

## Seasonal Time Series

TS with periodic patterns, e.g., quarterly earnings

Multiplicative model

Airline model (for quarterly series)

Form:

$$p_t - p_{t-1} - p_{t-4} + p_{t-5} = a_t - \theta_1 a_{t-1} - \theta_4 a_{t-4} + \theta_1 \theta_4 a_{t-5}$$

or

$$(1 - B)(1 - B^4)p_t = (1 - \theta_1 B)(1 - \theta_4 B^4)a_t.$$

Define the differenced series  $w_t$  as

$$w_t = p_t - p_{t-1} - p_{t-4} + p_{t-5} = (p_t - p_{t-1}) - (p_{t-4} - p_{t-5}).$$

It is called regular and seasonal differenced series.

$$(1 - B)(1 - B^s)p_t = (1 - \theta B)(1 - \Theta B^s)a_t.$$

Define the differenced series  $w_t$  as

$$w_t = p_t - p_{t-1} - p_{t-s} + p_{t-s-1} = (p_t - p_{t-1}) - (p_{t-s} - p_{t-s-1}).$$

Thus,

$$w_t = (1 - \theta B)(1 - \Theta B^s)a_t.$$

$s$  --- seasonal period.

ACF of  $w_t$  has a nice symmetric structure (see the text).

Let  $w_t = (1 - B)(1 - B^s)p_t$ . Then the autocovariance of  $w_t$  can be found to be

$$\begin{aligned}\gamma_0 &= (1 + \theta^2)(1 + \Theta^2)\sigma_a^2, \\ \gamma_1 &= -\theta(1 + \Theta^2)\sigma_a^2, \\ \gamma_{s-1} &= \theta\Theta\sigma_a^2, \\ \gamma_s &= -\Theta(1 + \theta^2)\sigma_a^2, \\ \gamma_{s+1} &= \theta\Theta\sigma_a^2, \\ \gamma_j &= 0, \quad \text{otherwise.}\end{aligned}$$

The ACF becomes:

$$\begin{aligned}\rho_1 &= \frac{-\theta}{1 + \theta^2}, \\ \rho_{s-1} &= \frac{\theta\Theta}{(1 + \theta^2)(1 + \Theta^2)} = \rho_{s+1}, \\ \rho_s &= \frac{-\Theta}{1 + \Theta^2}, \\ \rho_j &= 0, \quad \text{otherwise.}\end{aligned}$$

Usually, we can take  $w_t = (1 - B)(1 - B^s)p_t$ .

$w_t$  is called (pure) seasonal ARMA( $P, Q$ )<sub>s</sub> model if

$$\Phi_P(B^s)w_t = \phi_0 + \Theta_Q(B^s)a_t,$$

where  $\Phi_P(B^s) = 1 - \sum_{i=1}^P \Phi_i B^{is}$  and  $\Theta_Q(B^s) = 1 - \sum_{i=1}^Q \Theta_i B^{is}$ .

Box-Jenkins multiplicative seasonal ARMA model:

$$\Phi_P(B^s)\phi_p(B)w_t = \phi_0 + \theta_q(B)\Theta_Q(B^s)a_t,$$

where  $\phi_p(B) = 1 - \sum_{i=1}^p \phi_i B^i$  and  $\theta_q(B) = 1 - \sum_{i=1}^q \Theta_i B^i$ .

This model is widely applicable to many many seasonal time series.

Forecasts: exhibit the same pattern as the observed series.

**Empirical Example.** International Airline Passengers data in Box and Jenkins (1976).

$X_t$  = the number of Passengers in the  $t$ -th month.

**Step 1.** Make a transformation:  $p_t = \log(X_t)$ .

**Step 2.** Remove nonstationary or seasonal components:

$$\begin{aligned} w_t &= (1 - B)p_t, \text{ or} \\ w_t &= (1 - B^{12})p_t, \text{ or} \\ w_t &= (1 - B)(1 - B^{12})p_t. \end{aligned}$$

**Step 3.** Note that  $w_t = (1 - B)(1 - B^{12})p_t$  is stationary. So we use the seasonal ARMA model to fit the data:

$$\Phi_P(B^{12})\phi_p(B)w_t = \theta_q(B)\Theta_Q(B^{12})a_t,$$

or

$$\Phi_P(B^{12})\phi_p(B)w_t = \theta_q(B)\Theta_Q(B^{12})a_t.$$

Now, the problem is how to find  $p, q, P$  and  $Q$  !!!

**Step 4.** Look at the ACF and PACF of  $W_t$  or try some different  $p, q, P$  and  $Q$ .

For example, we try the model:

$$(1 - \phi B)(1 - \Phi B^{12})w_t = a_t.$$

**Step 5.** Estimate the parameters in the model.

How to estimate? CLSE, ULSE or MLE methods. The results are:

$$\phi = -0.38, \quad \Theta = -0.5.$$

**Step 6.** Diagnostic checking.

Calculate the residuals:

$$e_t = (1 + 0.38B)(1 + 0.52B^{12})w_t.$$

As for ARMA model, if  $\{e_t\}$  are white noises, the model is correct.

**Step 7.** If the model is wrong, we should try other models. Even if it is correct, we still need to try some possible models.

For example, we try another model:

$$w_t = (1 - \theta B)(1 - \Theta B^{12})a_t.$$

Through **Step 5**, we obtain:

$$\theta = 0.4, \quad \Theta = 0.61.$$

Through **Step 6**, we know this model is correct, too.

**Step 8.** Model selection: **AIC**, **BIC**, or **SBC**.

The final model is:

$$w_t = (1 - 0.40B)(1 - 0.61B^{12})a_t,$$

or

$$(1 - B)(1 - B^{12}) \log(X_t) = (1 - 0.40B)(1 - 0.61B^{12})a_t.$$

**Step 9.** Forecasting.