Math 301 (Real Analysis)  

Midterm (Version 1)  

Directions: This is a closed book exam. Work must be shown to receive credits. Answers alone are worth very little.

Notations: $\mathbb{R}$ denotes the set of all real numbers.

1. (5 marks) State the definition of a function $F: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$.

(15 marks) Define the function $F: \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x, y) = \begin{cases}  
  x + \cos x & \text{if } y = x \\
  x + 1 & \text{if } y \neq x 
\end{cases}$$

Determine with proof if $F$ is differentiable at $(0, 0)$ or not.

2. (25 marks) Consider the system of equations

$$\cos(uv) + uw = 0 \quad \text{and} \quad ue^{uw} + u^5w = 0.$$ 

Show that near $p = (u, v, w) = (1, 0, -1)$, $(v, w)$ can be expressed as a differentiable function of $u$ and find the value of $\frac{dv}{du}$ and $\frac{dw}{du}$ at $p$.

3. (20 marks) Prove that

$$\int_0^1 \sqrt{x} \cos(\sqrt{x}) \, dx = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)! (2k + 3)}.$$ 

4. (20 marks) If $a$ and $b$ are real numbers with $a < b$, then prove that $\sum_{k=1}^{\infty} \frac{(-1)^k \sin \frac{x}{k}}{k}$ converges uniformly on $(a, b)$.

5. (15 marks) Let $\{b_n\}$ be a sequence of real numbers such that for all positive integers $m$ and $n$, we have $b_{m+n} \leq b_m + b_n$. Prove that for every $m$, $\limsup_{k \to \infty} \frac{b_k}{k} \leq \frac{b_m}{m}$ and $\lim_{k \to \infty} \frac{b_k}{k}$ exists in $[-\infty, \infty]$.

Math 301 (Real Analysis)  

Midterm (Version 2)  

Directions: This is a closed book exam. Work must be shown to receive credits. Answers alone are worth very little.

Notations: $\mathbb{R}$ denotes the set of all real numbers.

1. (5 marks) State the definition of a function $F: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$.

(15 marks) Define the function $F: \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x, y) = \begin{cases}  
  x + \sin y & \text{if } y = x \\
  x + y & \text{if } y \neq x 
\end{cases}$$

Determine with proof if $F$ is differentiable at $(0, 0)$ or not.
2. (25 marks) Consider the system of equations
\[ x = v \cos u + \sin v \quad \text{and} \quad y = \sin u + v \cos u. \]
Show that near \( p = (u, v, x, y) = (0, 0, 0, 0) \), \((u, v)\) can be expressed as a differentiable function of \((x, y)\) and find the value of \( \frac{\partial u}{\partial y} \) and \( \frac{\partial v}{\partial x} \) at \( p \).

3. (20 marks) Prove that
\[\int_0^1 \sqrt{x} \ln(1 + x) \, dx = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(2k + 1)} \cdot \sum_{k=0}^{\infty} c_k x^k \text{ converges pointwise on } [0,1) \to \text{ a bounded function } f(x). \]

4. (15 marks) Let \( \{c_n\} \) be a sequence of nonnegative real numbers. If \( \sum_{k=0}^{\infty} c_k x^k \) converges pointwise on \([0,1)\) to a bounded function \( f(x) \). Prove that \( \sum_{k=0}^{\infty} c_k \) converges.

5. (20 marks) Let \( \{a_n\} \) be a sequence of positive real numbers. Prove that
\[ \limsup_{n \to \infty} \left( \frac{n(a_1 + a_{n+1})}{(n + 1)a_n} \right)^n < 1 \text{ is impossible!} \]

Math 301 (Real Analysis) Fall 2004
Midterm for Lecture 1

Directions: This is a closed book exam. No calculators are allowed. Work must be shown to receive credits. Answers alone are worth very little.

Notation: \( \mathbb{R} \) denotes the real numbers.

1. (20 marks) Define the function \( F : \mathbb{R}^2 \to \mathbb{R} \) by
\[ F(x, y) = \begin{cases} \frac{(x^4 + 2y^4)^{1/4}}{\sqrt{x^4 + y^4}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}. \]
Determine with proof if \( F \) is differentiable at \((0,0)\) or not.

2. (20 marks) Consider the system of equations
\[ x = ue^v + ve^u \quad \text{and} \quad y = ue^u - ve^v. \]
Show that near \((u, v, x, y) = (0, 0, 0, 0)\), \((u, v)\) can be expressed as a differentiable function of \((x, y)\) and find the values of \( \frac{\partial u}{\partial x} (0,0) \) and \( \frac{\partial v}{\partial x} (0,0) \).

3. (20 marks) Prove that
\[ \int_0^{1/2} \frac{\text{Arctan} \, 2x - \text{Arctan} \, x}{x} \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left( 1 - \frac{1}{2^{2k+1}} \right). \]
4. (a) (5 marks) State the definition of a sequence of functions \( S_n : E \to \mathbb{R} \) converging uniformly on \( E \) to a function \( S : E \to \mathbb{R} \).

(b) (15 marks) If \( \{ P_n(x) \} \) is a sequence of polynomials with real coefficients converging uniformly on \( \mathbb{R} \) to a function \( P(x) \), then prove that \( P(x) \) is a polynomial.

5. (20 marks) Let \( a_k \geq 0 \) for \( k = 1, 2, 3, \ldots \) and \( \sum_{k=1}^{\infty} a_k \) converges. Prove that \( \limsup_{k \to \infty} k a_k = 0 \). Give an example to show that in the above situation, it is possible that \( \limsup_{k \to \infty} k a_k = 1 \).

Math 301 (Real Analysis) Fall 2004

Midterm for Lecture 2

Directions: This is a closed book exam. No calculators are allowed. Work must be shown to receive credits. Answers alone are worth very little.

Notation: \( \mathbb{R} \) denotes the real numbers.

1. (20 marks) Define the function \( F : \mathbb{R}^2 \to \mathbb{R} \) by

\[
F(x, y) = \begin{cases} 
xy(1 + \cos(xy))/\sqrt{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0) 
\end{cases}
\]

Determine with proof if \( F \) is differentiable at \((0, 0)\) or not.

2. (20 marks) Consider the system of equations

\[
x = u \cos(uv) + v \sin(uv) \quad \text{and} \quad y = u \sin(uv) - v \cos(uv).
\]

Show that near \((u, v, x, y) = (0, 0, 0, 0)\), \((u, v)\) can be expressed as a differentiable function of \((x, y)\) and find the values of \( \frac{\partial u}{\partial x}(0, 0) \) and \( \frac{\partial v}{\partial x}(0, 0) \).

3. (20 marks) Prove that

\[
\int_0^1 \sin(e^x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k (e^{2k+1} - 1)}{(2k + 1)!(2k + 1)}.
\]

4. (a) (5 marks) State the definition of a sequence of functions \( S_n : E \to \mathbb{R} \) converging uniformly on \( E \) to a function \( S : E \to \mathbb{R} \).

(b) (15 marks) If \( \{ P_n(x) \} \) is a sequence of polynomials with real coefficients converging uniformly on \( \mathbb{R} \) to a function \( P(x) \), then prove that \( P(x) \) is a polynomial.

5. (20 marks) Let \( a_k \geq 0 \) for \( k = 1, 2, 3, \ldots \) and \( \limsup_{k \to \infty} k^{1/\ln k} a_k < \frac{1}{e} \). Prove that \( \sum_{k=1}^{\infty} a_k \) converges.
Directions: This is a closed book exam. No notes are allowed. Work must be shown to receive credits. Answers alone worth very little. No talking.

Notation: \( \mathbb{R} \) denotes the real numbers.

1. (20 marks) Define the function \( F : \mathbb{R}^2 \to \mathbb{R} \) by
   \[
   F(x, y) = \begin{cases} 
   xy + x + y & \text{if } xy \neq 0 \\
   x + y & \text{if } xy = 0.
   \end{cases}
   \]
   Prove that \( F \) is differentiable at \((0, 0)\) and write down \( DF(0, 0) \).

2. (20 marks) Consider the system of equations
   \[
   x = e^{uv} - 2 \sin v - 1 \quad \text{and} \quad y = u + \sin(uv).
   \]
   Show that near \((u, v, x, y) = (0, 0, 0, 0)\), \((u, v)\) can be expressed as a differentiable function of \((x, y)\) and find the values of \( \frac{\partial v}{\partial x}(0, 0) \) and \( \frac{\partial u}{\partial y}(0, 0) \).

3. (20 marks) Prove that
   \[
   \int_1^2 \frac{dx}{e^x - 1} = \sum_{k=1}^{\infty} \frac{e^{-k} - e^{-2k}}{k}.
   \]

4. (20 marks) Define \( S_n(x) = e^{-n^2x^2} \) for \( x \in \mathbb{R} \). Prove that \( S_n(x) \) converges uniformly on \( \mathbb{R} \), \( S'_n(x) \) converges pointwise on \( \mathbb{R} \) and \( S'_n(x) \) does not converge uniformly on any open interval containing 0.

5. (20 marks) Let \( \{a_n\} \) be a sequence of real numbers. Show that
   \[
   \limsup_{n \to \infty} \frac{a_n}{n^2} \leq \limsup_{n \to \infty} \frac{a_{n+1} - a_n}{2n + 1}.
   \]
   (Hint: Consider \( r \) greater than the right side.)
2. (20 marks) Consider the system of equations

\[ x = ue^v + ve^u \quad \text{and} \quad y^2 = v - u. \]

Show that near \((u, v, x, y) = (0, 0, 0, 0)\), \((u, v)\) can be expressed as a differentiable function of \((x, y)\) and find the values of \(\frac{\partial u}{\partial x}(0, 0)\) and \(\frac{\partial v}{\partial x}(0, 0)\).

3. (20 marks) Define \(S_n(x) = \frac{1}{1 + n^2 x^2}\) for \(x \in [0, 1]\). Prove that \(S_n(x)\) converges pointwise on \([0, 1]\) and

\[ \int_0^1 \lim_{n \to \infty} S_n(x) \, dx = \lim_{n \to \infty} \int_0^1 S_n(x) \, dx. \]

Determine if \(S_n(x)\) converge uniformly on \([0, 1]\) or not. Provide reasons.

4. (20 marks) Find \(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)! (2k + 2)}\) by considering \(f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k)!(2k+2)}\). Show work.

5. (20 marks) Find \(\lim\sup_{k \to \infty} \sqrt[k]{2^k + \frac{k^4}{k!}}\). Show work. (Hint: Take reciprocal.)

Math 301 (Real Analysis) Fall 2002

Midterm for Lecture 1

Directions: This is a closed book exam. No notes are allowed. Work must be shown to receive credits. Answers alone worth very little. No talking. You may use nonprogrammable calculators.

Notation: \(\mathbb{R}\) denotes the real numbers.

1. (20 marks) Consider the system of equations

\[ \sin x + 2 \sin y + 3 \sin z = 0 \quad \text{and} \quad x \cos y + y \cos z + z \cos x = 0. \]

Show that near \((x, y, z) = (0, 0, 0)\), \((x, y)\) can be expressed as a differentiable function of \(z\) and find the value of \(\frac{dx}{dz}(0)\) and \(\frac{dy}{dz}(0)\).

2. (15 marks) Define \(S_n(x) = \sin \left( \frac{2x}{x^2 + n^2} \right)\) for \(x \in [0, +\infty)\). Explain whether the sequence of functions \(S_n(x)\) converges pointwise on \([0, +\infty)\) or not. Explain whether \(S_n(x)\) converges uniformly on \([0, +\infty)\) or not.

3. (20 marks) Show that \(\int_0^1 \cos(e^x) \, dx = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (e^{2k} - 1)}{(2k)!(2k)}\).

4. (a) (5 marks) Give the definition of a function \(F : \mathbb{R}^n \to \mathbb{R}^m\) differentiable at \(a \in \mathbb{R}^n\).
(b) (5 marks) Give the definition of a function \( F : \mathbb{R}^n \to \mathbb{R}^m \) continuously differentiable (or \( C^1 \)) at \( a \in \mathbb{R}^n \).

(c) (15 marks) Explain whether the function \( F : \mathbb{R}^2 \to \mathbb{R} \) defined by \( F(x, y) = (\sin \sqrt{x^2 + y^2})^2 \) is \( C^1 \) at \((0, 0)\) or not. Explain whether \( F \) is differentiable at \((0,0)\) or not.

5. (20 marks) Let \( \{a_n\}, \{b_n\} \) be sequences of real numbers. Show that

\[
\liminf_{n \to \infty} a_n + \limsup_{n \to \infty} b_n \geq \liminf_{n \to \infty}(a_n + b_n),
\]

provided the left side is not of the form \(-\infty + \infty\). Give an example where the two sides are not equal in the above inequality.
Midterm for Lecture 3

Directions: This is a closed book exam. No notes are allowed. Work must be shown to receive credits. Answers alone worth very little. No talking. You may use nonprogrammable calculators.

Notation: \( \mathbb{R} \) denotes the real numbers.

1. (a) (5 marks) Give the definition of a sequence of functions \( S_n : E \to \mathbb{R} \) converges pointwise on a set \( E \) to a function \( S : E \to \mathbb{R} \).

(b) (5 marks) Explain whether the sequence of functions \( S_n : \mathbb{R} \to \mathbb{R} \) defined by \( S_n(x) = |\cos x|^n \) converges pointwise on \( \mathbb{R} \) to a function \( S : \mathbb{R} \to \mathbb{R} \) or not.

(c) (5 marks) Give the definition of a sequence of functions \( S_n : E \to \mathbb{R} \) converges uniformly on a set \( E \) to a function \( S : E \to \mathbb{R} \).

(d) (5 marks) Explain whether the sequence of functions \( S_n : \mathbb{R} \to \mathbb{R} \) defined by \( S_n(x) = |\cos x|^n \) converges uniformly on \( \mathbb{R} \) to a function \( S : \mathbb{R} \to \mathbb{R} \) or not.

2. (20 marks) Define \( F : \mathbb{R}^2 \to \mathbb{R} \) by

\[
F(x, y) = \begin{cases} 
\frac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]

Explain whether \( F \) is differentiable at \((1, 1)\) or not. Explain whether \( F \) is differentiable at \((0, 0)\) or not.

3. (20 marks) Consider the system of equations

\[
x = ve^u + u \quad \text{and} \quad y = ue^v - v.
\]

Show that near \((u, v, x, y) = (0, 0, 0, 0), (u, v)\) can be expressed as a differentiable function of \((x, y)\) and find the value of \( \frac{\partial u}{\partial x}(0, 0) + \frac{\partial u}{\partial y}(0, 0) + \frac{\partial v}{\partial x}(0, 0) + \frac{\partial v}{\partial y}(0, 0) \).

4. (20 marks) Show that the series \( 1 - \frac{1}{4} + \frac{1}{7} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{3k + 1} \) converges and find its sum.

(Hint: \( \frac{1}{1 + t^3} = \frac{1}{3} \left( \frac{1}{t + 1} - \frac{t - 2}{t^2 - t + 1} \right) \) and \( \int \frac{1}{t^2 - t + 1} \, dt = \frac{2}{\sqrt{3}} \arctan \frac{2t - 1}{\sqrt{3}} + C. \))

5. (20 marks) Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be sequences of real numbers such that \( c_n = \max(a_n, b_n) \) for \( n = 1, 2, 3, \ldots \). Show that \( \limsup c_n = \max \left( \limsup a_n, \limsup b_n \right) \).

Show that \( \liminf c_n = \max \left( \liminf a_n, \liminf b_n \right) \) is false in general by providing a counterexample.

Math 301 (Real Analysis) Fall 2001

Midterm for Lecture 1

Directions: Work must be shown to receive credits. Answers alone are worth very little. No talking.
1. (a) (5 marks) Give the definition of a vector-valued function \( F : \mathbb{R}^n \to \mathbb{R}^m \) differentiable at \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \).

(b) (10 marks) Is the function \( F : \mathbb{R}^2 \to \mathbb{R} \) defined by \( F(x, y) = \sqrt{|x y|} \) differentiable at \((x, y) = (0, 0)\)? Be sure to give a proof of your answer.

2. (25 marks) Consider the system of equations
\[
x = \theta + r \cos \theta \quad \text{and} \quad y = \theta + r \sin \theta.
\]
Show that near \((r, \theta, x, y) = (0, 0, 0, 0)\), \((r, \theta)\) can be expressed as a differentiable function of \((x, y)\). At \((x, y) = (0, 0)\), find \( \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x} \) and \( \frac{\partial \theta}{\partial y} \).

3. (a) (5 marks) Find \( \lim_{x \to 0} \frac{\sin(x) - x}{x^3} \). Show work.

(b) (20 marks) Prove that \( \int_0^1 \frac{\sin(x) - x}{x^3} \, dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)(2k-1)} \). Be sure to give reasons to support your argument.

4. (a) (15 marks) Let \( f : [-1, 1] \to \mathbb{R} \) be continuous and \( x_1, x_2, x_3, \ldots \) be a sequence of real numbers in \([-1, 1]\). Show that \( \liminf_{n \to \infty} f(x_n) \leq f(\liminf_{n \to \infty} x_n) \).

(b) (5 marks) Give an example of a continuous function \( f : [-1, 1] \to \mathbb{R} \) and sequence \( x_1, x_2, x_3, \ldots \) in \([-1, 1]\) such that \( \liminf_{n \to \infty} f(x_n) < f(\liminf_{n \to \infty} x_n) \). Show work.

5. (15 marks) Let \( f_n : [0, 1] \to \mathbb{R} \) be continuous and converge uniformly on \([0, 1]\) to a function \( f : [0, 1] \to \mathbb{R} \). Prove that \( \lim_{n \to \infty} f_n\left(\frac{1}{n}\right) = f(0) \).

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Math 301 (Real Analysis)  
Fall 2001

Midterm for Lecture 2

Directions: Work must be shown to receive credits. Answers alone are worth very little. No talking.

Notation: \( \mathbb{R} \) denotes the real numbers.

1. (a) (5 marks) Give the definition of a vector-valued function \( F : \mathbb{R}^n \to \mathbb{R}^m \) continuously differentiable (or \( C^1 \)) at \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \).

(b) (10 marks) Is the function \( F : \mathbb{R}^2 \to \mathbb{R} \) defined by \( F(x, y) = \sin(|xy|) \) \( C^1 \) at \((x, y) = (0, 0)\)? Be sure to give a proof of your answer.

2. (25 marks) Consider the system of equations
\[
u = e^x - xy \quad \text{and} \quad v = x + y + e^y.
\]
Show that near \((x, y, u, v) = (0, 0, 1, 1), (x, y)\) can be expressed as a differentiable function of \((u, v)\). At \((u, v) = (1, 1)\), find \(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}\) and \(\frac{\partial y}{\partial v}\).

3. (a) (5 marks) Find \(\lim_{w \to 0} \frac{w - \ln(1 + w)}{w^2}\). Show work.

(b) (20 marks) Prove that \(\int_0^1 \frac{x^2 - \ln(1 + x^2)}{x^4} \, dx = \sum_{k=2}^{\infty} \frac{(-1)^k}{k(2k - 3)}\). Be sure to give reasons to support your argument.

4. (a) (15 marks) Let \(f : [-1, 1] \to \mathbb{R}\) be continuous and \(x_1, x_2, x_3, \ldots\) be a sequence of real numbers in \([-1, 1]\). Show that \(\limsup_{n \to \infty} f(x_n) \geq f(\limsup_{n \to \infty} x_n)\).

(b) (5 marks) Give an example of a continuous function \(f : [-1, 1] \to \mathbb{R}\) and sequence \(x_1, x_2, x_3, \ldots\) in \([-1, 1]\) such that \(\limsup_{n \to \infty} f(x_n) > f(\limsup_{n \to \infty} x_n)\). Show work.

5. (15 marks) Let \(f_n : [0, 1] \to \mathbb{R}\) be continuous and converge uniformly on \([0, 1]\) to a function \(f : [0, 1] \to \mathbb{R}\). Prove that \(\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 f(x) \, dx\).

Math 301 (Real Analysis) Fall 2001

Midterm for Lecture 3

Directions: Work must be shown to receive credits. Answers alone are worth very little. No talking.

Notation: \(\mathbb{R}\) denotes the real numbers.

1. (a) (5 marks) Give the definition of a vector-valued function \(F : \mathbb{R}^n \to \mathbb{R}^m\) differentiable at \(a = (a_1, \ldots, a_n) \in \mathbb{R}^n\).

(b) (10 marks) Is the function \(F : \mathbb{R}^2 \to \mathbb{R}\) defined by \(F(x, y) = x|y| + y|x|\) differentiable at \((x, y) = (0, 0)\)? Be sure to give a proof of your answer.

2. (25 marks) Consider the system of equations

\[ s = (\cos x) - e^y \quad \text{and} \quad t = e^x + \sin y. \]

Show that near \((x, y, s, t) = (0, 0, 0, 1), (x, y)\) can be expressed as a differentiable function of \((s, t)\). At \((s, t) = (0, 1)\), find \(\frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial s}\) and \(\frac{\partial y}{\partial t}\).

3. (a) (5 marks) Find \(\int_0^1 x \cos x \, dx\). Show work.

(b) (20 marks) Show that \(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)! (2k + 2)}\) converges and find its sum. Be sure to give reasons to support your argument.
4. (a) (15 marks) Give an example of real numbers $a_{j,k}$ (for $j, k = 1, 2, 3, \ldots$) such that 
\[ \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} a_{j,k} \right) = 0 \]
and 
\[ \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} a_{j,k} \right) = 1. \]

(b) (5 marks) Give an example of a continuous function $f : [-1, 1] \rightarrow \mathbb{R}$ and a sequence $x_1, x_2, x_3, \ldots$ of real numbers in $[-1, 1]$ such that 
\[ \liminf_{n \rightarrow \infty} f(x_n) \neq f\left( \liminf_{n \rightarrow \infty} x_n \right). \]

5. (15 marks) Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous and $f(1) = 0$. Prove that the sequence of functions $f_n : [0, 1] \rightarrow \mathbb{R}$ defined by $f_n(x) = x^n f(x)$ converge uniformly on $[0, 1]$. 

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Directions: This is a closed book exam. Work must be shown to receive credits. Answers alone are worth very little.

Notations: \( \mathbb{R} \) denotes the set of all real numbers.

1. (20 marks) Consider the system of equations
   \[
   x = e^{uv} - \cos(u + v) + 2v \quad \text{and} \quad y = e^{u-v} - \cos(3uv).
   \]
   Show that near \( p = (u, v, x, y) = (0, 0, 0, 0) \), \((u, v)\) can be expressed as a differentiable function of \((x, y)\) and find the value of \( \frac{\partial u}{\partial y} \) and \( \frac{\partial v}{\partial x} \) at \( p \).

2. (a) (5 marks) State the Lebesgue Dominated Convergence Theorem.
   (b) (15 marks) Determine \( \lim_{n \to \infty} \int_{[1,n]} e^{-nx} \frac{1}{1 + x^2} dm \) with proof.

3. (15 marks) Let \( n \) be a fixed positive integer and \( E_1, E_2, E_3, \ldots, E_n \) be measurable subsets of \([0, 1]\) such that
   \[
   \sum_{k=1}^{n} m(E_k) > n - 1.
   \]
   Prove that \( m\left( \bigcap_{k=1}^{n} E_k \right) > 0 \).

4. (15 marks) Give an example of a nonmeasurable function \( h : \mathbb{R} \to \mathbb{R} \) such that for every \( c \in \mathbb{R} \), \( h^{-1}(\{c\}) \) is a measurable set on \( \mathbb{R} \).
   Be sure to provide details why your \( h \) satisfies the required conditions.

5. (15 marks) Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function such that \( f(x+1) = f(x) \) almost everywhere on \( \mathbb{R} \).
   Prove that there is a measurable function \( g : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = g(x) \) almost everywhere on \( \mathbb{R} \) and \( g(x+1) = g(x) \) for every \( x \in \mathbb{R} \).

6. (15 marks) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. For every \( \varepsilon > 0 \), there exists a continuous function \( g : \mathbb{R} \to \mathbb{R} \) such that \( S = \{x : f(x) \neq g(x)\} \) is measurable with \( m(S) < \varepsilon \). Prove that \( f \) is a measurable function.
Show that near \( p = (u, v, w) = (0, 0, 0) \), \((v, w)\) can be expressed as a differentiable function of \( u \) and find the value of \( \frac{dv}{du} \) and \( \frac{dw}{du} \) at \( p \).

2. (a) (5 marks) State the Lebesgue Dominated Convergence Theorem.

(b) (15 marks) Determine \( \lim_{n \to \infty} \int_{[-n,n]} \frac{\sin(x/n)}{1 + x^2} \, dm \) with proof.

3. (15 marks) Let \( E_1, E_2, E_3, \ldots \) be measurable subsets of \([0, 1]\) such that \( \lim_{k \to \infty} m(E_k) = 1 \). Prove that there is a subsequence \( E_{k_1}, E_{k_2}, E_{k_3}, \ldots \) of the \( E_k \)'s such that

\[
m\left( \bigcap_{n=1}^{\infty} E_{k_n} \right) > \frac{1}{2}.
\]

4. (15 marks) Prove that the intersection of a collection of measurable subsets in \( \mathbb{R} \) can be a nonmeasurable set in \( \mathbb{R} \).

5. (15 marks) Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function such that \( f(x+1) = f(x) \) almost everywhere on \( \mathbb{R} \). Prove that there is a measurable function \( g : \mathbb{R} \to \mathbb{R} \) such that \( f(x) = g(x) \) almost everywhere on \( \mathbb{R} \) and \( g(x+1) = g(x) \) for every \( x \in \mathbb{R} \).

6. (15 marks) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. For every \( \varepsilon > 0 \), there exists a continuous function \( g : \mathbb{R} \to \mathbb{R} \) such that \( S = \{ x : f(x) \neq g(x) \} \) is measurable with \( m(S) < \varepsilon \). Prove that \( f \) is a measurable function.

Math 301 (Real Analysis)  
Final Examination  
Fall 2004

Directions. You must show work to receive credits. Answers are worth very little. No calculator is allowed.

Notation: \( \mathbb{R} \) denotes the set of real numbers.

1. (20 marks) Consider the system of equations

\[
x^2 + \sin(xy) + e^{yz} = 5 \quad \text{and} \quad e^z + \cos(zy) + (xyz)^2 = 2.
\]

Show that near \((x, y, z) = (2, 0, 0)\), \((x, z)\) can be expressed as a differentiable function of \( y \). At \((x, y, z) = (2, 0, 0)\), find \( \frac{dx}{dy} \) and \( \frac{dz}{dy} \).

2. (a) (5 marks) State the Lebesgue Dominated Convergence Theorem.

(b) (15 marks) Determine \( \lim_{n \to \infty} \int_{0}^{1} \frac{\ln(2 + \frac{x}{n})}{1 + x} \, dx \) with proof.

3. (15 marks) Give an example of a continuous function \( f : [0, +\infty) \to \mathbb{R} \) such that \( f \) is improper Riemann integrable on \([0, +\infty)\), but \( f \) is not Lebesgue integrable on \([0, +\infty)\).
4. Let $S$ be a nonempty measurable set such that $S \neq \mathbb{R}$. Let $r$ be a positive real number. Let $S_r$ be the set of all $x$ such that $x \in S$ and there exists $y \notin S$ satisfying $|x - y| < r$.

(a) (5 marks) Prove that $S_r$ is measurable for every $r > 0$.

(b) (5 marks) Prove that if $S$ is open, then $S_r$ is open for every $r > 0$.

(c) (5 marks) If $S$ is closed, must $S_r$ be closed for every $r > 0$? Give a proof or a counterexample.

5. (15 marks) Let $W$ be a nonempty subset of $\mathbb{R}$. Define $f : \mathbb{R} \to [0, +\infty)$ by letting $f(x)$ be the greatest lower bound of $\{|x - w| : w \in W\}$. Prove that $f$ is measurable. (Hint: Try some examples of $W$ to find out what type of functions $f$ should be. Then prove your guess.)

6. (15 marks) Let $A$ be an uncountable set. For every $a \in A$, let $J_a$ be an interval in $\mathbb{R}$ such that $m(J_a) > 0$. Prove that $S = \bigcup_{a \in A} J_a$ is a measurable set.
(a) (5 marks) Prove that $S = [0, +\infty) \setminus C$ is a measurable set.

(b) (10 marks) If $f$ is continuous at every $x \in S = [0, +\infty) \setminus C$, then prove that the function $f : [0, +\infty) \to \mathbb{R}$ is a measurable function.

6. (a) (5 marks) State the structure theorem for open sets.

(b) (15 marks) Let $E$ be a bounded measurable set in $\mathbb{R}$ such that $m(E \cap I) \leq \frac{1}{2}m(I)$ for every interval $I$. Prove that $m(E) = 0$.

Math 301 (Real Analysis) Fall 2002

Final Examination

Directions. In general, work must be shown to receive credits. Answers alone are worth very little. While working on a part of a problem, you may assume the results of earlier parts.

Notations: $\mathbb{R}$ denotes the set of all real numbers; $m(X)$ denotes the Lebesgue measure of $X$.

1. (10 marks) Consider the equation $y + e^{xy} + x \sin y = 2$. Prove that near $(x, y) = (0, 1)$, the equation can be expressed as $y = f(x)$ for some differentiable function $f$ of $x$.

2. (a) (5 marks) State the Lebesgue Dominated Convergence Theorem.

(b) (10 marks) Find $\lim_{n \to +\infty} \int_{1}^{e} e^{-x^n} \, dx$. Give reason to support your answer.

3. (a) (7 marks) Let $f, g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be functions such that $f + g$, $fg$ and $\frac{f}{g}$ are measurable functions. If $f(x) + g(x) \neq 0$ for all $x \in \mathbb{R}$, must at least one of $f$ or $g$ be measurable? Give reasons to support your answer.

(b) (8 marks) Let $f, g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be functions such that $f + g$, $fg$ and $\frac{f}{g}$ are measurable functions. Must at least one of $f$ or $g$ be measurable? Give reasons to support your answer.

4. (15 marks) Let $X$ be a measurable set in $\mathbb{R}$ and $g : X \to [0, +\infty)$ be a measurable function. If $\int_{X} g \, dm < \infty$ and $\int_{X} g^3 \, dm < \infty$, then prove that $\int_{X} g^2 \, dm < \infty$.

5. (a) (5 marks) State the structure theorem for open sets.

(b) (15 marks) Let $S$ be a nonempty open set. Prove that not every subset of $S$ is measurable.

(c) (5 marks) Give an example of an infinite compact set $W$ in $\mathbb{R}$ such that every subset of $W$ is measurable. Be sure to give reasons to support your example.

6. (20 marks) Let $B$ be a bounded subset of $\mathbb{R}$. Prove that $B$ is a measurable set if and only if for every $\varepsilon > 0$, there exist a compact set $K$ and a bounded open set $S$ such that $K \subseteq B \subseteq S$ and $m(S \setminus K) < \varepsilon$.

Math 301 (Real Analysis) Fall 2001
Final Examination

Directions. In general, work must be shown to receive credits. Answers alone are worth very little. While working on a part of a problem, you may assume the results of earlier parts.

Notations: \( \mathbb{R} \) denotes the real numbers; \( \mathbb{Q} \) denotes the rational numbers.

1. (20 marks) For the system \( wx + xy + yz + zw = w^5 + x^5 + y^5 + z^5 \) and \( wxyz = xy + wz - yw \), show that \((x, y)\) can be expressed as a differentiable function of \((w, z)\) near the solution \( p = (w, x, y, z) = (1, 1, 1, 1) \).
   Find \( \frac{\partial x}{\partial w} (1, 1) \) and \( \frac{\partial y}{\partial w} (1, 1) \).

2. (a) (5 marks) State the Lebesgue Dominated Convergence Theorem.
   (b) (15 marks) Find \( \lim_{n \to \infty} \int_{(0,1]} \frac{\cos(x/n)}{1 + x^2} \, dm \). Be sure to give reasons, not just answer.

3. (5 marks) Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function. Show that the function \( h : \mathbb{R} \to \mathbb{R} \) defined by \( h(x) = \int_0^{f(x)} e^{-t^2} \, dt \) is a measurable function.

4. (a) (5 marks) State the definition of a measurable set in case it is an unbounded subset of \( \mathbb{R} \).
   (b) (15 marks) Let \( f : \mathbb{R} \to \mathbb{R} \) be a measurable function. Show that \( \{ x : f(x) \in \mathbb{Q} \} \) and \( \{ x : f(x) \in \mathbb{R} \setminus \mathbb{Q} \} \) are measurable sets.

5. (20 marks) Prove that \( \int_{(0,1]} \frac{\sin x}{x} \, dm = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(2k+1)} \).

6. Let \( A \) be a nonempty open set and \( A \subseteq [-1, 1] \).
   (a) (10 marks) Show that \( B = \{ x : \sin x \in A \} \) and \( C = \{ \sin x : x \in A \} \) are measurable sets.
   (b) (15 marks) Show that \( m(C) \leq m(A) \leq m(B) \).

7. (15 marks) Let \( D \) be a (possibly nonmeasurable) subset of \([0, 1]\). Show that there exists a measurable set \( E \) such that \( E \subseteq D \) and \( m_*(D) = m(E) \).