DISCONTINUOUS GALERKIN BGK METHOD FOR VISCOUS FLOW EQUATIONS: ONE-DIMENSIONAL SYSTEMS

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Abstract. This paper is about the construction of a BGK Navier–Stokes (BGK-NS) solver in the discontinuous Galerkin (DG) framework. Since in the DG formulation the conservative variables and their slopes can be updated simultaneously, the flow evolution in each element involves only the flow variables in the nearest neighboring cells. Instead of using the semidiscrete approach in the Runge–Kutta discontinuous Galerkin (RKDG) method, the current DG-BGK method integrates the governing equations in time as well. Due to the coupling of advection and dissipative terms in the gas-kinetic formulation, the DG-BGK method solves the viscous governing equations directly. Numerical examples for the one-dimensional compressible Navier–Stokes solutions will be presented.

Key words. discontinuous Galerkin, gas-kinetic schemes, Navier–Stokes equations

AMS subject classifications. 65M06, 82C40, 35L64, 76N10

DOI. 10.1137/S1064827502416113

1. Introduction. In the past decades, the finite volume (FV) and discontinuous Galerkin (DG) finite element methods have both been successfully developed for the compressible viscous flow simulation. Most FV schemes use piecewise constant representation for the flow variables, resorting to the reconstruction techniques to obtain high accuracy. These methods are effective when the mesh has a good structure but may have difficulties when the mesh is irregular, such as an unstructured one. Basically, higher-order FV schemes face the problem of reconstruction on arbitrary triangulation in multidimensional cases. The schemes higher than second order seem to have limited application in engineering problems with complicated geometry and flow structures. In order to get rid of this difficulty, one possible way is to develop compact schemes. For example, third-order compact schemes for the Navier–Stokes equations in the one-dimensional case have been constructed using Marquina’s piecewise hyperbolic approach [12].

For the DG method the higher accuracy is obtained by means of higher-order polynomial approximation within elements. The DG method can deliver higher accuracy without relying on reconstruction techniques or large stencils. Since both FV and DG methods use the Riemann solution in the flux construction at a cell interface, the physics of wave propagation are accounted for in both schemes. Different from the traditional Galerkin finite element method, the reason for the DG method’s ability to capture shock without oscillation is mainly due to the fact that a discontinuous flow distribution is introduced at a cell interface. The artificial kinematic numerical dissipation is added through the averaging process inside each element, where the kinetic energy is transferred into the thermal one [22]. The DG method can automatically generate the appropriate slopes in the smooth flow region, and the slopes (dissipation) can be subsequently limited (added) through a nonlinear limiter in the regions with

*Received by the editors October 15, 2002; accepted for publication (in revised form) October 29, 2003; published electronically May 25, 2004. This work was supported by the Hong Kong Research Grant Council.

http://www.siam.org/journals/sisc/25-6/41611.html

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discontinuities [5]. The DG method seems to be emerging as a potential higher-order method for a broad class of engineering problems [1].

The excellent results given by the Runge–Kutta DG (RKDG) methods for purely convective problems prompted several authors to try to extend them to the viscous flow equations. As realized by Lowrie [11], theoretically, there is no easy method for adding viscous terms to any version of DG methods, including the semidiscrete version. The fundamental problem is that the viscous terms (second-order derivative) are not defined, even in the weak sense for a discontinuous interpolant. At the same time, the interface flux for the viscous part of the Navier–Stokes (NS) equations is not available in the Riemann solution. However, in the past few years with the reformulation of the original governing equations as a first-order system, the DG method, originally conceived for purely advective problems, has been extended to treat advection-diffusion problems as well. The DG method has proved very successful in the numerical solution of NS equations in high Reynolds number cases. Several schemes for the discretization of the viscous terms have been proposed in the literature, such as those by Bassi and Rebay [2], Cockburn and Shu [7], and Baumann and Oden [3].

The aim of this paper is to construct a gas-kinetic Bhatnagar–Gross–Krook (BGK) method in the DG framework. The difference between DG-BGK and the previous DG methods for the viscous equations is that due to the relaxation gas-kinetic model, there exists a kinetic flux function which includes both inviscid and viscous terms, even with initially discontinuous flow distribution. Instead of using the Runge–Kutta method for temporal discretization [8, 14], the time accuracy of the current scheme is obtained by integrating the equation in time directly. Similar to the other DG methods, the DG-BGK scheme is characterized by a very compact stencil since all unknowns associated with an element $e$ are coupled only with the unknowns associated with the elements which share a face with $e$. In the numerical part, the high nonequilibrium NS shock structure and the shock tube problem are tested. The newly developed DG-BGK method could give accurate solutions in both high and low Reynolds number flow simulations.

2. DG-BGK scheme. For compressible flow simulations, the FV gas-kinetic BGK scheme has been developed and applied to many physical and engineering problems [21, 10, 17, 15]. Similar to many other FV methods, the gas-kinetic scheme is mainly concerned with the flux evaluation at a cell interface from the MUSCL-type reconstructed initial data. Here a nonlinear limiter is used to avoid over- and under-shoot in the initial data. Due to the intrinsic connection between the gas-kinetic BGK model and the NS equations, an NS flux function is obtained automatically from the BGK scheme without splitting the inviscid and viscous terms [22]. However, in the FV framework it is not easy to extend the scheme to a higher order of accuracy. In this section, we will extend the gas-kinetic BGK scheme for the NS equations in the DG framework and construct a DG-BGK(1) scheme with a second-order flow distribution inside each cell and a DG-BGK(2) scheme with third-order distribution.

2.1. DG-BGK method. For a one-dimensional flow, the BGK model in the $x$-direction is [4]

$$f_t + uf_x = \frac{g - f}{\tau},$$

where $u$ is the particle velocity, $f$ is the gas distribution function, and $g$ is the equilibrium state approached by $f$. The particle collision time $\tau$ is related to the viscosity
and heat conduction coefficients. The equilibrium state is a Maxwellian distribution,

$$
g = \rho \left( \frac{\lambda}{\pi} \right)^{K+1} e^{-\lambda \left( u - U \right)^2 + \xi^2},
$$

(2)

where $\rho$ is the density, $U$ is the macroscopic velocity, and $\lambda$ is related to the gas temperature $m/2kT$. The total number of degrees of freedom $K$ of $\xi$ is equal to $(5 - 3\gamma)/(\gamma - 1)$. In the above equilibrium state $g$, $\xi^2$ is equal to $\xi_1^2 + \xi_2^2 + \ldots + \xi_K^2$. The relation between mass $\rho$, momentum $\rho U$, and energy $\rho E$ densities with the distribution function $f$ and the equilibrium state is

$$
\mathbf{w} = (\rho, \rho U, \rho E)^T = \int \psi_\alpha f d\Xi = \int \psi_\alpha g d\Xi, \quad \alpha = 1, 2, 3,
$$

(3)

where $\psi$ is the vector of moments

$$
\psi = \left( 1, u, \frac{1}{2} (u^2 + \xi^2) \right)^T,
$$

and $d\Xi = dud\xi$ is the volume element in the phase space with $d\xi = d\xi_1 d\xi_2 \ldots d\xi_K$. Equation (3) is equivalent to the compatibility condition

$$
\int \psi_\alpha g - \frac{f}{\tau} d\Xi = 0,
$$

where $\tau$ is independent of the particle velocity. Based on the above BGK model, the NS equations can be derived [20]. The advantage of using the BGK equation instead of the NS equations is that it is a first-order differential equation.

Let us integrate (1) to the moments $\psi_\alpha$; due to the compatibility condition (3), we have

$$
\frac{\partial}{\partial t} \int \psi_\alpha f d\Xi + \int \frac{\partial}{\partial x} u \psi_\alpha f d\Xi = 0, \quad \alpha = 1, 2, 3,
$$

(4)

which is equivalent to

$$
\mathbf{w}_t + \mathbf{h}_x = 0,
$$

(5)

where $\mathbf{h} = \int u \psi_\alpha f d\Xi$ are the fluxes for the corresponding conservative flow variables $\mathbf{w} = \int \psi_\alpha f d\Xi$. Note that (4) does not describe the equilibrium limit ($\tau \to 0$), i.e., $f = g$ for the Euler equations. The zero on the right-hand side is due to the collisional invariants for the collision term in the Boltzmann equation. Since $f$ in (4) needs to be evaluated from (1), it will include both equilibrium and nonequilibrium parts in a viscous flow calculation.

Let us discretize the above equation in the spatial variable $x$. Let $I_j = (x_{j-1/2}, x_{j+1/2})$ be the cell $j$, and let $I = \bigcup_j I_j$ be a partition of the real line. Denote $\Delta x_j = x_{j+1/2} - x_{j-1/2}$ as the cell size. For a finite element method, we suppose that the flow variables $\mathbf{w} = \int \psi_\alpha f d\Xi$ can be expanded as

$$
\mathbf{w}(x, t) = \sum_{l=0}^{k} q_l \mathbf{w}_j^{(l)} v_j^{(l)}(x) \quad \text{for} \quad x \in I_j,
$$

(6)
where \( v_i'(x) \) form a local orthogonal basis functions over \( I_j \), such as a Legendre polynomial basis,
\[
v_j^{(0)}(x) = 1, \quad v_j^{(1)}(x) = x - x_j, \quad v_j^{(2)}(x) = (x - x_j)^2 - \frac{1}{12}\Delta x_j^2, \ldots
\]
The coefficients \( a_i \) have the values
\[
a_0 = 1, \quad a_1 = 12/(\Delta x)_j, \quad a_2 = 180/(\Delta x)_j^2, \ldots
\]
and the degree of freedom \( w_j^{(l)} \) are defined by
\[
w_j^{(l)} = \frac{1}{\Delta x_j^{(l+1)}} \int_{I_j} w(x,t)v_j^{(l)}(x)dx.
\]
In order to determine the above degree of freedom, as given in [8], we can multiply (5) by the test function \( v_j^{(l)} \) and get the DG scheme
\[
\frac{d}{dt} w_j^{(l)} + \frac{1}{\Delta x_j^{l+1}} [\Delta_x v_j^{(l)}(x_{j-1/2})h_{j-1/2}] - \frac{1}{\Delta x_j^{l+1}} \int_{I_j} h(w(x,t)) \frac{d}{dx} v_j^{(l)}(x)dx = 0,
\]
where \( \Delta_x v_j^{(l)}(x_{j-1/2})h_{j-1/2} = v_j^{(l)}(x_{j+1/2})h_{j+1/2} - v_j^{(l)}(x_{j-1/2})h_{j-1/2} \). Here \( h_{j-1/2} \) is the flux at \( x_{j-1/2} \), i.e., \( h_{j-1/2} = \int u\psi_a f(x_{j-1/2},t,u,\xi)d\Xi \), and \( h(w(x,t)) \) is the flux inside each cell. Instead of using the Runge–Kutta time stepping method to solve (7), we can further integrate the above equation in time from \( t^n \) to \( t^{n+1} \).

More specifically, for the DG-BGK(2) scheme with third-order initial flow distribution, the flow variables inside cell \( I_j \) are expressed as
\[
w = w_j + s_j (x-x_j) + q_j [(x-x_j)^2 - (\Delta x_j)^2/12] \quad \text{for} \quad x \in [x_j - \Delta x_j/2, x_j + \Delta x_j/2],
\]
where \( w_j \) is the cell-averaged conservative variable, and \( s_j \) and \( q_j \) are the first- and second-order derivatives. With a uniform mesh, the time integration of (7) from \( t^n \) to \( t^{n+1} \) updates the parameters \( (w_j, s_j, q_j) \) in the following way.
(i) The cell averaged value is
\[
w_j^{n+1} = w_j^n + \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \int \psi_a f(x_{j-1/2},t,u,\xi) - f(x_{j+1/2},t,u,\xi)d\Xi dt.
\]
(ii) The linear slope is
\[
s_j^{n+1} = s_j^n - \frac{6}{(\Delta x)^2} \int_{t^n}^{t^{n+1}} \int \psi_a f(x_{j-1/2},t,u,\xi) + f(x_{j+1/2},t,u,\xi)d\Xi dt
\]
\[
+ \frac{12}{(\Delta x)^3} \int_{-\Delta x/2}^{\Delta x/2} \int_{t^n}^{t^{n+1}} \int \psi_a f(x,t,u,\xi)d\Xi dx dt.
\]
(iii) The quadratic coefficient is
\[
q_j^{n+1} = q_j^n + \frac{30}{(\Delta x)^3} \int_{t^n}^{t^{n+1}} \int \psi_a f(x_{j-1/2},t,u,\xi) - f(x_{j+1/2},t,u,\xi)d\Xi dt
\]
\[
+ \frac{360}{(\Delta x)^5} \int_{-\Delta x/2}^{\Delta x/2} \int_{t^n}^{t^{n+1}} \int \psi_a f(x,t,u,\xi)(x-x_j)d\Xi dx dt.
\]
Fig. 1. Schematic distribution of flow variables around a cell interface.

Note that \( f(x_{j+1/2}, t, u, \xi) \) is the time-dependent gas distribution function at the cell interface \( x_{j+1/2} \), and \( f(x, t, u, \xi) \) is the time-dependent distribution inside each cell \( I_j \). For the DG-BGK(1) scheme, as with the FV BGK-NS solver [22], the parameters evaluated inside each cell are \( w_j \) and \( s_j \) only. In the BGK-NS method, \( s_j \) is reconstructed by the MUSCL-type interpolation [19]. However, for the DG method, it is updated automatically according to (9). In the following, we are going to present the flux evaluation at the cell interface and the flux integration inside each cell based on the BGK scheme for the viscous flow equations.

2.2. BGK flux evaluation at a cell interface. As with the previous approach in the BGK-NS scheme [22], the flux at the cell interface in the BGK scheme is evaluated based on the integral solution \( f \) of the BGK model (1),

\[
(11) \quad f(x_{j+1/2}, t, u, \xi) = \frac{1}{\tau} \int_0^t g(x', t', u, \xi)e^{-\frac{(t-t')}{\tau}}dt' + e^{-\frac{t}{\tau}}f_0(x_{j+1/2} - ut),
\]

where \( x' = x_{j+1/2} - u(t-t') \) is the trajectory of a particle motion and \( f_0 \) is the initial gas distribution function at the beginning of each time step \( t = 0 \). In order to figure out the gas distribution function at a cell interface \( x_{j+1/2} \), two unknowns, \( g \) and \( f_0 \), in the above equation have to be specified. In the following, \( x_{j+1/2} = 0 \) will be used.

With the initial macroscopic flow distributions around a cell interface (see Figure 1) to the NS order, the initial gas distribution function \( f_0 \) is constructed as

\[
(12) \quad f_0 = \begin{cases} 
  g^l \left( 1 + a^l x - \tau (a^l u + A^l) \right), & x \leq 0, \\
  g^r \left( 1 + a^r x - \tau (a^r u + A^r) \right), & x \geq 0,
\end{cases}
\]

where \( g^l \) and \( g^r \) are the equilibrium states at the left and right sides of a cell interface. The Maxwellian distributions \( g^l \) and \( g^r \) have a unique correspondence with the
macroscopic variables there, i.e.,
\[ g^l = g^l\left(\textbf{w}_{j+1/2}\right) \quad \text{and} \quad g^r = g^r\left(\textbf{w}_{j+1/2}\right). \]

Note that the values of the flow variables \( \textbf{w}_{j+1/2}, \textbf{w}_{j+1/2}', \textbf{s}_{j+1/2}, \) and \( \textbf{s}_{j+1/2}' \) will depend on the order of the schemes, which will be presented in section 2.4. Here \( a^l \) and \( a^r \) in (12) are coming from the spatial derivative of a Maxwellian distribution function, i.e.,
\[ a^l = \frac{\alpha}{2} + \frac{\alpha^2}{2}(u^2 + \xi^2) \quad \text{and} \quad a^r = \frac{\alpha}{2} + \frac{\alpha^2}{2}(u^2 + \xi^2), \]
which can be evaluated uniquely from the slopes of the conservative variables \( \textbf{s}_{j+1/2} \) and \( \textbf{s}_{j+1/2}' \). The terms \( -\tau(a^l u + A')g' \) and \( -\tau(a^r u + A')g^r \) are the nonequilibrium parts obtained from the Chapman–Enskog expansion, which account for the dissipative effects. Since there is no contribution in the mass, momentum, and energy from the nonequilibrium parts, the parameters in \( A' = A_1 + A_2 u + \frac{1}{2} A_3 (u^2 + \xi^2) \) and \( A^r = A_1 + A_2 u + \frac{1}{2} A_3 (u^2 + \xi^2) \) are uniquely determined by the compatibility conditions
\[
\int \psi(x, (a^l u + A')g') d\xi = 0 \quad \text{and} \quad \int \psi(x, (a^r u + A')g^r) d\xi = 0.
\]

Even though the nonequilibrium part has no contribution to the conservative flow variables (moments of \( \psi \)), it will effect the flux (moments of \( u\psi \)).

The equilibrium state \( g \) around \( (x = 0, t = 0) \) in (11) is assumed to have the form
\[
g = g_0 \left(1 + (1 - H(x))\bar{a}^l x + H(x)\bar{a}^r x + \bar{A} t\right),
\]
where \( H(x) \) is the Heaviside function. Here \( g_0 \) is a local Maxwellian distribution function located at \( x = 0 \). The dependence of \( \bar{a}^l, \bar{a}^r, \) and \( \bar{A} \) on the particle velocities are also coming from a Taylor expansion of the Maxwellian, which have the forms \( \bar{a}^l = \bar{a}_1^l + \bar{a}_2^l u + \frac{1}{2} \bar{a}_3^l (u^2 + \xi^2), \bar{a}^r = \bar{a}_1^r + \bar{a}_2^r u + \frac{1}{2} \bar{a}_3^r (u^2 + \xi^2), \) and \( \bar{A} = A_1 + A_2 u + \frac{1}{2} A_3 (u^2 + \xi^2) \). The determination of the parameters in (13) is the following. The equilibrium state \( g_0 \) at the cell interface is constructed from the conservative flow variable \( \textbf{w}_0 \), i.e.,
\[
\textbf{w}_0 = \int \psi(x, g_0) d\xi = \int_{u > 0} \psi(x, g_0) d\xi + \int_{u < 0} \psi(x, g_0) d\xi,
\]
and \( \bar{a}^l \) and \( \bar{a}^r \) can be obtained from the corresponding slopes
\[
(\textbf{w}_0 - \textbf{w}_j)/(\Delta x/2) \quad \text{and} \quad (\textbf{w}_{j+1} - \textbf{w}_0)/(\Delta x/2).
\]
The time variation part \( A \) in (13) is determined uniquely from the compatibility condition at the cell interface, i.e., \( \int_0^{\Delta t} \int \psi(f - g) d\xi dt = 0 \), where \( f \) (14) and \( g \) (13) have the common \( \bar{A} \) term.

After substituting \( f_0 \) and \( g \) into the integral solution of the BGK model (11), the gas distribution function \( f \) at a cell interface becomes
\[
f(x_{j+1/2}, t, u, \xi) = (1 - e^{-t/\tau})g_0 + \tau(t/\tau - 1 + e^{-t/\tau})\bar{A} g_0 + (1 - u(t + \tau)a^l)H[u]g^l + (1 - u(t + \tau)a^r)(1 - H[u])g^r
\]
\[
+ \left(\frac{\tau}{1 + e^{-t/\tau}} + e^{-t/\tau}\right)(\bar{a}^l H[u] + \bar{a}^r (1 - H[u])) u g_0
\]
\[
+ e^{-t/\tau}(1 - u(t + \tau)a^l)H[u]g^l + (1 - u(t + \tau)a^r)(1 - H[u])g^r
\]
\[
+ e^{-t/\tau}(-\tau A_1^l H[u] g^l - \tau A^r (1 - H[u]) g^r),
\]
from which the numerical fluxes across a cell interface can be constructed:

$$h(x_{j+1/2}, t) = \int \psi_\alpha f(x_{j+1/2}, t, u, \xi) d\xi$$

for $\alpha = 1, 2, 3$.  

Then the time integration of the above flux can be evaluated explicitly. Similar to an FV scheme, based on the above flux function, the cell averaged conservative variables $w_j$ in the DG method can be updated by

$$w_j^{n+1} = w_j^n + \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} (h(x_{j-1/2}, t, u, \xi) - h(x_{j+1/2}, t, u, \xi)) dt,$$

which is the same equation as (8). The above flux function can be used in (9) and (10) for the updates of $s_j$ and $q_j$ as well.

For the NS solutions, the viscosity and heat conduction coefficients are related to the particle collision time $\tau$ in (14). With a given dynamical viscosity law $\mu$, the general collision time can be determined by

$$\tau = \mu/p + \Delta t \frac{|p^l - p^r|}{|p^l + p^r|},$$

where $p$ is the pressure. The second term on the right-hand side of the above equation accounts for the artificial dissipation in case of underresolution of the viscous solution by the cell size. For the shock structure calculation in the numerical experiments section, the second term on the right-hand side of $\tau$ is absent. In order to get the heat conduction correct, the energy flux in (15) can be modified according to any realistic Prandtl number [22].

2.3. Flux integral inside a cell. Inside each cell $I_j$, the flow variables are distributed smoothly. To the NS order, the gas distribution function has the form

$$f(x, t, u, \xi) = g(x_j, t^n, u, \xi)[1 - \tau(a u + A) + At + a(x - x_j)],$$

where $a = a_1 + a_2 u + \frac{1}{2} a_3 (u^2 + \xi^2)$ are determined from the slopes of the macroscopic variables. The parameter $A$ is subsequently obtained by the requirement $\int \psi_\alpha (a u + A) g d\xi = 0$ for the nonequilibrium part in the gas distribution function. As shown by Ohwada [13], the above distribution (17) does correspond to the NS solution. In the above equation, the equilibrium state $g(x_j, t^n, u, \xi)$ at the cell center is determined from the conservative variables there, i.e., $w_j - q_j (\Delta x)^2/12$, and the linear slope $a$ is evaluated from $s_j$. Therefore, the integration of $f$ inside each cell, such as

$$\int_{-\Delta x/2}^{\Delta x/2} \int_{t^n}^{t^{n+1}} \int w_\alpha f(x, t, u, \xi) d\xi dt dx, \quad \alpha = 1, 2, 3,$$

and

$$\int_{-\Delta x/2}^{\Delta x/2} \int_{t^n}^{t^{n+1}} \int w_\alpha f(x, t, u, \xi)(x - x_j) d\xi dt dx, \quad \alpha = 1, 2, 3,$$

in (9) and (10) can be calculated explicitly.
2.4. Limited slopes. In the above BGK flux evaluation, the cell interface values $w_{j+1/2}^l$ and $w_{j+1/2}^r$ and their slopes $s_{j+1/2}^l$ and $s_{j+1/2}^r$ depend on the order of the schemes. The DG-BGK(1) scheme keeps only the averaged flow variables $w_j$ and their linear slopes $s_j$ inside each cell. Therefore, the initial condition of the macroscopic variables at a cell interface at $x_{j+1/2}$ (see Figure 1) is

$$w_{j+1/2}^l = w_j + s_j \Delta x / 2$$

and

$$w_{j+1/2}^r = w_{j+1} - s_{j+1} \Delta x / 2.$$ 

The corresponding slopes at $x_{j+1/2}$ are $s_j$ and $s_{j+1}$, respectively.

For the DG-BGK(2) scheme, $w_j$, $s_j$, and $q_j$ are kept inside each cell. Therefore, the initial cell interface flow variables become

$$w_{j+1/2}^l = w_j + s_j \Delta x / 2 + q_j (\Delta x)^2 / 6$$

and

$$w_{j+1/2}^r = w_{j+1} - s_{j+1} \Delta x / 2 + q_{j+1} (\Delta x)^2 / 6.$$ 

In the DG-BGK schemes, only linearly distributed flow variables are used in the flux evaluation at a cell interface. If a higher-order polynomial is directly followed in the gas evolution process based on the BGK equation, it will give a solution different from the NS equations. For example, the BGK solution from an initial parabola (cubic) distribution will give the Burnett (super-Burnett) solutions [18, 23]. So, in order to simulate the NS solutions, in the BGK scheme even with the higher-order initial interpolation, the time evolution of the gas distribution function at a cell interface should only follow the evolution of linearly distributed macroscopic flow variables. The different order of the scheme will be represented from the different values of the initial data, such as $w_{j+1/2}^l$, $w_{j+1/2}^r$, and their slopes $s_{j+1/2}^l$ and $s_{j+1/2}^r$.

For the compressible flow simulation, the direct update of $s_j$ and $q_j$ will basically generate oscillatory solutions across strong shock waves. In other words, the DG method without using a nonlinear limiter on the updated slopes will behave similarly to the traditional central schemes, like the Lax–Wendroff method. In order to eliminate these oscillations, numerical dissipation has to be added. The additional kinematic dissipation can be introduced through the modification of the slopes [22, 6]. Therefore, the nonlinear limiter, usually used in the FV scheme, has to be used in the DG method as well. Based on (8)–(10), we have updated slopes $s_j$ and $q_j$. In this paper, their modification will be based on the following van Albada limiter. Define

$$s_1 = (w_j - w_{j-1}) / \Delta x, \quad s_2 = (w_{j+1} - w_j) / \Delta x,$$

and the limited slope inside cell $j$ is

$$\bar{s}_j = (s_j|s_is_2| + s_1|s_is_2| + s_2|s_js_1|) / (|s_j|s_1| + |s_1|s_2| + |s_is_2|).$$

After limiting the linear slope, the quadratic slope can be limited similarly,

$$\bar{q}_j = (q_j|q_1q_2| + q_1|q_1q_2| + q_2|q_2q_1|) / (|q_j|q_1| + |q_1|q_2| + |q_1q_2|),$$

where $q_1 = (s_j - \bar{s}_{j-1}) / \Delta x$ and $q_2 = (\bar{s}_{j+1} - \bar{s}_j) / \Delta x$. Therefore, the flow variables in Figure 1 will be based on the above limited slopes. The above limiters are used in all test cases in the following numerical experiments. In order to compare the above
DG-BGK method with the BGK-NS scheme in [22], a similar nonlinear limiter is used for the initial data reconstruction. For example, for the FV BGK-NS scheme, the linear slope $\bar{s}$ is obtained based on

$$\bar{s}_j = (s_1|s_2| + s_2|s_1|)/(|s_1| + |s_2|).$$

It is certainly true that different limiters have different effects on the resolution of the shock and contact discontinuity waves, especially in the case that the cell size cannot properly resolve the corresponding physical thickness. In order to crisply capture the discontinuity solution, new limiters are continuously being invented. But it seems that the crisp capturing of a discontinuity transition and the robustness of the scheme are antithetical, and a good compromise between them has to be achieved. For example, it is observed that a sine wave can be changed into a square wave by a “compressed” limiter. In a high nonequilibrium flow calculation, i.e., the high Mach number shock structure, a compressed limiter can easily destabilize the shock structure.

3. Numerical experiments. The DG-BGK schemes will be tested in the NS shock structure and shock tube cases. In order to compare the FV and DG methods, the results from the BGK-NS solver of [22] will also be presented in some cases.

The first case is the shock structure calculation. Even though the shock structure is well resolved in this case, due to the high nonequilibrium state inside the shock layer, its correct calculation requires the robustness of any scheme. The problem of determining the profile of a normal shock wave, and the correct capturing of the viscous stress and heat conduction inside the shock layer, represents an extreme test condition for the viscous flow solver, where the corresponding Reynolds number is on the order 1. Although it is well known that the NS equations do not give the physically realistic shock wave profile in the high Mach number cases, it is still useful in establishing and testing a valid NS method. Also, the effect of the order of the accuracy of the scheme on the shock structure construction can be tested.

The shock structure calculation is for a monotonic gas with $\gamma = 5/3$ and a viscosity coefficient $\mu \sim T^{0.8}$, where $T$ is the temperature. The upstream Mach number used is $M = 3.0$, which is a pretty strong shock with a high nonequilibrium state inside the shock layer. In order to test the validity of the DG-BGK scheme for any Prandtl number, $Pr = 2/3$ is used in the following calculations. The reference solution is obtained by directly integrating the steady state NS equations (ODEs) using the Matlab programs, which are provided in Appendix C of [22]. In all cases, the dynamic viscosity coefficient at the upstream keeps a constant value $\mu_{-\infty} = 0.0005$. Because the normal stress and the heat flux seem to show the greatest numerical sensitivity, these are selected also to display. Therefore, the temperature $T$ and the fluid velocity $U$ profiles across the shock layer, as well as the normal stress and the heat flux defined by

$$\tau_{nn} = \frac{4}{3} \mu u_x^2/2p, \quad q_x = -\frac{5}{4} \frac{\mu}{Pr} T_x/pc,$$

versus fluid velocity $u/U_{-\infty}$, are presented.

Figure 2 presents the results calculated with the DG-BGK(1) scheme without a limiter. In other words, the slopes $s_j$ used here are updated directly. The mesh size used is $\Delta x = 800$ and there are about 10 points inside the shock layer. Figure 2 shows that there is significant deviation in the temperature and velocity distributions in the upstream region. With the implementation of the nonlinear limiter on the
The viscosity coefficient $\mu \sim T^{0.8}$ with $\mu_{-\infty} = 0.0005$ and $Pr = 2/3$.

modification of $s_j$, the results from the DG-BGK(1) are shown in Figure 3, where the limiter slightly improves the results of Figure 2 in the upstream region. Figure 4 shows the results from the DG-BGK(2) scheme with a limiter. From this figure, we can realize the importance of incorporating the higher-order terms in the viscous
Fig. 4. DG-BGK(2) with limiter. $M = 3.0$ shock structure calculation with $\Delta x = 1/800$. The viscosity coefficient $\mu \sim T^{0.8}$ with $\mu_{-\infty} = 0.0005$.

Fig. 5. BGK-NS scheme with limiter [22]. $M = 3.0$ shock structure calculation with $\Delta x = 1/800$. The viscosity coefficient $\mu \sim T^{0.8}$ with $\mu_{-\infty} = 0.0005$.

calculations. Even with 10 grid points in the shock layer, the NS solution is perfectly obtained. As a comparison, Figure 5 shows the results from the BGK-NS scheme [22], where the slope is reconstructed instead of updated. Basically, the BGK-NS scheme gives the results similar to those from the DG-BGK(1) method, and both schemes
Fig. 6. DG-BGK(1) without limiter. $M = 3.0$ shock structure calculation with $\Delta x = 1/1600$. The viscosity coefficient $\mu \sim T^{0.8}$ with $\mu_{-\infty} = 0.0005$.

Fig. 7. DG-BGK(1) with limiter. $M = 3.0$ shock structure calculation with $\Delta x = 1/1600$. The viscosity coefficient $\mu \sim T^{0.8}$ with $\mu_{-\infty} = 0.0005$.

are second-order accurate in the flow distribution inside each cell. The simulation results from the DG-BGK and BGK-NS methods when we reduce the cell size to $\Delta x = 1/1600$ are shown in Figures 6–9. The same conclusion can be drawn as in the coarse mesh case. Figures 10–13 present the convergence results for the temperature
Fig. 8. DG-BGK(2) with limiter. $M = 3.0$ shock structure calculation with $\Delta x = 1/1600$. The viscosity coefficient $\mu \sim T^{0.8}$ with $\mu_{-\infty} = 0.0005$.

Fig. 9. BGK-NS scheme [22]. $M = 3.0$ shock structure calculation with $\Delta x = 1/1600$. The viscosity coefficient $\mu \sim T^{0.8}$ with $\mu_{-\infty} = 0.0005$.

and velocity distributions from the DG-BGK and BGK-NS schemes with mesh sizes $\Delta x = 1/800, 1/1600,$ and $1/3200$. As shown in these figures, the nonlinear limiter does help the convergence of the DG-BGK schemes to the NS solutions, and the higher-order method performs much better than the lower-order ones. Based on the above
shock structure data, we also estimate the orders of accuracy for the DG-BGK(1) and DG-BGK(2) schemes. With the mesh refinement, the $L_1$ and $L_2$ errors, as well as the orders of accuracy, are shown in Tables 1 and 2. It is surprising that in the viscous and heat conducting shock structure calculations, we get only first-order accuracy.
Fig. 11. Temperature and velocity distributions for $M = 3.0$ shock structure with $\Delta x = 1/800, 1/1600, 1/3200$. DG-BGK(1) with limiter.

for the DG-BGK(1) scheme and second-order accuracy for the DG-BGK(2) scheme. Usually when estimating the order of accuracy, linear or linear diffusion equations are used. For these equations, the BGK-NS code does have a second-order accuracy. However, for the current high nonequilibrium flow with Reynolds number on the
Fig. 12. Temperature and velocity distributions for $M = 3.0$ shock structure with $\Delta x = 1/800, 1/1600, 1/3200$. DG-BGK(2) scheme.

order 1, both DG-BGK(1) and DG-BGK(2) schemes are one order lower than what we expected. The reason for this will be further investigated. Hopefully, the current shock structure case can also be used by other authors to test different higher-order DG schemes.
In order to further test the DG-BGK scheme in the capturing of the NS solutions in the unsteady case, in the following we will calculate the Sod test directly by solving the NS equations with $Pr = 1.0$. The cell size used here is $\Delta x = 1/200$. Figure 14 presents the density and velocity distributions in the case with a physical kinematic
viscosity coefficient equal to \( \nu = 0.0005/\rho \sqrt{\lambda} \), where \( \lambda \) is related to the temperature in the local equilibrium distribution function \( g_0 \). In this case, due to large viscosity coefficient, both the shock structure and contact wave can be well resolved by this cell size and both waves are captured accurately. Figure 14 shows excellent results from the DG-BGK(1) and DG-BGK(2) schemes. The solid lines here are the reference solutions obtained by solving the NS equations with a much refined mesh size \( \Delta x = 1/1200 \). Figure 15 is the enlarged view around the shock and contact discontinuity. When the viscosity coefficient is reduced to a value \( \nu = 0.000005/\rho \sqrt{\lambda} \), both NS shock and contact waves cannot be resolved by the current cell size \( \Delta x = 1/200 \) and both schemes become shock-capturing methods. The results are shown in Figure 16. Here the shock and contact transitions are purely constructed from the numerical dissipations, which are much wider than the physical ones determined from the above physical viscosity. In Figure 16, the DG-BGK(1) scheme gives sharper transition in the discontinuity regions than the DG-BGK(2) scheme. This means that the artificial dissipation introduced in DG-BGK(1) is less than that in DG-BGK(2). The origin of the artificial dissipation is mainly the result of the limited initial data, where the kinetic energy is implicitly transferred into thermal energy. In the underresolved regions it is possible that more dissipation is introduced in a higher-order interpolation, where the initial interpolated data becomes smoother. As a last test case, Figure 17 shows the Shu–Osher case with \( \Delta x = 1/400 \) and \( \nu = 0.000005/\rho \sqrt{\lambda} \) [16]. In this case, the DG-BGK(1) and DG-BGK(2) schemes perform similarly in resolving the shock sound wave interaction, where the solid line is the reference Euler solution.

From these examples, we can see that the DG-BGK schemes basically get accurate NS solutions, even in the high nonequilibrium case. The performance of the DG-BGK(1) scheme is almost identical to that of the BGK-NS method based on the MUSCL-type interpolation directly. For all cases, the accuracy of the schemes depends on the limiter used, especially in the underresolved regions. For the crisp capturing of a discontinuous solution, the Huynh limiter can be a good alternative choice [9].

4. Conclusion. In this paper, we extend the BGK-NS scheme to the DG framework, where not only the conservative variables, but also their slopes, are updated. Due to the coupling of the inviscid and viscous terms in the gas-kinetic approach, the NS solutions are automatically obtained in the DG-BGK method. In the DG-BGK formulation, the high-order temporal accuracy is obtained by integrating the governing equations in time directly. The high nonequilibrium viscous shock structure calculation, where the corresponding Reynolds number is on the order of 1, shows that

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the nonlinear limiter used on the updated slopes could not only increase the stability of the scheme but also improve the accuracy of the method. Also, it is noticed that the higher-order DG-BGK(2) scheme does give a more accurate solution than the lower-order DG-BGK(1) method, especially in the resolved case. In this paper, we
also realize that the DG-BGK(1) and the MUSCL-type FV BGK-NS schemes behave similarly. The DG framework is a good method for increasing the order of accuracy of a scheme, which is very important and effective for the viscous calculation when the flow structures are just resolved, such as those with 8 to 10 points in the shock or boundary layers. In conclusion, it should be emphasized that in the Mach 3 shock...
Fig. 16. Shock tube problem for the NS equations with kinematic viscosity coefficient $\nu = 0.000005/\rho \sqrt{\lambda}$ and $P_r = 1$. The cell size used is $\Delta x = 1/200$ and the solid line is the exact solution. The results from the DG-BGK(1) and DG-BGK(2) schemes are presented.

structure calculations, we get only first-order accuracy for the DG-BGK(1) scheme and second-order accuracy for the DG-BGK(2) scheme. This is different from what we had expected. For the linear advection and linear diffusion equations, we did get second-order accuracy for the DG-BGK(1) scheme. The reason for the reduction of
order of accuracy in the current nonlinear viscous and heat conducting shock structure calculations is still unknown, and it will be further investigated. Hopefully, other higher-order DG schemes can also be tested in high nonequilibrium flows.

Acknowledgments. The author would like to thank the reviewers for their helpful comments and constructive suggestions, which improved the manuscript greatly, and Mr. Hongwei Liu for his help in estimating the order of accuracy of the DG-BGK schemes.

REFERENCES