GENERALIZED GAS DYNAMIC EQUATIONS WITH MULTIPLE TRANSLATIONAL TEMPERATURES

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Based on a multiple stage BGK-type collision model and the Chapman–Enskog expansion, the corresponding macroscopic gas dynamics equations in three-dimensional space will be derived. The new gas dynamic equations have the same structure as the Navier–Stokes equations, but the stress strain relationship in the Navier–Stokes equations is replaced by an algebraic equation with temperature differences. In the continuum flow regime, the new gas dynamic equations automatically recover the standard Navier–Stokes equations. The current gas dynamic equations are natural extension of the Navier–Stokes equations to the near continuum flow regime and can be used for near continuum flow study.

Keywords: Multiple translational temperature; extended Navier–Stokes equations.

1. Introduction

One of the alternative approach to simulate the non-equilibrium flow is those based on the moment closures. The well-known moment system is Levermore’s 10 moment closure, which follows his hierarchy of non-perturbative moment closures with many desirable mathematical properties. These equations don’t suffer from the closure-breakdown deficiencies, and they always give physically realizable solution due to non-negative gas distribution function. However, the 10 moment Gaussian closure has no heat flux even though it is proved that Navier–Stokes viscous terms can be recovered in the continuum flow regime. Recently, McDonald and Groth took a Chapman–Enskog-type expansion about either the moment equations or the kinetic equation, and introduced the heat flux into Levermore’s 10 moment closures and obtained extended fluid dynamic equations for non-equilibrium flow simulation. The new system present improved results in the transition flow regime where the heat transfer has a significant effect. In this paper, based on the multiple stage collision model, we are going to derive generalized gas dynamic equations. Surprisingly, the obtained new equations become regularization of Levermore’s 10 moment
2. Two Stage Gas-Kinetic Collision Model

The gas-kinetic Bhatnagar-Gross-Krook (BGK) model has the form,

\[
\partial_t f + u_i \partial_i f = (f_{eq} - f)/\tau, \tag{1}
\]

where the particle distribution function \( f \) is a function of time \( t \), spatial location \( x_i \), and particle velocity \( u_i \). The left hand side of the above equation represents the free streaming of molecules in space, and the right side denotes the simplified collision term of the Boltzmann equation. In the BGK model, the collision operator involves a single relaxation time \( \tau \) for a non-equilibrium state to evolve to an equilibrium one \( f_{eq} \), which is an isotropic Gaussian

\[
f_{eq} = \frac{\rho}{(2\pi RT_{eq})^{3/2}} \exp \left[ -\frac{(u_i - U_i)(u_i - U_i)}{2RT_{eq}} \right],
\]

where \( \rho \) is the density, \( T_{eq} \) is the equilibrium temperature, and \( U_i \) is the averaged macroscopic fluid velocity. Traditionally, based on the above BGK model, the Navier-Stokes and higher-order equations, such as Burnett and Super-Burnett, can be derived.\(^2\)\(^5\) Unfortunately, these higher-order equations have intrinsic physical and mathematical problems in the transitional flow regime. In order to extend the capacity of the BGK model to the non-equilibrium flow regime, we can re-write the collision term of the BGK model into two physical sub-processes,

\[
\partial_t f + u_i \partial_i f = (g - f)/\tau + (f_{eq} - g)/\tau, \tag{2}
\]

where \( g \) is a middle state between \( f \) and \( f_{eq} \), see figure 1. In the above equation, the term \( (f_{eq} - g)/\tau \) has no direct connection with \( f \), therefore, we can consider it as a source term in the above generalized BGK (GBGK) model,

\[
\partial_t f + u_i \partial_i f = (g - f)/\tau + Q, \tag{3}
\]
where \( Q = (f^{eq} - g) / \tau \) for the monatomic gas, and the state \( g \) between \( f \) and \( f^{eq} \) is the Gaussian distribution,

\[
g = \frac{\rho}{\sqrt{\det(2\pi RT_{ij})}} \exp\left(-\frac{1}{2}(u_i - U_i)(RT_{ij})^{-1}(u_j - U_j)\right),
\]

where \( T_{ij} \) is the positive definite temperature matrix.


The generalized BGK model includes two relaxation processes. One is from \( f \) to the Gaussian \( g \), and the other is from \( g \) to an equilibrium state \( f^{eq} \). The solution \( f \) around the Gaussian \( g \) is constructed using the iterative expansion to the 1st-order,

\[
f = g - \tau(\partial_t g + u_i \partial_i g) + \tau Q,
\]

where in the kinetic scheme \( \partial_t g \) is determined using the compatibility condition

\[
\int \psi(\partial_t g + u_i \partial_i g) du = \int \psi Q du,
\]

which is exactly the 10 moment closure of Levermore. Substituting the distribution function \( f \) in (4) into the BGK model (3), the equation becomes

\[
\partial_t g + u_i \partial_i g = \tau(\partial_t^2 g + 2u_i \partial_i g + u_i u_j \partial_j g) + Q - \tau(\partial_t Q + u_i \partial_i Q).
\]

Taking the moments \( \psi \) to the above equation and using Eq.(5) to express the time derivative in terms of the spatial derivative, we can get the following macroscopic equations,

\[
\partial_t \rho + \partial_k(\rho U_k) = 0,
\]

\[
\partial_t(\rho U_i) + \partial_k[\rho(U_i U_k + RT_{ik})] = \partial_k[\rho R(T^{eq} \delta_{ki} - T_{ki})],
\]

\[
\partial_k[\rho(U_i U_j + RT_{ij})] + \partial_k[\rho( U_i U_j U_k + RU_k T_{ij} + RU_i T_{jk} + RU_j T_{ki})]
\]

\[
= \frac{2}{\tau} \rho R(T^{eq} \delta_{ij} - T_{ij})
\]

\[
+ \partial_k\{\rho R[U_k(T^{eq} \delta_{ij} - T_{ij}) + U_j(T^{eq} \delta_{jk} - T_{jk}) + U_j(T^{eq} \delta_{ki} - T_{ki})] \}
\]

\[
+ \partial_k\{\tau \rho R^2 \frac{1}{Pr} (T_{ki} \partial_i T_{ij} + T_{ji} \partial_i T_{jk} + T_{j}\partial_i T_{ki}) \}.
\]

The above equations have been written in a similar way as the Navier–Stokes equations. The differences between the above equations and the 10 moment closure are the additional terms appeared on the right hand sides of the momentum and energy equations. It is interesting to see that the corresponding heat conduction term derived above has the same form as that obtained by McDonald and Groth, even though they are obtained through different considerations. Based on the BGK-type collision model, a unit Prandtl number is obtained for the heat conduction term. However, since we believe that up to the NS order the structure of the gas dynamic equations will not be changed due to the BGK collision term or the exact
Boltzmann collision model, we add the Prandtl number in the above corresponding heat flux term. Since the viscosity and heat conduction coefficients are the concepts for the continuum flow, the particle collision time $\tau$ in the above equations is defined according to the result in the continuum regime,$$
abla = \frac{\mu}{(\rho RT_{eq})},$$where $\mu$ is the dynamical viscosity coefficient.

4. Conclusion

Based on the multiple stage particle collision BGK model and the Gaussian distribution function as the middle state, the generalized gas dynamic equations have been derived. In the non-equilibrium flow regime, the randomness of the particle distribution indeed depends on the spatial orientation. The new gas dynamic equations have the same structure as the Navier–Stokes equations, but the NS constitutive relationship,$$
abla = \rho RT_{ij} \delta_{ij} + \mu (\partial_i U_j + \partial_j U_i - \frac{2}{3} \partial_k U_k \delta_{ij}),$$is replaced by$$\nabla = -\rho RT_{ij} + \rho R(T_{eq} \delta_{ij} - T_{ij}).$$At the same time, the heat flux in the $k-$direction for the transport of thermal energy $\rho RT_{ij}$ becomes$$q_{kij} = \frac{\tau \rho R^2}{Pr} (T_{kl} \partial_l T_{ij} + T_{li} \partial_l T_{jk} + T_{lj} \partial_l T_{ki}).$$In the continuum flow regime, the generalized constitutive relationship and the heat flux term go back to the corresponding Navier–Stokes formulations. More detailed analysis of the above equations will be presented in.\textsuperscript{6}

References