Extra Examples for the Chapter on Differentiation

① If $\{x_n\}$ is Cauchy, and $x_n \ge 0$ for all n, then prove $\{\sqrt[5]{x_n}\}$ is Cauchy.

<u>Solution.</u> Since $\{x_n\}$ is Cauchy, for every $\varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $m, n \ge K \Rightarrow |x_m - x_n| < \varepsilon^5$.

Then $|\sqrt[5]{x_m} - \sqrt[5]{x_n}| \le \sqrt[5]{|x_m - x_n|} < \sqrt[5]{\varepsilon^5} = \varepsilon$.

$\therefore \{\sqrt[5]{x_n}\}$ is Cauchy.

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② $f : \mathbb{R} \to \mathbb{R}$ is three-times differentiable.

If $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$, then prove that $\sup\{f^{(3)}(x) : -1 < x < 1\} \ge 3$.

<u>Thoughts</u> $f^{(3)}, f', f$ suggest Taylor's theorem

c should be -1, 1 or 0, more likely $c = 0$.

<u>Solution</u> By Taylor's theorem, using $c = 0$,

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(\theta_x)}{6}(x-0)^3$$
$$= f(0) + \frac{f''(0)}{2}x^2 + \frac{f'''(\theta_x)}{6}x^3$$

for some θ_x between x and 0. Setting $x = 1, -1$,

$1 = f(1) = f(0) + \frac{f''(0)}{2} + \frac{f'''(\theta_1)}{6}$ for some $\theta_1 \in (0,1)$

$0 = f(-1) = f(0) + \frac{f''(0)}{2} - \frac{f'''(\theta_{-1})}{6}$ for some $\theta_{-1} \in (-1,0)$.

Subtracting yields $f'''(\theta_1) + f'''(\theta_{-1}) = 6$.

$\sup\{f^{(3)}(x) : -1 < x < 1\} \ge \max\{f'''(\theta_1), f'''(\theta_{-1})\} \ge 3$.

③ Let f be twice differentiable on $[0,2]$.

$\forall x \in [0,2]$, $|f(x)| \le 1$, $|f''(x)| \le 1$.

Prove that $\forall x \in [0,2]$, $|f'(x)| \le 2$.

<u>Solution</u> By Taylor's theorem, let $x \in [0,2]$, $a \in [0,2]$

$$f(a) = f(x) + f'(x)(a-x) + \frac{f''(\theta_a)}{2}(a-x)^2$$

for some θ_a between a and x. Setting $a = 0, 2$,

$f(0) = f(x) - f'(x)x + \frac{f''(\theta_0)}{2}x^2$ for some $\theta_0 \in (0,x)$.

$f(2) = f(x) + f'(x)(2-x) + \frac{f''(\theta_2)}{2}(2-x)^2$ for some $\theta_2 \in (x,2)$.

Subtracting these, we get

$$f(2) - f(0) = 2f'(x) + \frac{f''(\theta_2)}{2}(2-x)^2 - \frac{f''(\theta_0)}{2}x^2.$$

Solving for $f'(x)$, we see

$$|f'(x)| = \frac{1}{2}\left|f(2) - f(0) + \frac{f''(\theta_0)}{2}x^2 - \frac{f''(\theta_2)}{2}(2-x)^2\right|$$
$$\le \frac{1}{2}\left(1 + 1 + \frac{1}{2}x^2 + \frac{1}{2}(2-x)^2\right)$$
$$= \frac{1}{2}(x^2 - 2x + 4)$$
$$\le \frac{1}{2}((x-1)^2 + 3)$$
$$\le \frac{1}{2}(1+3) = 2$$

$\left\{\begin{array}{l} x \in [0,2] \\ |x-1| \le 1 \end{array}\right.$

④ Let $f:[1,2]$ be continuous and f be differentiable on $(1,2)$. Prove there exists $\theta\in(1,2)$ such that

$$f(2)-f(1)=\tfrac{1}{2}\theta^2 f'(\theta).$$

(Note mean value theorem only gives $f(2)-f(1) = f'(\theta_0)(2-1)= f'(\theta_0)$ for some $\theta_0\in(1,2)$.)

Solution Write

$$\theta^2 f'(\theta)=\frac{f'(\theta)}{1/\theta^2}=\frac{f'(\theta)}{g'(\theta)}, \text{ where } g(\theta)=-\frac{1}{\theta}.$$

By generalized mean value theorem,

$$\frac{f(2)-f(1)}{g(2)-g(1)}=\frac{f'(\theta)}{g'(\theta)} \text{ for some } \theta\in(1,2).$$

This is just

$$\frac{f(2)-f(1)}{-\frac{1}{2}-(-1)}=\frac{f'(\theta)}{1/\theta^2} \iff f(2)-f(1)=\tfrac{1}{2}\theta^2 f'(\theta).$$

⑤ Let $0<a<b$ and let $f:[a,b]\to\mathbb{R}$ be continuous with f differentiable on (a,b). Prove $\exists\,\theta\in(a,b)$ such that

$$\frac{1}{b-a}\begin{vmatrix} f(a) & f(b)\\ a & b\end{vmatrix}=f(\theta)-\theta f'(\theta).$$

Solution. Expanding the left side, we get

$$\frac{1}{b-a}\begin{vmatrix} f(a) & f(b)\\ a & b\end{vmatrix}=\frac{bf(a)-af(b)}{b-a}=\frac{f(a)/a-f(b)/b}{1/a-1/b}$$

dividing by ab in numerator + denominator.

This suggest we consider $F(x)=f(x)/x$, $G(x)=1/x$.

By the generalized mean value theorem,

$$\frac{f(a)/a-f(b)/b}{1/a-1/b}=\frac{F(a)-F(b)}{G(a)-G(b)}=\frac{F'(\theta)}{G'(\theta)} \text{ for some } \theta\in(a,b)$$

$$=\frac{\theta f'(\theta)-f(\theta)}{\theta^2}\Big/\left(-\frac{1}{\theta^2}\right)$$

$$=f(\theta)-\theta f'(\theta).$$

The result follows.

Landau's Big-Oh and Little-Oh Notations

Definitions Let $c \in \mathbb{R}$ or $c = +\infty$ or $c = -\infty$.
Let I be an interval containing c or with c as an endpoint.
Let $f(x)$ and $g(x)$ be functions on I, $g(x) \neq 0$ on $I \setminus \{c\}$

① We write $f(x) = O(g(x))$ iff $\exists A \in \mathbb{R}$ such that $\forall x \in I$, $|f(x)| \leq A\,|g(x)|$.
So $f(x) = b(x)g(x)$ for some $b(x)$ bounded on I $\quad b(x) = f(x)/g(x)$
Example, $(\sin x)\ln x = O(\ln x)$ on $I = (0, +\infty)$.

② We write $f(x) = o(g(x))$ as $x \to c$ iff $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = 0$.
So $f(x) = s(x)g(x)$ for some $s(x)$ with $\lim\limits_{x\to c} s(x) = 0$.
Example, $(\sin x)e^x = o(e^x)$ as $x \to 0$.

③ We write $f(x) = O^*(g(x))$ as $x \to c$ iff $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = k$ is a nonzero number.
So $f(x) = kg(x) + s(x)g(x)$ for some $s(x)$ with $\lim\limits_{x\to c} s(x) = 0$.
Example, $\sin 2x = O^*(x)$ as $x \to 0$.

④ We write $f(x) \sim g(x)$ as $x \to c$ iff $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = 1$.
Example, $\sin x \sim x$ as $x \to 0$.

Read "$\sim$" as "asymptotically equal to"
or "asymptotically equivalent to"

CAUTION In equations involving big-oh and little-oh notations, $A = B$ means "A is of type B", which does **not** imply "B is of type A".
Example For every $g(x)$, $g(x) = O^*(g(x))$ because $\lim\limits_{x\to c} \frac{g(x)}{g(x)} = 1 \neq 0$, but $O^*(g(x)) \neq g(x)$ since the $O^*(g(x))$ function on the left may be $2g(x)$ and $2g(x) \neq g(x)$ in general.

Remarks ① The phrase "as $x \to c$" may be omitted when the context is clear.
② For sequences a_n, b_n as $n \to \infty$, there are similar concepts and notations.
③ $f(x) = O(1)$ means $f(x)$ is bounded on I
$f(x) = o(1)$ means $\lim\limits_{x\to c} f(x) = 0$.
④ Extending notations, we write
$f(x) = h(x) + O(g(x))$ to mean $f(x) - h(x) = O(g(x))$
$f(x) = h(x) + o(g(x))$ to mean $f(x) - h(x) = o(g(x))$.

Basic Facts As $x \to c$,

① $O(g(x)) \pm O(g(x)) = O(g(x))$ (meaning if $f_1(x) = O(g(x))$ and $f_2(x) = O(g(x))$, then $f_1(x) \pm f_2(x) = O(g(x))$ since
$$\left.\begin{array}{l} |f_1(x)| \le A_1 |g(x)| \\ |f_2(x)| \le A_2 |g(x)| \end{array}\right\} \Rightarrow |f_1(x) \pm f_2(x)| \le (A_1 + A_2)|g(x)|\ )$$

② $o(g(x)) \pm o(g(x)) = o(g(x))$; $o(g(x)) \pm O^*(g(x)) = O^*(g(x))$

③ $O(g_1(x))O(g_2(x)) = O(g_1(x)g_2(x))$
$O^*(g_1(x))O^*(g_2(x)) = O^*(g_1(x)g_2(x))$
$$\left.\begin{array}{l} o(g_1(x))\,o(g_2(x)) \\ o(g_1(x))O^*(g_2(x)) \\ o(g_1(x))O(g_2(x)) \\ o(g_1(x))\,g_2(x) \end{array}\right\} = o(g_1(x)g_2(x))$$

$\forall p > 0$, $O(g(x))^p = O(g(x)^p)$

$\forall r \in \mathbb{R}$, $O^*(g(x))^r = O^*(g(x)^r)$

④ $o(o(g(x))) = o(g(x))$; $O^*(o(g(x))) = o(g(x))$
$o(O^*(g(x))) = o(g(x))$

⑤ $a < b \Rightarrow O^*(x^a) \pm O^*(x^b) = O^*(x^a)$ as $x \to 0$
$o(x^n) \pm o(x^m) = o(x^{\min(n,m)})$ as $x \to 0$

$$\lim_{x\to 0} \frac{1-e^x}{-x} = \lim_{x\to 0} \frac{-e^x}{-1} = 1$$
$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$
$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1/(1+x)}{1} = 1$$

Remainder of Taylor Expansion

$$R_n(x) = \frac{f^{(n)}(\xi)}{n!}(x-c)^n = \begin{cases} O((x-c)^n) & \text{if } f^{(n)}(x) \text{ is bounded on } I \\ o((x-c)^{n-1}) & \text{as } x \to c \\ O^*((x-c)^n) & \text{as } x \to c \end{cases} \left.\vphantom{\begin{cases}a\\b\end{cases}}\right\} \begin{array}{l} f^{(n)}(x) \text{ contin. at } x=c \\ \text{if } f^{(n)}(c) \ne 0. \end{array}$$

Taylor Expansions at $c=0$ $\boxed{\text{As } x \to 0,}$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_{n+1}(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + R_{n+1}(x)$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + R_{2n+2}(x)$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+3}(x)$$
$$(1+x)^a = 1 + \sum_{k=1}^{n} \frac{a(a-1)\cdots(a-k+1)}{k!} x^k + R_{n+1}(x)$$
$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots + \frac{(-1)^{n-1} x^n}{n!} + R_{n+1}(x)$$
$$\operatorname{Arctan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + R_{2n+3}(x)$$
$$\operatorname{Arcsin} x = x + \sum_{k=1}^{n} \underbrace{\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)}}_{\frac{(2k-1)!!}{(2k)!!}} \frac{x^{2k+1}}{2k+1} + R_{2n+3}(x)$$

Notation:
$$m!! = \begin{cases} 1\cdot 3\cdots m & \text{if } m \text{ is odd} \\ 2\cdot 4\cdots m & \text{if } m \text{ is even} \end{cases}$$

Remarks We have $1-e^x \sim -x$ as $x \to 0$, also note $\sin x \sim x$, $\ln(1+x) \sim x$, $\tan x \sim x$, $\operatorname{Arctan} x \sim x$, $\operatorname{Arcsin} x \sim x$ as $x \to 0$. These are often useful.

Examples

① Consider the convergence of $\sum_{n=1}^{\infty}\left(\sin\frac{1}{n} - \text{Arctan}\left(\frac{1}{n}\right)\right)$

<u>Solution</u> Let $x = \frac{1}{n} \in [0,1] = I$. We have

$$\sin x = x - \frac{1}{6}x^3 + O(x^5)$$

$$\text{Arctan}\, x = x - \frac{1}{3}x^3 + O(x^5)$$

$$\sin x - \text{Arctan}\, x = \frac{1}{6}x^3 + \underbrace{O(x^5) - O(x^5)}_{\overset{\shortparallel}{O}(x^5)}$$

$$= O(x^3) + \overset{\shortparallel}{O}(x^5) \quad \text{by Fact ①}$$

$$= O(x^3) \qquad \overset{\shortparallel}{O}(x^3) \quad \text{by box on left}$$

$\boxed{\text{On } I = (0,1] \quad O(x^5) \le C|x^5| \le C|x^3|}$

$$\sum_{n=1}^{\infty}\left|\sin\left(\frac{1}{n}\right) - \text{Arctan}\left(\frac{1}{n}\right)\right| = \sum_{n=1}^{\infty} O\left(\frac{1}{n^3}\right) \le \sum_{n=1}^{\infty} A\frac{1}{n^3} < \infty$$

$\therefore \sum_{n=1}^{\infty}\left(\sin\left(\frac{1}{n}\right) - \text{Arctan}\left(\frac{1}{n}\right)\right)$ Converges absolutely.

For later solution, we record the fact in the box above

<u>Fact 6</u> On $I = [-1,1]$, if $0 \le a < b$, then

$$O(x^a) + O(x^b) = O(x^a).$$

<u>Proof</u> $f(x) = O(x^b) \Leftrightarrow |f(x)| \le C|x^b|$

On $I = [0,1]$ $\quad \Rightarrow |f(x)| \le C|x^a|$

$|x| \le 1 \Rightarrow |x^b| \le |x^a| \quad \Leftrightarrow f(x) = O(x^a)$

by fact 1.

So $O(x^a) + O(x^b) = O(x^a) + O(x^a) = O(x^a)$

② Let $p \in \mathbb{R}$ and $a_n = \left(e - \left(1+\frac{1}{n}\right)^n\right)^p$. For which p, will $\sum_{n=1}^{\infty} a_n$ converges ?

<u>Solution</u> As $n \to \infty$, $x = \frac{1}{n} \to 0$, $O^*\left(\frac{1}{n}\right) = \frac{1}{n}\,\frac{O^*\left(\frac{1}{n}\right)}{\frac{1}{n}}$, $0 \times K = 0$

① $\ln(1+x) = x + O^*(x^2)$

② $1 - e^x \sim -x$ since $\lim_{x\to 0}\frac{1-e^x}{-x} = 1$.

$$\left(1+\frac{1}{n}\right)^n = e^{n\ln\left(1+\frac{1}{n}\right)} = e^{n\left(\frac{1}{n} + O^*\left(\frac{1}{n^2}\right)\right)} \text{ by ①}$$

$$e - \left(1+\frac{1}{n}\right)^n = e - e^{1 + O^*\left(\frac{1}{n}\right)} = e\left(1 - e^{O^*\left(\frac{1}{n}\right)}\right)$$

$$\underset{\text{by②}}{\sim} e\left(-O^*\left(\frac{1}{n}\right)\right) = O^*\left(\frac{1}{n}\right)$$

Fact ③

$$a_n = \left(e - \left(1+\frac{1}{n}\right)^n\right)^p \sim O^*\left(\frac{1}{n}\right)^p = O^*\left(\frac{1}{n^p}\right) = b_n \qquad \frac{b_n}{\frac{1}{n^p}} \to K' \ne 0$$

$$a_n \sim b_n \Leftrightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = 1 \Rightarrow \lim_{n\to\infty}\frac{a_n}{1/n^p} = K' \ne 0$$

$\underset{\text{by limit comparison test}}{\Rightarrow}$ $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty}\frac{1}{n^p}$ both converges or both diverges

$\underset{\text{by p-test}}{\Rightarrow}$ $\sum_{n=1}^{\infty} a_n$ Converges iff $p > 1$

③ $\lim\limits_{x\to\infty}\left(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5}\right)$

$= \lim\limits_{x\to\infty} x\left(\sqrt[6]{1+\frac{1}{x}}-\sqrt[6]{1-\frac{1}{x}}\right)$

$= \lim\limits_{x\to\infty} x\left(\left(1+\frac{1}{x}\right)^{1/6}-\left(1-\frac{1}{x}\right)^{1/6}\right)$

As $x\to\infty$, $w=\frac{1}{x}\to 0$, $(1+w)^a = 1+aw+o(w)$

$= \lim\limits_{x\to\infty} x\left(1+\frac{1}{6x}+o\left(\frac{1}{x}\right)-1+\frac{1}{6x}+o\left(\frac{1}{x}\right)\right)$, by fact ③

$x\,o\left(\frac{1}{x}\right)=o(1)$

$= \lim\limits_{x\to\infty}\left(\frac{1}{3}+o(1)\right)=\frac{1}{3}$

④ Find $\lim\limits_{x\to 0}\dfrac{\cos(\sin x)-\cos x}{\sin^4 x}$.

As $x\to 0$, $\cos x = 1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^4)$

$\sin x = x-\frac{1}{6}x^3+o(x^3)$.

$$\cos(\sin x) = 1-\frac{1}{2}\sin^2 x+\frac{1}{24}\sin^4 x+\underbrace{o(\sin^4 x)}_{\substack{=o(O^*(x^4))\\ =o(x^4)\text{ by fact ④}}}$$

$$= 1-\frac{1}{2}\left(x-\frac{1}{6}x^3+o(x^3)\right)^2 + \frac{1}{24}\left(x-\frac{1}{6}x^3+o(x^3)\right)^4+o(x^4)$$

$$= 1-\frac{1}{2}\left(x^2-\frac{1}{3}x^4+o(x^4)\right)+\frac{1}{24}\left(x^4+o(x^4)\right)+o(x^4)$$

$$= 1-\frac{1}{2}x^2+\frac{5}{24}x^4+o(x^4)$$

So $\cos(\sin x)-\cos x = \frac{1}{6}x^4+o(x^4)$

$$\frac{\cos(\sin x)-\cos x}{\sin^4 x} = \frac{\frac{1}{6}x^4+o(x^4)}{x^4}\,\frac{x^4}{\sin^4 x}\to\left(\frac{1}{6}+0\right)1=\frac{1}{6}.$$

Proof of facts ①, ②, ③ follow easily from the definitons of big-oh, little-oh notations. We will leave them as exercises later.

Proofs of Facts ④ and ⑤

$o(o(g(x)))=o(g(x))$ Let $f(x)=o(o(g(x)))$, then $f(x)=o(h(x))$, where $h(x)=o(g(x))$. So we have

$$\lim_{x\to c}\frac{f(x)}{h(x)}=0,\ \lim_{x\to c}\frac{h(x)}{g(x)}=0 \Rightarrow \lim_{x\to c}\frac{f(x)}{g(x)}=0 \Leftrightarrow f(x)=o(g(x))$$

$O^*(o(g(x)))=o(g(x))$ Let $f(x)=O^*(o(g(x)))$, then $f(x)=O^*(h(x))$, where $h(x)=o(g(x))$. So we have

$$\lim_{x\to c}\frac{f(x)}{h(x)}=k\neq 0,\ \lim_{x\to c}\frac{h(x)}{g(x)}=0 \Rightarrow \lim_{x\to c}\frac{f(x)}{g(x)}=0 \Leftrightarrow f(x)=o(g(x)).$$

$o(O^*(g(x)))=o(g(x))$ Let $f(x)=o(O^*(g(x)))$, then $f(x)=o(h(x))$, where $h(x)=O^*(g(x))$. So we have

$$\lim_{x\to c}\frac{f(x)}{h(x)}=0,\ \lim_{x\to c}\frac{h(x)}{g(x)}=k\neq 0 \Rightarrow \lim_{x\to c}\frac{f(x)}{g(x)}=0 \Leftrightarrow f(x)=o(g(x)).$$

$a<b \Rightarrow O^*(x^a) \pm O^*(x^b) = O^*(x^a)$ as $x \to 0$

Let $f_1(x) = O^*(x^a)$, $f_2(x) = O^*(x^b)$, then we have

$$\lim_{x\to 0} \frac{f_1(x)}{x^a} = k_1^{\neq 0}, \quad \lim_{x\to 0} \frac{f_2(x)}{x^b} = k_2^{\neq 0}$$

$b-a>0$ since $a<b$

$$\Rightarrow \lim_{x\to 0} \frac{f_1(x) \pm f_2(x)}{x^a} = \lim_{x\to 0}\left(\frac{f_1(x)}{x^a} \pm x^{b-a}\frac{f_2(x)}{x^b}\right) = k_1 \pm 0 \cdot k_2 = k_1 \neq 0$$

$$\Leftrightarrow f_1(x) + f_2(x) = O^*(x^a)$$

$o(x^n) \pm o(x^m) = o(x^{\min(n,m)})$ as $x \to 0$

Let $f_1(x) = o(x^n)$, $f_2(x) = o(x^m)$, then we have

$$\lim_{x\to 0} \frac{f_1(x)}{x^n} = 0, \quad \lim_{x\to 0} \frac{f_2(x)}{x^m} = 0.$$

Suppose $\min(n,m) = n$.

$m-n \geq 0$

Then $$\lim_{x\to 0} \frac{f_1(x) \pm f_2(x)}{x^n} = \lim_{x\to 0}\left(\frac{f_1(x)}{x^n} \pm x^{m-n}\frac{f_2(x)}{x^m}\right) = 0 \pm 0 = 0$$

$$\Leftrightarrow f_1(x) + f_2(x) = o(x^n).$$

Related to fact 4, we also have

$O(o(g(x))) = o(g(x))$ Let $f(x) = O(o(g(x)))$, then $f(x) = O(h(x))$, where $h(x) = o(g(x))$. So we have

$$|f(x)| \leq A|h(x)|, \quad \lim_{x\to c}\frac{h(x)}{g(x)} = 0 \Rightarrow \left|\frac{f(x)}{g(x)}\right| \leq A\left|\frac{h(x)}{g(x)}\right| \to 0 \text{ as } x\to c$$

$$\Rightarrow \lim_{x\to c}\frac{f(x)}{g(x)} = 0 \Leftrightarrow f(x) = o(g(x)).$$

$o(O(g(x))) = o(g(x))$ Let $f(x) = o(O(g(x)))$, then $f(x) = o(h(x))$, where $h(x) = O(g(x))$. So we have

$$\lim_{x\to c}\frac{f(x)}{h(x)} = 0, \quad |h(x)| \leq A|g(x)| \Rightarrow \left|\frac{f(x)}{g(x)}\right| \leq A\left|\frac{f(x)}{h(x)}\right| \to 0 \text{ as } x\to c.$$

$$\Rightarrow \lim_{x\to c}\frac{f(x)}{g(x)} = 0 \Leftrightarrow f(x) = o(g(x)).$$

Stolz' Theorem

Let $b_1, b_2, b_3, \cdots$ be a strictly monotone sequence.

If either $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$

or $\lim_{n\to\infty} b_n = \pm\infty$,

then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$

provided the right side exists as a number or $\pm\infty$.

Examples ① Find $\lim_{n\to\infty} \frac{1^p+2^p+\cdots+n^p}{n^{p+1}}$, where p is a positive integer.

$a_n = 1^p+2^p+\cdots+n^p$, $b_n = n^{p+1} \nearrow +\infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}} = \lim_{n\to\infty} \frac{(n+1)^p}{(p+1)n^p+o(n^p)}$$

$$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})^p}{p+1+o(1)} = \frac{1}{p+1}. \quad \therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{p+1}.$$

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② If $\lim_{n\to\infty} x_n = x$, then prove that $\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = x$.

(Here x is a number or $\pm\infty$).

$a_n = x_1+x_2+\cdots+x_n$, $b_n = n \nearrow +\infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{x_{n+1}}{1} = x. \quad \therefore \lim_{n\to\infty} \frac{a_n}{b_n} = x.$$

---

③ Let $a_n = 1-2+3-4+\cdots+(-1)^{n-1}n$ and $b_n = n^2 \nearrow +\infty$.

Then $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{(-1)^n(n+1)}{2n+1}$ doesn't exist, but $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

since $|a_n| \le \frac{n+1}{2}$.

④ Let $0 < x_1 < 1$ and $x_{n+1} = x_n(1-x_n)$ for $n=1,2,3,\cdots$

Prove that $x_1, x_2, x_3, \cdots$ decreases to 0 and $\lim_{n\to\infty} n x_n = 1$.

We will show $0 < x_n < 1$ and $x_{n+1} < x_n$ by math induction. We are given $0 < x_1 < 1$. Now

$0 < x_n < 1 \Rightarrow 0 < 1-x_n < 1 \Rightarrow 0 < x_{n+1} = x_n(1-x_n) < 1$,

completing induction step for $0 < x_n < 1$.

Next $x_{n+1} = x_n - x_n^2 < x_n$.

By monotone sequence theorem, $\lim_{n\to\infty} x_n = x$ exists.

$x_{n+1} = x_n(1-x_n) \Rightarrow x = x(1-x) = x - x^2 \Rightarrow x = 0$.

By Stolz' theorem, since $1/x_n \nearrow +\infty$,

$$\lim_{n\to\infty} n x_n = \lim_{n\to\infty} \frac{n}{1/x_n} = \lim_{n\to\infty} \frac{1}{1/x_{n+1} - 1/x_n}$$

$$= \lim_{n\to\infty} \frac{x_{n+1}x_n}{x_n - x_{n+1}} = \lim_{n\to\infty} \frac{x_n(1-x_n)x_n}{x_n^2}$$

$$= \lim_{n\to\infty} (1-x_n) = 1.$$

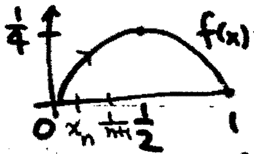
Remarks For example ④, alternatively we can show $\lim_{n\to\infty} nx_n = 1$ by the $\frac{0}{0}$ version of Stolz' Theorem as follow. As before, we have $x_n \searrow 0$.

First we show nx_n is increasing and bounded above.

Note
$$(n+1)x_{n+1} - nx_n = (n+1)x_n(1-x_n) - nx_n$$
$$= x_n\big((n+1)(1-x_n) - n\big)$$
$$= x_n\big(1-(n+1)x_n\big) \ge 0$$
$$\iff 1-(n+1)x_n \ge 0 \iff x_n \le \frac{1}{n+1}$$

Next, we show $x_n \le \frac{1}{n+1}$ for $n>1$ by math induction.

$$x_2 = x_1(1-x_1) = \frac14 - \left(x_1-\frac12\right)^2 \le \frac14 < \frac13$$



Suppose $x_n \le \frac{1}{n+1} < \frac12$.

Then
$$x_{n+1} = x_n(1-x_n) \le \frac{1}{n+1}\left(1-\frac{1}{n+1}\right)$$
$$\le \frac{1}{n+1}\left(1-\frac{1}{n+2}\right) = \frac{1}{n+1}\,\frac{n+1}{n+2}$$
$$= \frac{1}{n+2}.$$

$f(x)=x(1-x)$ is increasing on $[0,\frac12]$

Now $x_n \le \frac{1}{n+1} \Rightarrow nx_n \le (n+1)x_n < 1$. So $\lim_{n\to\infty} nx_n$ exists.

Let $L = \lim_{n\to\infty} nx_n$. Then recall $x_n \searrow 0$. So

$$L = \lim_{n\to\infty} nx_n = \lim_{n\to\infty} \frac{x_n}{1/n} \overset{\frac{0}{0}}{=} \lim_{n\to\infty} \frac{x_{n+1}-x_n}{1/(n+1) - 1/n}$$
$$= \lim_{n\to\infty} -n(n+1)(x_{n+1}-x_n) = \lim_{n\to\infty} n(n+1)x_n^2$$
$$= \lim_{n\to\infty} (nx_n)^2\,\frac{n+1}{n} = L^2 \Rightarrow L = 1.$$

Proof of Stolz' Theorem Suffices to consider $\underbrace{b_n < b_{n+1}}_{(*)}$ for $n=1,2,3,\dots$. Let $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L$.

First, suppose $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$. Then $b_n \nearrow 0$.

Case $L \in \mathbb{R}$

① $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ n \ge K \Rightarrow L-\varepsilon < \frac{a_{n+1}-a_n}{b_{n+1}-b_n} < L+\varepsilon$.

② by (*), $b_{n+1}-b_n > 0$
$$\Longrightarrow (L-\varepsilon)(b_{n+1}-b_n) < a_{n+1}-a_n < (L+\varepsilon)(b_{n+1}-b_n)$$

③
$$\Rightarrow (L-\varepsilon)\underbrace{\sum_{j=n}^{\infty}(b_{j+1}-b_j)}_{=0-b_n} < \underbrace{\sum_{j=n}^{\infty}(a_{j+1}-a_j)}_{=0-a_n} < (L+\varepsilon)\underbrace{\sum_{j=n}^{\infty}(b_{j+1}-b_j)}_{=0-b_n}$$

$b_n \nearrow 0 \Rightarrow b_n < 0$

④ $-b_n > 0$
$$\Longrightarrow L-\varepsilon < \frac{a_n}{b_n} < L+\varepsilon. \quad \therefore \lim_{n\to\infty}\frac{a_n}{b_n} = L.$$

Case $L = +\infty$ $\forall r \in \mathbb{R}\ \exists K,\ n \ge K \Rightarrow r < \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$

Follow steps ② to ④ above to get $r < \frac{a_n}{b_n}$.

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = +\infty = L$.

Case $L = -\infty$ $\forall r \in \mathbb{R}\ \exists K,\ n \ge K \Rightarrow \frac{a_{n+1}-a_n}{b_{n+1}-b_n} < r$.

Again follow steps ② to ④ above to get $\frac{a_n}{b_n} < r$.

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = -\infty = L$.

Next, suppose $\lim_{n\to\infty} b_n = +\infty$ (Since $b_n \nearrow$, $-\infty$ is not possible).

Case $L \in \mathbb{R}$

Modify step ① with $\frac{\varepsilon}{2}$ replacing ε.

Modify step ③ with $\left\{\begin{matrix} m \text{ replacing } \infty \\ K \text{ replacing } n\end{matrix}\right\}$ to get

$$\left(L-\frac{\varepsilon}{2}\right)(b_m-b_K) < a_m - a_K < \left(L+\frac{\varepsilon}{2}\right)(b_m-b_K)$$

$$\Rightarrow \left|\frac{a_m-a_K}{b_m-b_K} - L\right| < \frac{\varepsilon}{2}.$$

By expansion, we can check that

$$\frac{a_m}{b_m} - L = \frac{a_K - Lb_K}{b_m} + \underbrace{\left(1-\frac{b_K}{b_m}\right)}_{<1 \text{ as } b_n\nearrow+\infty}\overbrace{\left(\frac{a_m-a_K}{b_m-b_K}-L\right)}^{\text{less than } \varepsilon/2}$$

doesn't depend on m

As $b_m \nearrow +\infty$, $\exists M$, $m\ge M \Rightarrow \frac{2}{\varepsilon}|a_K - Lb_K| < b_m$.

Then $\left|\frac{a_m}{b_m} - L\right| < \frac{\varepsilon}{2} + 1\cdot\frac{\varepsilon}{2} = \varepsilon$. $\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = L$.

Case $L=+\infty$ $\quad \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \to +\infty \Rightarrow \frac{a_{n+1}-a_n}{b_{n+1}-b_n} > 1$ for n large

$\Rightarrow a_{n+1}-a_n > b_{n+1}-b_n > 0$

$a_n - a_1 > b_n - b_1 \to \infty$ $\quad a_n > a_1 + b_n - b_1 \to \infty$

$\Rightarrow a_n \nearrow +\infty$ since $\lim_{n\to\infty} b_n - b_1 = \sum_{n=1}^{\infty} (b_{n+1}-b_n)$ is $+\infty$.

Now $\lim_{n\to\infty} \frac{b_{n+1}-b_n}{a_{n+1}-a_n} = 0 \Rightarrow \lim_{n\to\infty} \frac{b_n}{a_n} = 0$ by above $\therefore$

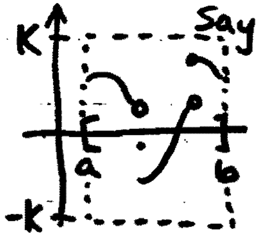
$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$.

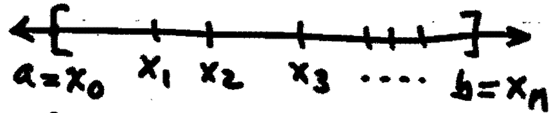
Case $L=-\infty$ is similar to Case $L=+\infty$.

Chapter 9 Riemann Integral

Proper Integral

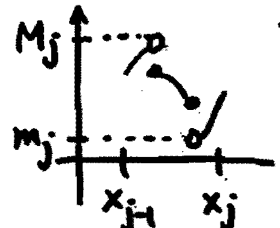
Setting: Let $f(x)$ (may be discontinuous) be a **bounded** function on a **closed and bounded** interval $[a,b]$, say $\exists K>0$, $\forall x\in[a,b]$, $|f(x)|\le K$.





$P=\{x_0,x_1,\cdots,x_n\}$ is a **partition** of $[a,b]$ iff $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$.

Let $\Delta x_j = x_j - x_{j-1}$. $\|P\| = \max\{\Delta x_1,\cdots,\Delta x_n\}$ is called the **mesh** of P.

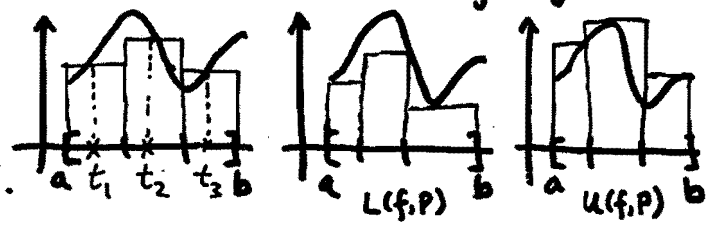


Since $f(x)$ may be discontinuous on $[a,b]$, we consider

$$m_j = \inf\{f(x) : x\in[x_{j-1},x_j]\}$$
$$M_j = \sup\{f(x) : x\in[x_{j-1},x_j]\}.$$

Definitions

Let $t_j\in[x_{j-1},x_j]$ for $j=1,2,\cdots,n$.



$S=\sum_{j=1}^{n} f(t_j)\Delta x_j$ is a **Riemann sum** with respect to P.

$L(f,P)=\sum_{j=1}^{n} m_j\Delta x_j$ is the **lower Riemann sum** w.r.t. P.

$U(f,P)=\sum_{j=1}^{n} M_j\Delta x_j$ is the **upper Riemann sum** w.r.t. P.