2} = R_{i+1/2} W_{i+1/2} \). If \( c < 0 \) and \( c_0 \), we will make the following computation.

\[
\beta_1 = \sqrt{c_1^2 + \epsilon + c_s}, \quad \beta_2 = \sqrt{c_2^2 + \epsilon - c_s}, \quad w_s = \beta_i \sum_{p=1}^{2} \beta_p (s = 1, 2),
\]

\[
W_{x,i+1/2} = w_1 \tilde{W}_{x,i+1/2} + w_2 \tilde{W}_{x,i+1/2}, \quad W_{i+1/2} = R_{i+1/2} W_{i+1/2}
\]

After the construction of macroscopic flow variables, the equilibrium distribution function \( g^* \) is a Maxwellian corresponding to \( W_{i+1/2} = (\rho, \rho W, E) \),

\[
g^* = \rho^* \left( \frac{c_1}{\pi} \right)^{3/2} \exp \left[ \frac{c_1}{\rho^*} (u^t - u)^2 + \frac{c_2}{\rho^*} \right],
\]

and the spatial derivatives of a Maxwellian, i.e., \( a = a_1 + a_2 u + a_3 (u^t + \frac{c_1}{\rho^*}) \), can be determined by

\[
M(a\phi_s) = \frac{dW_x}{dx} \bigg|_{i+1/2}.
\]

In two-dimensional space, the WENO reconstruction is performed as

1. In \( x \)-direction, use the cell average \( W_{i+1/2,j} = (W_{ij} + W_{i+1,j})/2 \) to calculate the left and right eigenvector matrix \( L_{i+1/2,j} \) and \( R_{i+1/2,j} \) and the characteristic value \( c_s(x = 1, 2, 3) \).
2. Project the conservative variables \( W_{i+1/2,j} = (1, \ldots, 2) \) to characteristic one \( \tilde{W}_{i+1/2,j} \) by

\[
\tilde{W}_{i+1/2,j} = L_{i+1/2,j} W_{i+1/2,j} (l = -1, \ldots, 2).
\]

3. Make the same interpolation procedure as in one-dimensional case. In 2D case, with the variation of the flow variables in the vertical direction, the resulting line integrals of the conservative variables in this direction become

\[
W_{x,i+1/2,j} = \int_{y_{i-1/2}}^{y_{i+1/2}} \tilde{W}_{x,i+1/2,j} dy, \quad W_{y,i+1/2,j} = \int_{y_{i-1/2}}^{y_{i+1/2}} \frac{dW_x(x_{i+1/2,j})}{dx} dy.
\]

When needed, the computed \( W_{x,i+1/2,j} \) will be used to calculate \( p_{x,i+1/2,j} \) and to determine the artificial viscosity

\[
\tau_{x,i+1/2,j} = \left( \chi + \sqrt{\frac{\rho_{x}^* - \rho_{x}^*}{\rho_{x}^* + \rho_{x}^*}} \right) \Delta t.
\]

This artificial viscosity is only used in the Mach 3 forward step flow.

4. The same interpolation is done in \( y \)-direction to obtain \( W_{y,i+1/2,j} \), \( W_{y,i+1/2,j} \), and \( \tau_{y,i+1/2,j} \).

5. For the cell interface \( (x_{i-1/2}, y \in [y_{i-1/2}, y_{i+1/2}]) \), use \( W_{p,i+1/2,j} \) to compute the left and right eigenvector matrix \( L_{i+1/2,j} \) and \( R_{i+1/2,j} \) and the characteristic value \( c_s(x = 1, 2, 3) \).

6. Project \( W_{p,i+1/2,j} \) to characteristic and into characteristic space and then interpret the boundary characteristic value, where least square method is used

\[
\tilde{W}_{x,i+1/2} = \frac{436W_{x,i+1/2} + 18W_{x,i+1,2} + \tilde{W}_{x,i+1,2} + \tilde{W}_{x,i+1,2} - 2W_{x,i+1,2}}{436},
\]

\[
\tilde{W}_{y,i+1/2} = \frac{436W_{y,i+1/2} + 18W_{y,i+1,2} + \tilde{W}_{y,i+1,2} + \tilde{W}_{y,i+1,2} - 2W_{y,i+1,2}}{436},
\]

from which \( W_{x,i+1/2} \) through (34) can be computed. The derivatives are interpolated by

\[
\frac{dW_x}{dx} \bigg|_{i+1/2} = W_{x,i+1/2,j}, \quad \frac{dW_y}{dy} \bigg|_{i+1/2} = \frac{W_{y,i+1/2,j+1} - W_{y,i+1/2,j-1}}{2}.
\]

The scheme constructed so far with both the 3rd-order WENO reconstruction and the kinetic flux is defined as the new gaskinetic scheme (NGKS-W3).
4. Numerical experiments

4.1. Stability test

The new kinetic scheme is for the Euler and Navier--Stokes equations. Both the scheme and the equations are highly nonlinear. It is not easy to obtain the stability property of the scheme theoretically. Hence we explore the stability property numerically by a test case for the Euler equations with initial conditions

\[
\rho = 1 + \frac{1}{4}\sin\left(\frac{2\pi \xi}{5}\right), \quad U = 1, \quad p = 1,
\]

(42)

where \(\rho\) is density, \(U\) is velocity, and \(p\) is pressure. Periodic boundary conditions are set at both ends in the computational domain \(x \in [-5, 5]\). We use different interpolations and test the maximum stable CFL number numerically. The detailed procedures are:

1. For a coarse grid \(N = 20\), set a CFL number, implement the computation, and calculate the maximum density \((\rho_{\text{max}})\) at every time step.
2. If the maximum density decrease to 90% of initial wave amplitude, i.e., \(\rho_{\text{max}} < 1.225\), the scheme can be regarded as a stable one. If the maximum density increases to 110% of the initial wave amplitude, i.e., \(\rho_{\text{max}} > 1.275\), the scheme is defined as unstable.
3. If the maximum density locates within the interval \([1.225, 1.275]\), continuous computation will take place.

The numerical experiments show that

1. For the 1st-order up-wind scheme (characteristic variables) with

\[
\tilde{W}_{i+1/2} = \tilde{W}_i, \quad \tilde{W}_{i+1/2} = \tilde{W}_{i+1}, \quad \frac{dW}{dx}\big|_{i+1/2} = 0,
\]

(43)

and with (34) for the interpolation, the obtained maximum stable CFL number is \(CFL = 1.05\).
2. For the 2nd-order central interpolation (conservative variables)

\[
W_{i+1} = \frac{W_i + W_{i+1}}{2}, \quad \frac{dW}{dx}\big|_{i+1/2} = \frac{W_{i+1} - W_i}{\Delta x},
\]

(44)

the maximum stable value is \(CFL = 1.1\).
3. For the 3rd order upwind scheme with characteristic variables reconstruction

\[
\tilde{W}_{i+1/2} = -\frac{1}{6}\tilde{W}_{i-1} + \frac{1}{3}\tilde{W}_i + \frac{1}{6}\tilde{W}_{i+1},
\]

\[
\tilde{W}_{i+1/2} = -\frac{1}{6}\tilde{W}_{i-1} + \frac{1}{3}\tilde{W}_i + \frac{1}{6}\tilde{W}_{i+1},
\]

\[
\frac{dW}{dx}\big|_{i+1/2} = \frac{W_{i+1} - W_i}{\Delta x},
\]

(45)

the maximum stable value is \(CFL = 0.82\).
4. For the 4th-order central interpolation with conservative variable reconstruction

\[
W_{i+1/2} = \frac{7(W_i + W_{i+1}) - 15W_{i+1} - W_{i+2}}{12},
\]

\[
\frac{dW}{dx}\big|_{i+1/2} = \frac{W_{i+1} - W_{i+2} + 5(W_{i+1} - W_i)}{12\Delta x},
\]

(46)

the maximum CFL number is 0.68.
5. For the 3rd-order WENO interpolation with characteristic variable reconstruction, see Section 3.2, the maximum CFL number is 0.83.

From these results we can see that the current NGKS is stable for both up-wind and central interpolations. As we know, for the schemes solving the macroscopic equations directly the central schemes are apt to blow up. So in terms of stability the current NGKS is superior in comparison with central schemes.

4.2. Accuracy test

We use the flow case (42) to test both spatial and temporal accuracy of the schemes in the inviscid flow computations. For the 3rd-order upwind scheme (45), the spatial errors are shown in Table 1. For the 4th-order central scheme (46), the spatial errors are shown in Table 2.

The 4th-order central scheme is also used to test time accuracy, where we use a mesh with \(N = 320\) points and gradually decrease the time step. The error is shown in Table 3. Hence the time accuracy is of 2nd-order.
The error of NGKS-W3 are shown in Table 4. As the grid number \( N \) is large enough, the NGKS-W3 is a 3rd-order scheme.

4.3. Viscous and heat conducting tests

The Couette flow with a temperature gradient provides a good test for the current NGKS to describe the viscous heat conducting flow. We made the same comparisons as in [16]. With a fixed bottom wall, the top boundary is moving at a speed \( U \) in the horizontal direction. The temperatures at the bottom and top are fixed with values \( T_0 \) and \( T_1 \). Under the assumption of constant viscosity and heat conduction coefficients and in the incompressible limit, a steady state analytic temperature distribution can be obtained as

\[
\frac{T - T_0}{T_1 - T_0} = \frac{y}{H} + \frac{PrEc}{2} \frac{y}{H} \left(1 - \frac{y}{H}\right),
\]

where \( H \) is the height of the channel, \( Pr \) is the Prandtl number, \( Ec = U^2/|C_p(T_1 - T_0)| \) is the Eckert number, and \( C_p \) is the specific heat ratio at constant pressure.

We set up the simulation as a two-dimensional problem in the domain \((x, y) \in [0, 1] \times [0, 1]\). The grid is \( 20 \times 20 \). The moving velocity at the upper boundary in the \( y \)-direction is \( U(y = 1) = 1.0 \) and at the down wall \( U(y = 0) = 0.0 \). For \( x \)-direction we use periodic boundary condition at \( x = 0 \) and \( 1 \). The initial density and Mach number of the gas inside the channel are 1.0 and 0.1, respectively. So the fluid in the channel is almost incompressible. The isothermal no-slip boundary conditions are implemented at both walls. We have tested the current NGKS-W3 with a wide range of parameters: (i) specific heat ratio \( \gamma = 7/5, 5/3, 2.0 \), (ii) different Prandtl number \( Pr = 0.5, 0.72, 1.0, 1.5, 2.0 \). With the variations of all these parameters, the simulation results recover the exact solutions very well.

Fig. 1 presents the solutions in a few cases with different Prandtl numbers and Eckert numbers. These figures clearly show that the Prandtl number fix does modify the heat conduction term correctly.

In the above case, the velocity in the channel is linearly distributed. For a highly dissipative flow, an analytic expression in terms of \( y/H \) is not available. In the following, we are going to test the NGKS-W3 when the stress and heat flux are large. When the velocity of the upper wall is large enough, the compressibility of the fluid becomes appreciable. Under the conditions of \( \mu \sim T^\omega \) with \( \omega = 1 \) and of adiabatic lower wall condition \( q_w = 0 \), there is an analytic solution in the compressible case [8].

\[
\frac{\tau_{w}y}{\mu_\infty U} = \frac{u}{U} + \frac{Pr}{2} \left( \frac{\gamma - 1}{2} \right) \frac{M^2_\infty}{\left( \frac{u}{U} - \frac{1}{3} \right)^3},
\]

where \( U \) is the horizontal velocity of the upper wall and \( M_\infty \) is the corresponding Mach number. In order to further test the NGKS-W3, we have set up the upper and lower boundary conditions with \( M_\infty = 3.0 \), \( q_w = 0.0 \), and Prandtl number \( Pr = 2/3 \). The viscosity coefficient is set to be \( \mu \sim T \). The same number of 20 mesh points is used. The simulation result for the velocity distribution in the channel is shown in Fig. 2, where the solids line is the exact solution from the above formula. In order to see clearly the compressibility, the density distribution in the channel is also included in Fig. 2, from which we can observe the large density variation close to the upper boundary. Also, as a reference, the velocity distribution at a low Mach number case \( M_\infty = 0.1 \) is included, which reduces to a nearly linear profile. From these experiments we can see that NGKS-W3 is accurate enough in describing the viscous and heat conducting flows.

4.4. Time efficiency and accuracy test for discontinuous flows

For flow with discontinuous structures, we test NGKS-W3 and compare the results with classical finite volume 3rd-order WENO (WENO3) scheme [4], where in one-dimensional cases the reconstruction of cell interface values is exactly the same. In the Following test cases the CFL number 0.5 is used for different schemes. The efficiency of the schemes is obtained by comparing the computer time to run the codes ten steps on the same machine.

Example 4.1. Lax problem

Solve the Euler equation with the Riemann initial condition:

\[
\begin{align*}
\langle \rho, U, p \rangle &= (0.445, 0.698, 3.528) & \text{for } x \in [-5, 0], \\
\langle \rho, U, p \rangle &= (0.5, 0, 0.571) & \text{for } x \in (0, 5).
\end{align*}
\]

<table>
<thead>
<tr>
<th>( N )</th>
<th>20</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
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<td>2.77</td>
<td>2.97</td>
<td>2.99</td>
<td>2.99</td>
</tr>
</tbody>
</table>
The computation is run up to $t = 1.3$. Set the cell number $N = 200$. In Fig. 3, the left figure is density obtained by NGKS-W3 (square), and the right one is obtained by WENO3 (square). The solid curves in Figs. 3 and 4 are exact Riemann solutions [13]. Fig. 4 gives the velocity and pressure against the accurate solution. We can see that NGKS-W3 has higher resolution than WENO3 in capturing the discontinuities, the numerical oscillations have been effectively suppressed.

**Example 4.2.** Blast wave problem

Solve Woodward–Colella problem [14] in domain $x \in [-5, 5]$ with initial condition

$$
\rho = 1; \quad U = 0; \quad p = \begin{cases} 
1000, & x \in [-5, -4), \\
0.01, & x \in [-4, 4), \\
100, & x \in [4, 5].
\end{cases}
$$

The computation time is $t = 0.38$. The density profiles on grids $N = 200$ and $N = 400$ are compared with the results of WENO3 on Fig. 5. The reference solution is obtained by fifth-order WENO scheme with $N = 2000$ grids. We can see that NGKS-W3 has better resolution than WENO3.

---

**Table 2**
The error of 4th-order center NGKS.

<table>
<thead>
<tr>
<th>N</th>
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<th>80</th>
<th>160</th>
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<tbody>
<tr>
<td>$L_1$ error</td>
<td>9.33e−3</td>
<td>5.82e−4</td>
<td>2.57e−5</td>
<td>1.13e−6</td>
<td>7.10e−8</td>
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<td>4.50</td>
<td>4.00</td>
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<tr>
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<td>1.11e−7</td>
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<tr>
<td>Order</td>
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</tr>
</tbody>
</table>

**Table 3**
The time error of 4th-order center NGKS.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>7.92e−3</th>
<th>5.28e−3</th>
<th>2.64e−3</th>
<th>1.32e−3</th>
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</thead>
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<td>2.04</td>
<td>2.21</td>
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<tr>
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<tr>
<td>Order</td>
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</table>

**Table 4**
The error of 3rd-order NGKS-W3.

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<th>160</th>
<th>320</th>
<th>640</th>
<th>1280</th>
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<td>2.72e−2</td>
<td>1.12e−2</td>
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<td>4.24e−4</td>
<td>4.95e−5</td>
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<tr>
<td>Order</td>
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<td>1.88</td>
<td>1.27</td>
<td>2.20</td>
<td>2.53</td>
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<td>3.80</td>
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<td>9.13e−3</td>
<td>2.75e−3</td>
<td>5.86e−4</td>
<td>6.05e−5</td>
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<tr>
<td>Order</td>
<td>-</td>
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<td>1.34</td>
<td>1.46</td>
<td>1.73</td>
<td>2.23</td>
<td>3.28</td>
</tr>
</tbody>
</table>

---

**Fig. 1.** Temperature ratio $(T - T_0) / (T_1 - T_0)$ in the Couette flow in the low Mach number case $M = 0.1$. Solid line: analytical solution (47); Circle: NGKS-W3. Left: $Ec = 40, Pr = 2.5, 1.0, 0.72$; Right: $Pr = 0.5$. $Ec = 40.0, 20.0, 4.0$. The computation is run up to $t = 1.3$. Set the cell number $N = 200$. In Fig. 3, the left figure is density obtained by NGKS-W3 (square), and the right one is obtained by WENO3 (square). The solid curves in Figs. 3 and 4 are exact Riemann solutions [13]. Fig. 4 gives the velocity and pressure against the accurate solution. We can see that NGKS-W3 has higher resolution than WENO3 in capturing the discontinuities, the numerical oscillations have been effectively suppressed.

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<td>2.05</td>
<td>2.21</td>
</tr>
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</table>

**Table 4**
The error of 3rd-order NGKS-W3.

<table>
<thead>
<tr>
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**Example 4.2.** Blast wave problem

Solve Woodward–Colella problem [14] in domain $x \in [-5, 5]$ with initial condition

$$
\rho = 1; \quad U = 0; \quad p = \begin{cases} 
1000, & x \in [-5, -4), \\
0.01, & x \in [-4, 4), \\
100, & x \in [4, 5].
\end{cases}
$$

The computation time is $t = 0.38$. The density profiles on grids $N = 200$ and $N = 400$ are compared with the results of WENO3 on Fig. 5. The reference solution is obtained by fifth-order WENO scheme with $N = 2000$ grids. We can see that NGKS-W3 has better resolution than WENO3.
Fig. 2. Velocity $u/U$ and density $\rho$ distributions in high speed Couette flow case for a gas with $Pr = 2/3$ and $\mu \sim T$, where the up-plate is moving with speed of $M = 3.0$ and the lower boundary has $q_w = 0$. The solid line is the analytic solution given by (48). As a reference, the solution for $M = 0.1$ is also included.

Fig. 3. The Lax problem. $t = 1.3$. NGKS-W3 (left) and WENO3 (right), $N = 200$ cells. Density $\rho$. Solid line: exact solution; squares: computed solution.

Fig. 4. The Lax problem. $t = 1.3$. Velocity (left) and pressure (right) obtained by NGKS-W3, $N = 200$ cells. Solid line: exact solution; squares: computed solution.
In 2009 Yamaleev and Carpenter have constructed an energy stable 3rd-order WENO scheme (ESWENO) [17], which is probably the best 3rd-order shock capturing scheme, where a new weight is devised. We use the same weight as that in the paper [17] and compute the blast wave problem on grid $N = 200$ and $N = 400$ and the results are included in Fig. 5.

The enlarged comparison of the results of NGKS-W3 with those of ESWENO is shown in Fig. 6, where the grid number is $N = 800$. The comparison shows that NGKS-W3 has better resolution than ESWENO.

**Example 4.3.** Shock acoustic-wave interaction

The Shu–Osher shock acoustic-wave interaction problem [11] is in a domain $x \in [-5, 5]$ with initial condition

$$
(p, u, \rho) = \begin{cases} 
(3.857134, 2.629369, 10.333333), & x < -4, \\
(1 + 0.2 \sin(5x), 0, 1), & x \geq -4.
\end{cases}
$$

The computation time is $t = 1.8$. Fig. 7 shows that the current NGKS-W3 scheme has better resolution for short waves than WENO3, especially when Yamaleev et al.’s weight [17] is used.

We use the above Examples 4.1, 4.2, 4.3 to test the time efficiency of the current NGKS-W3. In these experiments, the grid number is fixed with $N = 200$. We compare the CPU time cost for ten steps using NGKS-W3 and WENO3 schemes for different test cases in Table 5. From this comparison we can see that current NGKS-W3 is about 2 times faster than WENO3.

4.5. Two dimensional cases

Now we validate the new gas-kinetic scheme in two-dimensional space.

**Example 4.4.** Convection of isotropic vortex for two-dimensional Euler equations.

This example is used to test NGKS-W3 for two-dimensional Euler solutions. The periodic vortex problem is set up in a computational domain $[0, 10] \times [0, 10]$. The boundary condition is periodic in both directions. The initial condition is given by

$$
(w_1(x, y, 0), w_2(x, y, 0)) = (1, 1) + \frac{\epsilon}{8\pi} e^{0.5(1-r^2)}(-\bar{y}, \bar{x}),
$$

$$
T(x, y, 0) = 1 - \frac{(\bar{x})^2 - (\bar{y})^2}{6\pi^2} e^{-r^2}, \quad S(x, y, 0) = 1,
$$

where the temperature $T$ and the entropy $S$ are related to the density $\rho$ and the pressure $p$ by

$$
T = p/\rho, \quad S = p/\rho^\gamma,
$$

and $(\bar{x}, \bar{y}) = (x - 5, y - 5)$, $r^2 = \bar{x}^2 + \bar{y}^2$, and the vortex strength $\epsilon = 5$. The exact solution is convected with the speed $(1, 1)$ in the diagonal direction. We make the calculation up to $t = 10$, namely one time period.

The numerical experiment shows that the stable condition is $CFL \leq 0.64$. The $L_1$ and $L_\infty$ errors of NGKS-W3 are compared with those of finite difference WENO3, shown in Table 6.

NGKS-W3 is about 2.5th-order, not exactly 3rd-order, because in computing flux, the $y$-directional integration is a 2nd-order approximation. But the error of NGKS-W3 is smaller than that of WENO3. As to the speed, for the grid $31 \times 31$, with the same computer, the cost for ten steps calculation using NGKS-W3 takes $4.898 \times 2$ s, while finite difference WENO3 uses $1.264 \times 1$ s. Hence for two-dimensional Euler equations, the speed of NGKS-W3 is about 1.5 times faster than that of finite difference WENO3.
Example 4.5. Shock–vortex interaction

We compute the shock–vortex interaction problem [2]. On the computational domain \((x, y) \in [0.2, 0.1]\), a stationary shock front is positioned at \(x = 0.5\). The left upstream state is \((\rho, U, V, p) = (M^2, \sqrt{\gamma}, 0, 1)\), where \(\gamma\) is the specific heat ratio and \(M\) is the Mach number. A small vortex is a perturbation on the mean flow with the velocity \((U, V)\), temperature \((T = p/\rho)\), and entropy \((s = \ln(\rho/\rho_0))\), where the perturbation is

\[
\bar{U} = \kappa \eta \rho^{1-(1-\eta)} \sin \theta, \quad \bar{V} = -\kappa \eta \rho^{1-(1-\eta)} \sin \theta, \quad \bar{T} = -\frac{(\gamma - 1) \kappa^2 \rho^{2(1-\eta^2)}}{4 \mu \gamma}, \quad \bar{s} = 0,
\]

where \(\eta = r/r_c, \quad r = \sqrt{(x-x_c)^2 + (y-y_c)^2}, \quad (x_c, y_c) = (0.25, 0.5)\) is the center of the vortex, \(\eta\) and \(\mu\) control the strength and decay rate of the vortex, and \(r_c\) is the critical radius. Here we choose \(\kappa = 0.3, \quad r_c = 0.05\) and \(\eta = 0.204\).
Table 6
Euler equation; convection of isotropic vortex; periodic boundary conditions. Compare the errors of NGKS-W3 and WENO3.

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<tr>
<th>N</th>
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</tbody>
</table>

Fig. 8. Shock vortex interaction by NGKS-W3 on a grid $202 \times 101$. The pressure distributions at different time. In each figure, there are 60 contours from 0.8 to 1.4.
The gas is set to be a diatomic molecule with $\gamma = 1.4$ and $CFL = 0.5$. The grid number is $202 \times 101$. The reflected boundary condition is used on the top and bottom boundaries. The evolution of the flow is given in Fig. 8. There is no artificial viscosity in our scheme, but the numerical oscillation is effectively suppressed. We also give the detailed profiles of pressure and velocity at the horizontal symmetric line $y = 0.5$ at $t = 0.8$. Solid line: reference solution; circle: NGKS-W3.

Example 4.6. Mach 3 forward step flow

The Mach 3 step problem was first proposed by Woodward and Colella on [14]. The computational domain is $(x, y) \in [0, 3] \times [0, 1]$. A step with height 0.2 is located at $x = 0.55$. The upstream velocity is $(U, V) = (3, 0)$. The adiabatic slip Euler boundary condition is implemented at all boundaries. In this test case we found that the artificial viscosity (39) has to be used. We set $\alpha = 1.8$ and $\beta = 10.0$. The results on grid $120 \times 40$ are shown in Fig. 10.

Example 4.7. Two-dimensional cavity problem for the Navier–Stokes solutions
The cavity flow at low Mach number is a standard test case for validating incompressible or low speed Navier–Stokes flow solvers. Since the benchmark solution is for incompressible Navier–Stokes equations, we set Mach number 0 and the Reynolds number 3200 in the current calculation. The fluid is bounded in a unit square and is driven by a uniform translation velocity of the top boundary. The boundary used is isothermal and non-slip condition. In this problem in order to make the simulation more accurate we set the parameter \( \epsilon = 1e^{-3} \) in the WENO reconstruction.

Fig. 11 shows the streamlines plots of the flow calculated with 65 x 65 and 101 x 101 mesh points. The results of velocity and pressure along the vertical horizontal symmetric line at \( x = 0.5 \) and \( y = 0.5 \) are shown in Fig. 12. The benchmark solution is from [3].

The cavity flow at low Mach number is a standard test case for validating incompressible or low speed Navier–Stokes flow solvers. Since the benchmark solution is for incompressible Navier–Stokes equations, we set Mach number 0.1 and the Reynolds number 3200 in the current calculation. The fluid is bounded in a unit square and is driven by a uniform translation velocity of the top boundary. The boundary used is isothermal and non-slip condition. In this problem in order to make the simulation more accurate we set the parameter \( \epsilon = 1e^{-3} \) in the WENO reconstruction.

Fig. 11 shows the stream traces of the flow calculated with 65 x 65 and 101 x 101 mesh points. The results of velocity and pressure along the vertical horizontal symmetric line at \( x = 0.5 \) and \( y = 0.5 \) are shown in Fig. 12. The benchmark solution is from [3].

In summary, this section presents many numerical tests for the newly developed kinetic scheme. Due to the use of the continuous initial flow distribution at a cell interface, it will become difficult to theoretically define a valid collision time, which takes into account of both physical and numerical requirements, especially for the complicated flow computation. All examples presented in the current paper are relatively simple. Except the step problem, the collision time used in other cases is defined by the physical viscosity coefficient only. For the step problem, the numerical results are not very sensitive to the values of \( \alpha \) and \( \beta \) in Eq. (39). In the previous BGK-NS method, \( \tau \) is always defined with the inclusion of artificial dissipative term which is proportional to the cell interface pressure jump. However, in the current paper due to the upwind-biased WENO reconstruction, it seems that the introduction of additional dissipation is not necessary in most cases.

5. Conclusion

In this paper, based on the exact solution of the truncated Chapman–Enskog expansion of the kinetic BGK equation, a new gas-kinetic scheme for the compressible NS equations is constructed starting from a continuous initial condition which is reconstructed by the 3rd-order upwind-biased WENO method. Numerical studies show that the current scheme is accurate, stable, and non-oscillatory in capturing discontinuity. With the same initial reconstruction, the speed of current method is
several times faster than the finite difference schemes which discretize the macroscopic governing equations directly and use the Runge–Kutta method for the time accuracy.

In addition we devise a new 3rd-order WENO reconstruction for two-dimensional flows. Experiments show that this interpolation has higher resolution than the classical 3rd-order WENO reconstruction both in discontinuous and smooth regions.

Although the current scheme shows good performance in all numerical tests, the scheme has not achieved a fully 3rd-order accuracy in the two-dimensional space. A fully 3rd-order scheme in two and three dimensional space is under construction.

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