Comment on "Development of an Improved Gas-kinetic BGK Scheme for Inviscid and Viscous Flows" [J. Comput. Phys. 158, 1-27 (2000)]

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The paper by Chae, Kim, and Rho presents a modified gas-kinetic BGK scheme. In their approach, they attempted to improve the accuracy of the original BGK scheme in the viscous flow calculations by modifying EFM or KFVS flux using Mach number splitting technique, as shown in Eq.(31) of [1]. At the same time, they corrected unit Prandtl number in the BGK scheme. In order to demonstrate the improvement in the numerical results obtained from their new scheme, they also included the simulation results from the "original" BGK method [2], see Figures (12) and (14) in [1]. In this comment, we are going to point out that the paper [1] misinterprets the original BGK scheme. Also, similar test cases presented in [1] will be calculated by the real BGK method presented in [2].

The starting point of the modification of the BGK scheme in [1] is Eq.(19), which is pretended as the integral solution of the BGK model,

$$
\begin{aligned}
f(0,0,t,u,v,\xi) &= (1-e^{-t/\tau})g_0 + e^{-t/\tau} f_0(-ut,-vt) \\
&\quad +\tau(-1+e^{-t/\tau})(u\bar{a}+v\bar{b})g_0 \\
&\quad +te^{-t/\tau}(u\bar{a}+v\bar{b})g_0,
\end{aligned}
\tag{0.1}
$$

and the simulation results in Fig.(12) and (14) are supposed coming from the above equation along with the assumption of the initial gas distribution function Eq.(7a) and the equilibrium state (7b). If we compare the above equation with Eq.(3.5) in the original BGK scheme [2], we can see easily that Eq.(0.1) loses a few critically important terms, which are responsible for the Navier-Stokes solutions. At the cell interface $x = 0$, the original BGK scheme (in 2D extension) is based on the distribution function (3.5) in [2], which is

$$
\begin{aligned}
f(0,0,t,u,v,\xi) &= (1-e^{-t/\tau})g_0 + e^{-t/\tau} f_0(-ut,-vt) \\
&\quad +\tau(-1+e^{-t/\tau})(u\bar{a}+v\bar{b}+A)g_0 \\
&\quad +te^{-t/\tau}(u\bar{a}+v\bar{b})g_0 + tAg_0.
\end{aligned}
\tag{0.2}
$$

The main difference between Eq.(0.1) and Eq.(0.2) is that Eq.(0.1) ignores all terms related to A. The inconsistency of Eq.(0.1) with the distribution function of the Navier-Stokes solution can be observed in the following. From the gas-kinetic theory [3], the gas distribution function corresponding to the Navier-Stokes solution can be generally expressed as,

$$f_{ns} = g(1+\phi),$$

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where g is the equilibrium state and $g\phi$ is the part departuring from the equilibrium one. The non-equilibrium part $g\phi$ satisfies the conservation condition,

$$\int g\phi\Psi d\Xi = 0,$$

where $\Psi = [1, u, v, \frac{1}{2}(u^2 + v^2 + \xi^2)]^T$ and $d\Xi = d\xi_1 d\xi_2 ... d\xi_K dudv$. Eq.(0.1) basically does not satisfy the above conservation requirement. In other words, it does not correspond to the gas distribution function for the viscous flow. This can be clearly observed in the smooth viscous flow situation. In this situation, Eq.(0.1) goes to,

$$g = g_0(1 - \tau(u\bar{a} + v\bar{b})).$$

On the contrary, Eq.(0.2) of the original BGK scheme [2] goes to

$$g = g_0(1 - \tau(u\bar{a} + v\bar{b} + A) + tA),$$

which is proved to be the Chapman-Enskog expansion for the viscous solution [4]. Furthermore, even in the absence of the time evolution part (tA) in the above equation, the A term in $(u\bar{a} + v\bar{b} + A)$ still has to be kept because the conservation requirement can be only satisfied with the inclusion of this term,

$$\int (u\bar{a} + v\bar{b} + A)\Psi d\Xi = 0,$$

and this is precisely the equation to evaluate A in the smooth flow region in the original BGK scheme. So, from the original BGK method, the corresponding nonequilibrium part in the gas distribution function is $\phi = -\tau(u\bar{a} + v\bar{b} + A)$. But, the corresponding nonequilibrium part in [1] is $\phi = -\tau(u\bar{a} + v\bar{b})$, which cannot even satisfy the basic requirement of $\int g\phi\Psi d\Xi = 0$. Even for the steady state calculation, once there are spatial gradients in the flow variables, such that $\bar{a} \neq 0$ or $\bar{b} \neq 0$, the term A will not be zero and it must be included in the gas distribution function for the correct capturing of viscous term. In summary, the original BGK formulation (3.5) in [2] is different from the so-claimed BGK solution (19) in [1]. Eq.(19) does not correspond to the distribution for the Navier-Stokes solution, it is only a simplification of the BGK solution.

After having the correct integral solution (0.2) of the BGK model, we need still define the particle collision time τ. As shown in [4,2], there are two parts in the collision time,

$$\tau = \frac{\mu}{p} + \Delta t \frac{p^l - p^r}{p^l + p^r}, \tag{0.3}$$

where μ is the viscosity coefficient and p is the pressure. The first part in the above equation corresponds to the physical viscosity and the second one is the artificial dissipation. For the Euler calculations, there is no any explicit physical viscosity, the collision time is simplified as

$$\tau = 0.01\Delta t + \Delta t \frac{p^l - p^r}{p^l + p^r}, \tag{0.4}$$

where Δt is the time step. With the original BGK distribution (0.2) and the initial distribution functions Eq.(7a) and (7b) in [1], we calculate the similar test cases, i.e., shock boundary layer interaction with $M = 2.0$ and Re$= 2.96 \times 10^5$, boundary layer case with Re$= 10^5$, and the

Mach reflection case. In all above cases, the time step Δt is determined by the CFL condition with Courant number equal to 0.7. All steady state solutions are obtained by the integration of the unsteady flow code, and the criteria for the steadyness is that the flow variation is less than 10^{-4}. No speed-up technique for the steady state solution is implemented. All codes of the original BGK scheme for these test cases can be obtained from the author's web page (http://www.math.ust.hk/ ˜makxu). In the first two Navier-Stokes calculations, the collision time presented in Eq.(0.3) is used, and for the third Euler calculation the one in Eq.(0.4) is adapted. The simulation results are shown in Fig.(1)-(5). For the shock boundary layer interaction case, the cited experimental data in [1] seems different from the data in [6] and [7] even though they are supposed coming from the same experiment [5]. As shown in these figures, the results from the original BGK scheme [2] are totally different from those obtained using pseudo-BGK one in [1]. In terms of accuracy, the real improvement in [1], if there is any, is only marginally.

In conclusion, the paper [1] misinterprets the original BGK scheme and constructs an over-simplified distribution function. The BGK results presented in figures (12) and (14) of [1] are not from the original BGK scheme, i.e., Eq.(3.5) in [2]. In terms of viscous boundary layer, the original BGK scheme [2] and all further improvements [4] could all give accurate solutions. In fact, the real improvement necessary for the BGK scheme is the accurate calculation of the viscous shock structure [8] instead of the boundary layer or the shock boundary interaction.

References

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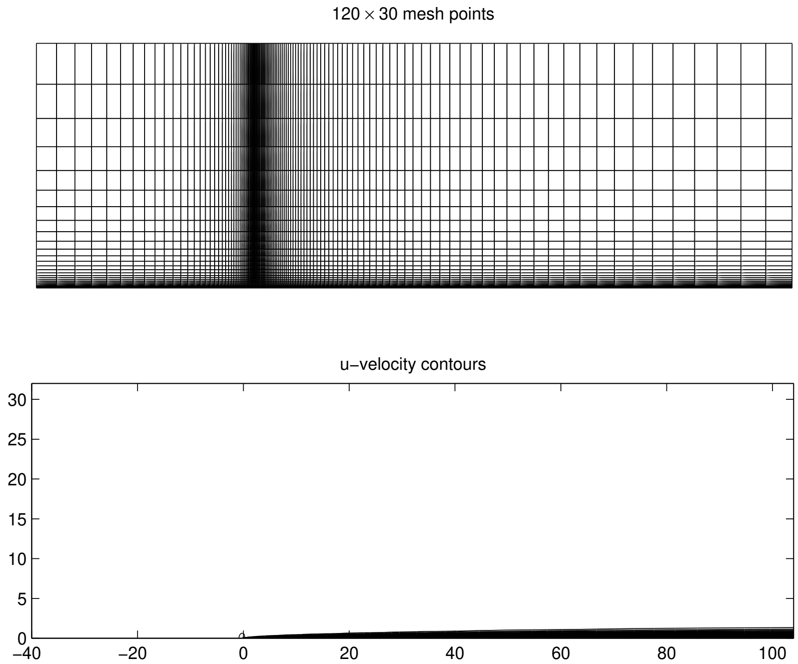

Figure 1: Upper: 120×30 grid points cover a computational domain $[-42, 120] \times [0, 32]$. In [1], a mesh according to the boundary layer solution (with fixed grid points everywhere in the boundary layer) is used, which is different from the current one. Lower: U-velocity contours calculated using the original BGK scheme [2] in the boundary case with Re$= 10^5$ and $M = 0.3$.

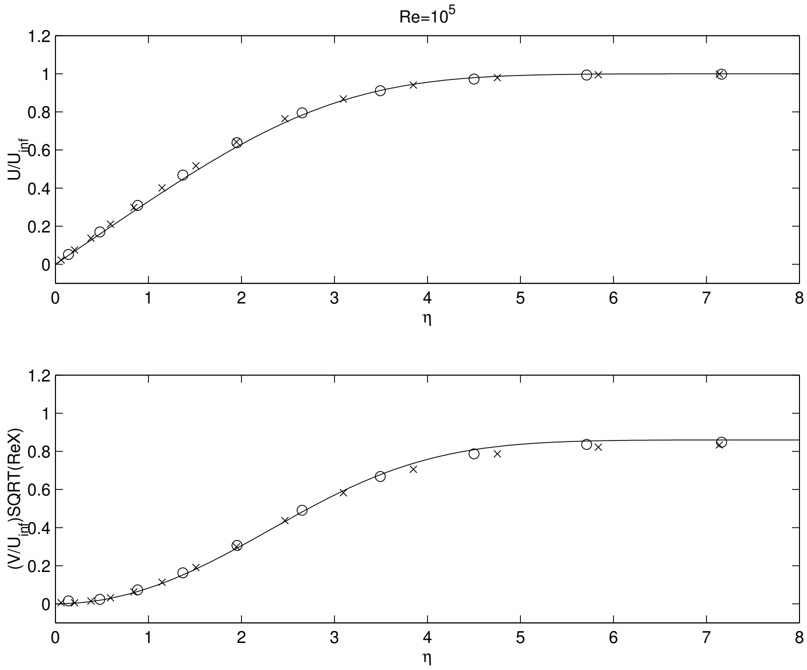

Figure 2: U and V velocity distributions at the locations $x = 6.438$ (circles) and $x = 34.469$ (cross sign). Both solid lines are the exact solutions.

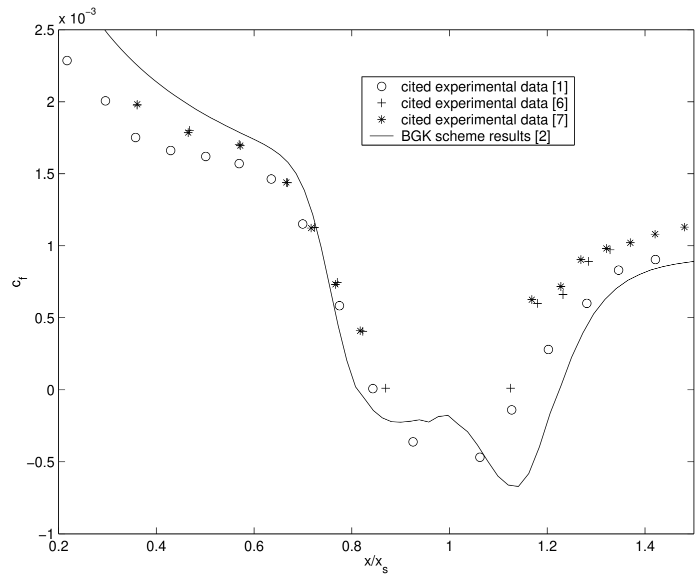

Figure 3: Shock boundary layer interaction case at $M = 2.0$ and Re$= 2.96 \times 10^5$ on a mesh of (106×66) grid points. Skin friction coefficient c_f along the flat plate is compared with the experimental data.

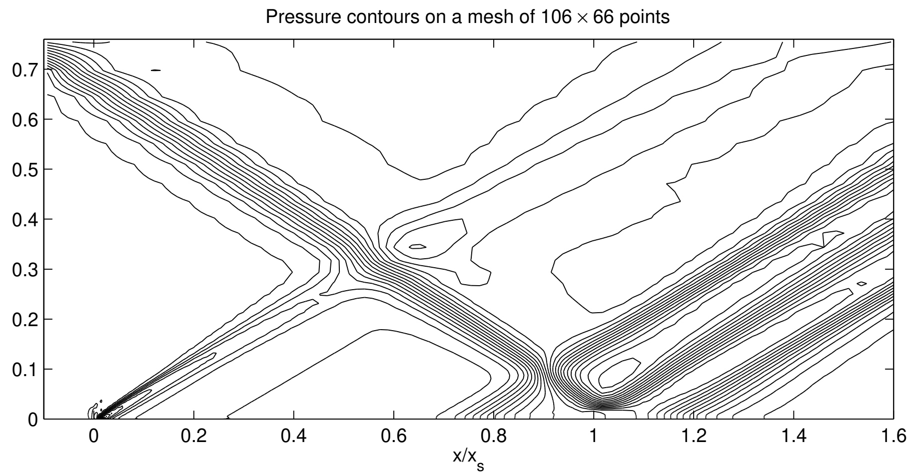

Figure 4: Pressure contours in the shock boundary layer interaction case.

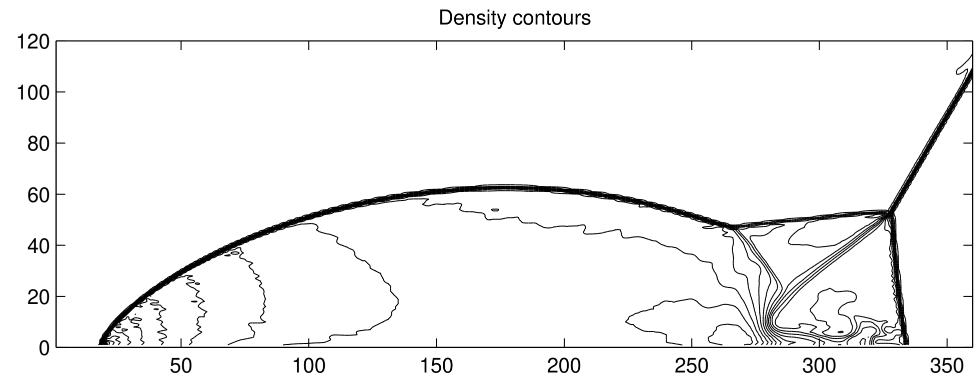

Figure 5: Density contours in the Mach reflection case on a computational mesh of (360×120) grid points.