Chapter 2: Probability

Definition on Sample Space:

1. Any process that generates a set of data is called an **experiment**.

2. The set of all possible outcomes of an experiment is called the **sample space**, denoted by S.

3. An element of the sample space is called a **sample point**.

Examples:

1. Experiment: Toss a coin. $S =$

2. Experiment: Toss a die once. $S =$

3. Experiment: Toss a coin one time. If a head occurs, then toss the coin again. If a tail occurs, then toss a die one time. $S =$

4. Experiment:

Take a point on the boundary or the interior of a circle of radius 2. S=

Definition on Event:

For any given experiment, we may be interested in the occurrence of certain **events** rather than one specific outcome.

Example. We play a game with the rule as follows: tossing a die, if the number is greater than or equal to 4, then you win, but if the number is less than 4, then you lose. So, you are concerned the **event** that the number is greater than or equal to 4.

Definition:

An **event** is the subset of a sample space.

An event may include all the elements in S. So S itself is an event.

When an event does not include any element in S, this event is called the **null** event, denoted by ϕ.

Why do we need this null event?

Experiment: Toss a coin.

$S = \{H, T\}$.

Let A be the event that a head or a tail appears in the experiment. $A = \{H, T\} = S$.

Let A be the event that a number 5 appears in the experiment. $A = \phi$.

Ex. 1-1. Given the sample space $S = \{H, T\}$.

List all the events in S that include H . Answer:

Ex. 1-2. Experiment: Toss a coin

Sample space $S = \{H, T\}$.

Let A be the event that gets head in the experiment. Answer:

Ex. 2-1. Given the sample space $S = \{1, 2, 3, 4, 5, 6\}$

(a). List all the events in S consisting of one score which is greater than 4. Answer:

(b). List all the events in S consisting of two scores which are greater than 4. Answer:

(c). List all the events in S in which all scores are even. Answer:

Ex. 2-2. Experiment: Toss a die once

Sample space $S = \{1, 2, 3, 4, 5, 6\}$

(a). Let A be the event that gets a score greater than 4 in the experiment. Answer:

(b). Let A be the event that gets an even score in the experiment. Answer:

Ex. 3. Experiment: Toss a coin one time. If a head occurs, then toss the coin again. If a tail occurs, then toss a die one time.

$S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$

The event of no head in the experiment $=\{T1, T2, T3, T4, T5, T6\}$

The event of no tail in the experiment $=\{HH\}$

The event of no number in the experiment $=\{HH, HT\}$.

Ex. 4. Given the sample space: $S = \{HH, HT, T1, T2, T3, T4, T5, T6\}$,

(a). there are many events of no head, e.g. $\{T1\}$, $\{T2\}$,$\{T1, T3\}$,$\cdots$.

(b). the events of no number are: $\{HH\}$, $\{HT\}$, $\{HH, HT\}$, $\{\phi\}$.

Definition: The **complement** of the event A with respect to the space S is the subset of all the elements in S that are not in A, denoted by A'.

Let A be the event that a head appears when we toss a coin. $A' =$

Let A be the event that an even number appears when we toss a die. $A' =$

Let A be the event that a head appears when we toss a die. $A' =$

Let A be the event that a head or tail appears when we toss a coin. $A' =$

Definition: The **intersection** of two events A and B, denoted by $A \cap B$, is the event containing all the elements that are common to A and B.

Let A be the event that an even number appears and B be the event that get a number greater than 4 when we toss a die. $A \cap B =$

Let A be the event that an even number appears and B that an odd number appears when we toss a die. $A \cap B =$

Definition: Two or more events are **mutually exclusive** if they are pairwise disjoint (if they cannot occur simultaneously).

Definition: The **union** of two events A and B, denoted by $A \cup B$, is the event containing all the elements that belongs to A or B or both.

Let A be the event that an even number appears and B be the event that get a number greater than 4 when we toss a die. $A \cup B =$

Let A be the event that an even number appears and B that an odd number appears when we toss a die. $A \cup B =$

Let A be the event that an even number appears and B that a head appears when we toss a die. $A \cup B =$

Probability of Event:

Simply, **Probability of Event** A is how much possibility that the event A occurs, denoted by $P(A)$.

To know $P(A) =?$, we need to know how much possibility that the outcome ω_i occurs in $S = \{\omega_1, \cdots, \omega_N\}$, i.e. $P(\{\omega_i\}) =?$.

Formally, let $P(\{\omega_i\}) = p_i$ and $A = \{\omega_{i_i}, \cdots, \omega_{i_n}\}$. Then we define

$$\begin{aligned} P(A) &= P(\{\omega_{i_1}\}) + \cdots + P(\{\omega_{i_n}\}) \\ &= p_{i_1} + \cdots + p_{i_n}, \end{aligned}$$

where $p_i \geq 0$ and $p_1 + \cdots + p_N = 1$ (?).

Properties:

(1). $0 \leq P(E) \leq 1$

(2). $P(\phi) = 0$ and $P(S) = 1$

(3). If A_1, A_2, $\cdots$, A_k are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + \cdots + P(A_k).$$

Theorem:

If an experiment has N equally likely outcomes in S and n of them are the event A, then the theoretical probability of event A occurring is

$$P(A) = \frac{n}{N}.$$

$P(A) = 0$ means that A is an impossibility.

$P(A) = 1$ means that A is a certainty.

Examples

1. Experiment: Toss a coin once. Note that $S =$

Let $A = H$, then $P(A) =$

2. Experiment: Toss a coin two times. Note that $S =$

Let A be the event of at least one heads, then $P(A) =$

3. Experiment: Toss a coin three times. Note that $S =$

Let A be the event of at least 2 heads, then $P(A) =$

4. Experiment: Toss a die once

$S =$

Let A be the event of a score greater than 4. Then $P(A) =$

Remark: If we cannot invoke the concept of equal likelihood, a priori evaluation of probability is difficult.

Empirical probability

The **empirical probability** of an event E is the relative frequency of occurrence of the event E, i.e. f/n.

Here, n is interpreted as the number of experiments and f is the number of occurrence of the event E among the n experiments.

$$\lim_{n\to\infty} \frac{f}{n} = P(E).$$

$$P(E) \approx \frac{f}{n} \textit{ as } n \textit{ is large enough.}$$

For example,

n (no of tosses) 10 100 1000

$f(E)$ (no of heads observed) 6 47 510

$f(E)/n$ (relative frequency of heads) 0.6 0.47 0.51

Rules of probability

1. **Addition rule**

If A_1 and A_2 are two events, then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

Example

Experiment: Toss a die once. Note that $S = \{1, 2, 3, 4, 5, 6\}$. Let $A = 1$ and $B = 2$.

$$P(\{1, 2\}) = P(A \cup B) =$$