

Estimation of Change-points in ARMA-GARCH/IGARCH
and General Time Series Models
Supplementary Appendix: Proofs of Lemmas 7.1-7.3

Shiqing Ling

Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong

We first gives one lemma, which is from Ling and Li (1997), see also Bougerol and Picard (1992).

Lemma 0.1. *Suppose $\{y_t\} \in M(\vartheta)$. If Assumptions 4.1-4.2 hold, then h_t has the expansion:*

$$h_t = \alpha_0 \left[1 + \sum_{j=1}^{\infty} u' \prod_{i=0}^{j-1} A_{t-i} \xi_{t-j} \right],$$

where $\xi_t = (\eta_t^2, 0, \dots, 0, 1, \dots, 0)_{(r+s) \times 1}'$ with the first component η_t^2 and the $(r+1)$ th component 1, $u = (0, \dots, 1, \dots, 0)_{(r+s) \times 1}'$ with the $(r+1)$ th component 1 and

$$A_t = \begin{pmatrix} \alpha_1 \eta_t^2 & \cdots & \alpha_r \eta_t^2 & \beta_1 \eta_t^2 & \cdots & \beta_s \eta_t^2 \\ I_{r-1} & O & O & O & \cdots & O \\ \alpha_1 & \cdots & \alpha_r & \beta_1 & \cdots & \beta_s \\ O & & O & I_{s-1} & \cdots & O \end{pmatrix},$$

Proof of Lemma 7.1. By Theorem 2.1 of Ling (2007b), Assumption 4.2 implies that, for any $\iota \in (0, 1)$, there exists an integer i_0 such that

$$E \left\| \prod_{i=0}^{i_0-1} A_{t-i} \right\|^{\iota} < 1.$$

Let $B_t = \xi_t + \sum_{j=1}^{i_0-1} \prod_{r=0}^{j-1} A_{t-r} \xi_{t-j}$ and $\tilde{A}_t = \prod_{i=0}^{i_0-1} A_{t-i}$. We rewrite h_t as

$$h_t = \alpha_0 \left[u' B_t + \sum_{J=1}^{\infty} u' \prod_{r=0}^{J-1} \tilde{A}_{t-i_0 r} B_{t-J i_0} \right],$$

where $u' \xi_t = 1$ is used. Let $k = ai_0 + b$, $b = 1, 2, \dots, i_0 - 1$, and $h_{ta} = \alpha_0 [u' B_t +$

$\sum_{J=1}^a u' \prod_{r=0}^{J-1} \tilde{A}_{t-i_0 r} B_{t-J i_0}$. Then $h_{ta} \in \mathcal{F}_k(t)$. Thus,

$$\begin{aligned} E|h_t - E[h_t|\mathcal{F}_k(t)]|^\iota &= E|(h_t - h_{ta}) - E[(h_t - h_{ta})|\mathcal{F}_k(t)]|^\iota \\ &\leq 2E|h_t - h_{ta}|^\iota \leq 2E \left| \sum_{J=a+1}^{\infty} u' \prod_{r=0}^{J-1} \tilde{A}_{t-i_0 r} B_{t-J i_0} \right|^\iota = O(\rho^k), \end{aligned}$$

because $\{\tilde{A}_t\}$ is an i.i.d. series. Thus, (a) holds. (b) and (c) are directly from (a). This completes the proof. \square

Proof of Lemma 7.2. We only give the proof of (b) since the proof of (a) is similar and simpler. By (2.5) of Ling (2007b) and Assumptions 4.1-4.2, we can show that

$$h_t(\vartheta_0) = \alpha_0 \beta^{-1}(1) + \sum_{j=1}^{\infty} v_{hj} \varepsilon_{t-j}^2 \text{ and } \frac{\partial h_t(\vartheta_0)}{\partial \gamma} = 2 \sum_{j=1}^{\infty} v_{hj} \varepsilon_{t-j} \frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}, \quad (0.1)$$

where $v_{hj} = O(\rho_1^j)$, $\rho_1 \in (0, 1)$ and

$$\frac{\partial \varepsilon_t(\vartheta_0)}{\partial \gamma_a} = \sum_{i=0}^{\infty} v_{ai} \varepsilon_{t-i-a}, \quad (0.2)$$

where $a = 1, \dots, p+q$, $v_{ai} = O(\rho^i)$ and $\rho \in (0, 1)$. Denote $A_t(k) = \sum_{i=0}^{k-1} v_{ai} \varepsilon_{t-i-a}$ and $\tilde{A}_t(k) = \sum_{i=k}^{\infty} v_{ai} \varepsilon_{t-i-a}$. Then, when $\iota \in (0, 1)$,

$$\begin{aligned} E \left| \frac{\tilde{A}_{t-j}(k)}{\sqrt{h_t}} - E \left[\frac{\tilde{A}_{t-j}(k)}{\sqrt{h_t}} \middle| \mathcal{F}_k(t) \right] \right|^{2-2\iota} &\leq C E \left| \frac{\tilde{A}_{t-j}(k)}{\sqrt{h_t}} \right|^{2-2\iota} \\ &\leq C \sum_{i=k}^{\infty} \rho^i E |\varepsilon_{t-i-a}|^{2-2\iota} = O(\rho^k). \quad (0.3) \end{aligned}$$

By Holder inequality with $p = (2-\iota)/(2-2\iota)$ and $q = (2-\iota)/\iota$, we have

$$\begin{aligned} E \left(A_{t-j}(k) \left| \frac{1}{\sqrt{h_t}} - E \left[\frac{1}{\sqrt{h_t}} \middle| \mathcal{F}_k(t) \right] \right| \right)^{2-2\iota} &\leq [E |A_t(k)|^{2-\iota}]^{\frac{1}{p}} \left(E \left| \frac{1}{\sqrt{h_t}} - E \left[\frac{1}{\sqrt{h_t}} \middle| \mathcal{F}_k(t) \right] \right|^{(2-2\iota)q} \right)^{\frac{1}{q}} \\ &\leq C \left(E \left| \frac{1}{\sqrt{h_t}} - E \left[\frac{1}{\sqrt{h_t}} \middle| \mathcal{F}_k(t) \right] \right|^{\iota} \right)^{\frac{1}{q}} \\ &\leq C \left(E |h_t - E[h_t|\mathcal{F}_k(t)]|^\iota \right)^{\frac{1}{q}} = O(\rho^k), \quad (0.4) \end{aligned}$$

where the last step holds by Lemma 7.1(a). When $j = 1, \dots, k$, by Lemma 7.1(b), we have

$$\begin{aligned} E \left\{ \frac{1}{\sqrt{h_t}} \left| A_{t-j}(k) - E[A_{t-j}(k)|\mathcal{F}_k(t)] \right| \right\}^{2-2\iota} \\ \leq O(1) \left\{ \sum_{i=1}^{k-1} \rho^i \left[E \left| \varepsilon_{t-j-i-a} - E[\varepsilon_{t-j-i-a}|\mathcal{F}_k(t)] \right|^{2-2\iota} \right]^{\frac{1}{2-2\iota}} \right\}^{2-2\iota} \leq O(\rho^{k-j}), \quad (0.5) \end{aligned}$$

for some $\rho \in (0, 1)$. By (0.2)-(0.5), we can show that show that

$$E\left\|\frac{1}{\sqrt{h_t}}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma} - E\left[\frac{1}{\sqrt{h_t}}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}\middle| \mathcal{F}_k(t)\right]\right\|^{2-2\iota} = O(\rho^{k-j}), \quad (0.6)$$

for some $\iota, \rho \in (0, 1)$, where $j = 1, \dots, k$.

Using Lemma 7.1, we can show that there exists a $\iota \in (0, 1)$ such that

$$E\left|\frac{\varepsilon_{t-j}}{\sqrt{h_t}} - E\left[\frac{\varepsilon_{t-j}}{\sqrt{h_t}}\middle| \mathcal{F}_k(t)\right]\right|^{2\iota} = O(\rho^{k-j}).$$

By (0.1), $\varepsilon_{t-j} \sqrt{h_t} \leq 1/\sqrt{v_{hj}}$. Thus, for any $u \geq \iota$, we have

$$\begin{aligned} E\left|\frac{\varepsilon_{t-j}}{\sqrt{h_t}} - E\left[\frac{\varepsilon_{t-j}}{\sqrt{h_t}}\middle| \mathcal{F}_k(t)\right]\right|^{2u} &\leq O(v_{hj}^{-(u-\iota)}) E\left|\frac{\varepsilon_{t-j}}{\sqrt{h_t}} - E\left[\frac{\varepsilon_{t-j}}{\sqrt{h_t}}\middle| \mathcal{F}_k(t)\right]\right|^{2\iota} \\ &= O(v_{hj}^{-(u-\iota)} \rho^{k-j}). \end{aligned} \quad (0.7)$$

By (0.6) and (0.7), and using Hölder's inequality, we can show that

$$\begin{aligned} &E\left\|\frac{\varepsilon_{t-j}}{h_t}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma} - E\left[\frac{\varepsilon_{t-j}}{h_t}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}\middle| \mathcal{F}_k(t)\right]\right\|^{2-2\iota} \\ &\leq O(1)E\left\|\frac{\varepsilon_{t-j}}{\sqrt{h_t}}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma} - E\left[\frac{\varepsilon_{t-j}}{\sqrt{h_t}}\middle| \mathcal{F}_k(t)\right]E\left[\frac{1}{\sqrt{h_t}}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}\middle| \mathcal{F}_k(t)\right]\right\|^{2-2\iota} \\ &\leq O(v_{hj}^{-(1-\iota)})E\left\|\frac{1}{\sqrt{h_t}}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma} - E\left[\frac{1}{\sqrt{h_t}}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}\middle| \mathcal{F}_k(t)\right]\right\|^{2-2\iota} \\ &\quad + O(1)E\left\{\left\|\frac{1}{\sqrt{h_t}}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}\right\| \left\|\frac{\varepsilon_{t-j}}{\sqrt{h_t}} - E\left[\frac{\varepsilon_{t-j}}{\sqrt{h_t}}\middle| \mathcal{F}_k(t)\right]\right\|^{2-2\iota}\right\} \\ &= O(\rho^{k-j} v_{hj}^{-(1-\iota)}), \end{aligned} \quad (0.8)$$

where $j = 1, \dots, k$.

Since $\varepsilon_{t-j}/\sqrt{h_t} \leq 1/\sqrt{v_{hj}}$, for any $\iota \in (0, 1/2)$, there exist an $\rho_1 \in (0, 1)$ such that

$$E\left\|\sum_{j=k}^{\infty} v_{hj}\frac{\varepsilon_{t-j}}{h_t}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}\right\|^{2-2\iota} \leq E\left\|\sum_{j=k}^{\infty} \sqrt{v_{hj}}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}\right\|^{2-2\iota} = O(\rho_1^k). \quad (0.9)$$

By (0.1), (0.8)-(0.9) and Hölder's inequality, we have

$$\begin{aligned} &E\left\|\frac{1}{h_t}\frac{\partial h_t(\vartheta_0)}{\partial \gamma} - E\left[\frac{1}{h_t}\frac{\partial h_t(\vartheta_0)}{\partial \gamma}\middle| \mathcal{F}_k(t)\right]\right\|^{2-2\iota} \\ &\leq \left[O(1)\sum_{j=1}^{k-1} v_{hj} \left(E\left\|\frac{\varepsilon_{t-j}}{h_t}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma} - E\left[\frac{\varepsilon_{t-j}}{h_t}\frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma}\middle| \mathcal{F}_k(t)\right]\right\|^{2-2\iota}\right)^{\frac{1}{2-2\iota}} + O(\rho_1^k)\right]^{2-2\iota} \\ &= O(1)\left(\sum_{j=1}^{k-1} v_{hj}^{\iota} \rho^{k-j} + \rho_1^k\right) = O(\rho^k), \end{aligned}$$

for some $\rho \in (0, 1)$. By Lemma A.5(ii) of Ling (2007b), we have

$$\left\| \sum_{j=1}^{\infty} v_{hj} \frac{\varepsilon_{t-j}}{h_t} \frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma} \right\| \leq O(1) \sum_{j=1}^{\infty} \rho^j \frac{1}{\sqrt{h_t}} \left\| \frac{\partial \varepsilon_{t-j}(\vartheta_0)}{\partial \gamma} \right\| \leq \frac{C \xi_{\rho t}^{1-\tilde{\iota}}}{\sqrt{h_t}} \leq C \xi_{\tilde{\rho} t}^{1-\tilde{\iota}}. \quad (0.10)$$

Taking ι small enough such that $(1+3\iota)(1-\tilde{\iota}) < 1$ and using (0.10)-(0.13) and Cauchy-Schwarz inequality, we can show that

$$\begin{aligned} E \left\| \frac{1}{h_t} \frac{\partial h_t(\vartheta_0)}{\partial \gamma} - E \left[\frac{1}{h_t} \frac{\partial h_t(\vartheta_0)}{\partial \gamma} \middle| \mathcal{F}_k(t) \right] \right\|^{2+2\iota} \\ \leq E \xi_{\tilde{\rho} t}^{(1+3\iota)(1-\tilde{\iota})} \left\| \frac{1}{h_t} \frac{\partial h_t(\vartheta_0)}{\partial \gamma} - E \left[\frac{1}{h_t} \frac{\partial h_t(\vartheta_0)}{\partial \gamma} \middle| \mathcal{F}_k(t) \right] \right\|^{1-\iota} = O(\rho^k), \end{aligned} \quad (0.11)$$

where $E \xi_{\tilde{\rho} t-1}^{2(1+3\iota)(1-\tilde{\iota})} < \infty$ since $E|\varepsilon_t|^{2\iota} < \infty$ for any $\iota \in (0, 1)$. Similarly, we can show that

$$E \left\| \frac{1}{h_t} \frac{\partial h_t(\vartheta_0)}{\partial \delta} - E \left[\frac{1}{h_t} \frac{\partial h_t(\vartheta_0)}{\partial \delta} \middle| \mathcal{F}_k(t) \right] \right\|^{2+2\iota} = O(\rho^k). \quad (0.12)$$

By (0.12)-(0.13), we can show that (b) holds. This completes the proof. \square

Lemma 0.2. *If Assumptions 4.1-4.2 hold, then, for any $\iota \in (0, 1)$, it follows that*

- (a) $E|h_{1t} - h_{2t}|^\iota = o(1)$,
- (b) $E|\varepsilon_{1t} - \varepsilon_{2t}|^{2\iota} = o(1)$,
- (c) $E|y_{1t} - y_{2t}|^{2\iota} = o(1)$,

as $\|d\| \rightarrow 0$, where $h_{1t} = h_t(\theta_{i0})$ and $\varepsilon_{1t} = \varepsilon_t(\theta_{i0})$, $i = 1, 2$.

Proof. Let A_{vt} be defined as A_t in Lemma 0.1 when $\vartheta = \theta_{v0} = (\alpha_{v00}, \alpha_{v10}, \dots, \alpha_{vr0}, \beta_{v10}, \dots, \beta_{vs0})'$, $v = 1, 2$. By Theorem 2.1 of Ling (2007b), Assumption 4.2 implies that, for any $0 < \iota < 1$, there exists a $i_0 > 0$ such that

$$E \left\| \prod_{i=0}^{i_0-1} A_{vt-i} \right\|^\iota < 1/C,$$

where $C > 1$ is any given constant. Let $B_{vt} = \xi_t + \sum_{j=1}^{i_0-1} \prod_{r=0}^{j-1} A_{vt-r} \xi_{t-j}$ and $\tilde{A}_{vt} = \prod_{i=0}^{i_0-1} A_{vt-i}$, $v = 1, 2$. Then h_{vt} , $v = 1, 2$, have the expansions [see Ling and Li (1997)]:

$$h_{vt} = \alpha_{v00} \left[u' B_{vt} + u' \tilde{A}_{vt} H_{vt-i_0} \right],$$

where $H_{vt} = (\varepsilon_{vt}^2, \dots, \varepsilon_{vt-r+1}^2, h_{vt}, \dots, h_{vt-s+1})'$.

$$\begin{aligned} E|h_{1t} - h_{2t}|^\iota &\leq E|\alpha_{100} u' B_{1t} - \alpha_{200} u' B_{2t}|^\iota + \alpha_{100}' E\|\tilde{A}_{1t}\|^\iota E\|H_{1t-i_0} - H_{2t-i_0}\|^\iota \\ &\quad + E\|\alpha_{100} u' \tilde{A}_{1t} - \alpha_{200} u' \tilde{A}_{2t}\|^\iota E\|H_{2t-i_0}\|^\iota. \end{aligned}$$

By the stationarity of H_{1t} and H_{2t} , we have $E\|H_{2t-i_0}\|^\ell = C$, a constant, and

$$\begin{aligned} E\|H_{1t-i_0} - H_{2t-i_0}\|^\ell &= E\|H_{1t} - H_{2t}\|^\ell \\ &\leq \sum_{i=0}^{r-1} E|\varepsilon_{1t-i}^2 - \varepsilon_{2t-i}^2|^\ell + \sum_{i=0}^{s-1} E|h_{1t-i} - h_{2t-i}|^\ell \\ &= (rE|\eta_t|^{2\ell} + s)E|h_{1t} - h_{2t}|^\ell. \end{aligned}$$

Thus, taking i_0 large enough such that $\alpha_{100}^\ell(rE|\eta_t|^{2\ell} + s)E\|\tilde{A}_{1t}\|^\ell < 1$. By the previous two inequalities, we have

$$E|h_{1t} - h_{2t}|^\ell \leq \frac{E|\alpha_{100}u'B_{1t} - \alpha_{200}u'B_{2t}|^\ell + CE\|\alpha_{100}u'\tilde{A}_{1t} - \alpha_{200}u'\tilde{A}_{2t}\|^\ell}{1 - \alpha_{100}^\ell(rE|\eta_t|^{2\ell} + s)E\|\tilde{A}_{1t}\|^\ell} \rightarrow 0,$$

as $\|d\| \rightarrow 0$, i.e., (a) holds. (b) is directly from (a). For (c), when $2\ell \in (1, 2)$, using the expansion: $y_{vt} = \sum_{i=0}^{\infty} a_{vi}\varepsilon_{vt-i}$, $v = 1, 2$, by the monotonic convergence theorem, we have

$$E|y_{1t} - y_{2t}|^{2\ell} \leq \left[\sum_{i=0}^{\infty} |a_{1i}|(E|\varepsilon_{2t-i} - \varepsilon_{1t-i}|^{2\ell})^{\frac{1}{2\ell}} + \sum_{i=0}^{\infty} |a_{2i} - a_{1i}|(E|\varepsilon_{2t-i}|^{2\ell})^{\frac{1}{2\ell}} \right]^{2\ell} \rightarrow 0,$$

as $\|d\| \rightarrow 0$. Thus, (c) holds. This completes the proof. \square

Proof of Lemma 7.3. By (2.5) of Ling (2007b) and Assumption 4.2, we have

$$h_{vt} = \alpha_{v00}\beta_{v00}^{-1}(1) + \sum_{j=1}^{\infty} v_{vhj}\varepsilon_{vt-j}^2 \text{ and } \frac{\partial h_{vt}(\theta_{v0})}{\partial \gamma} = 2 \sum_{j=1}^{\infty} v_{vhj}\varepsilon_{vt-j} \frac{\partial \varepsilon_{vt-j}(\theta_{v0})}{\partial \gamma}, \quad (0.13)$$

where $v_{vhj} = O(\rho_1^j)$ and $v = 1, 2$. Similar to Lemma 0.2(c), we can show that, for any $1 < \ell < 2$,

$$E\left\|\frac{\partial \varepsilon_{1t}(\theta_{10})}{\partial \gamma} - \frac{\partial \varepsilon_{2t}(\theta_{20})}{\partial \gamma}\right\|^\ell = o(1), \quad (0.14)$$

as $\|d\| \rightarrow 0$. By Lemma 0.2, we can show that

$$\begin{aligned} &E\left|\frac{v_{1hj}\varepsilon_{1t-j}}{h_{1t}} - \frac{v_{2hj}\varepsilon_{2t-j}}{h_{2t}}\right|^\ell \\ &\leq \left\{C|v_{1hj} - v_{2hj}| + |v_{2hj}|(E|\varepsilon_{1t-j} - \varepsilon_{2t-j}|^\ell)^{\frac{1}{\ell}} + |v_{2hj}|\left[E\left|\varepsilon_{2t-j}\left(\frac{1}{h_{1t}} - \frac{1}{h_{2t}}\right)\right|^\ell\right]^{\frac{1}{\ell}}\right\}^\ell = o(1), \end{aligned}$$

uniformly in j . Take $\iota_1 \in (\iota, 2)$ and $v > 0$ such that $v\iota_1/(\iota_1 - \iota) < \iota$. By the previous equation and Hölder's inequality, we have

$$\begin{aligned} &E\left\{\left|\frac{v_{1hj}\varepsilon_{1t-j}}{h_{1t}} - \frac{v_{2hj}\varepsilon_{2t-j}}{h_{2t}}\right|^v \left\|\frac{\partial \varepsilon_{2t}(\theta_{20})}{\partial \gamma}\right\|^\ell\right\} \\ &\leq \left\{E\left|\frac{v_{1hj}\varepsilon_{1t-j}}{h_{1t}} - \frac{v_{2hj}\varepsilon_{2t-j}}{h_{2t}}\right|^{\frac{v\iota_1}{\iota_1-\iota}}\right\}^{\frac{\iota_1-\iota}{\iota_1}} \left\{E\left\|\frac{\partial \varepsilon_{2t}(\theta_{20})}{\partial \gamma}\right\|^{\iota_1}\right\}^{\frac{\iota}{\iota_1}} = o(1), \quad (0.15) \end{aligned}$$

uniformly in j . By (0.14), $\varepsilon_{vt-j}/\sqrt{h_{vt}} \leq 1/\sqrt{v_{vhj}}$. Furthermore, when $\iota \in (1, 2)$, by (0.14)-(0.16), we have

$$\begin{aligned}
& E \left\| \frac{1}{h_{1t}} \frac{\partial h_{1t}(\theta_{10})}{\partial \gamma} - \frac{1}{h_{2t}} \frac{\partial h_{2t}(\theta_{20})}{\partial \gamma} \right\|^\iota \\
& \leq \left[C \sum_{j=1}^{\infty} \sqrt{v_{1hj}} \left(E \left\| \frac{\partial \varepsilon_{1t}(\theta_{10})}{\partial \gamma} - \frac{\partial \varepsilon_{2t}(\theta_{20})}{\partial \gamma} \right\|^\iota \right)^{\frac{1}{\iota}} \right. \\
& \quad \left. + C \sum_{j=1}^{\infty} \left(E \left\{ \left| \frac{v_{1hj} \varepsilon_{1t-j}}{h_{1t}} - \frac{v_{2hj} \varepsilon_{2t-j}}{h_{2t}} \right|^\iota \left\| \frac{\partial \varepsilon_{2t}(\theta_{20})}{\partial \gamma} \right\|^\iota \right\} \right)^{\frac{1}{\iota}} \right]^\iota \\
& = \left[o(1) + C \sum_{j=1}^{\infty} (\sqrt{v_{1hj}} + \sqrt{v_{2hj}})^{\frac{\iota-\nu}{\iota}} \left(E \left\{ \left| \frac{v_{1hj} \varepsilon_{1t-j}}{h_{1t}} - \frac{v_{2hj} \varepsilon_{2t-j}}{h_{2t}} \right|^\nu \left\| \frac{\partial \varepsilon_{2t}(\theta_{20})}{\partial \gamma} \right\|^\iota \right\} \right)^{\frac{1}{\iota}} \right]^\iota \\
& = o(1). \tag{0.16}
\end{aligned}$$

By (0.14) and Lemma 5.4(ii) of Ling (2007b), there exists an $\tilde{\iota} \in (0, 1)$ such that

$$\left\| \frac{1}{h_{vt}} \frac{\partial h_{vt}(\theta_{v0})}{\partial \gamma} \right\| = \frac{C}{\sqrt{h_{vt}}} \sum_{j=1}^{\infty} \sqrt{v_{vhj}} \left\| \frac{\partial \varepsilon_{vt-j}(\theta_{v0})}{\partial \gamma} \right\| \leq \frac{C \sum_{i=0}^{\infty} \rho^i |\varepsilon_{vt-j}|}{\sqrt{h_{vt}}} \leq \xi_{v\tilde{\rho}t-1}^{1-\tilde{\iota}}, \tag{0.17}$$

where $\xi_{v\tilde{\rho}t} = \sum_{i=0}^{\infty} \tilde{\rho}^i |\varepsilon_{vt-j}|$, $\tilde{\rho} \in (0, 1)$ and $v = 1, 2$. Taking a small δ such that $2(1-\tilde{\iota})(1+\delta) < 2$, we can show that $E\xi_{1\tilde{\rho}t}^{2(1+\delta)(1-\tilde{\iota})}(d) \rightarrow E\xi_{2\tilde{\rho}t}^{2(1+\delta)(1-\tilde{\iota})} < \infty$. Thus, by (0.17)-(0.18), we have

$$\begin{aligned}
& E \left\| \frac{1}{h_{1t}} \frac{\partial h_{1t}(\theta_{10})}{\partial \gamma} - \frac{1}{h_{2t}} \frac{\partial h_{2t}(\theta_{20})}{\partial \gamma} \right\|^2 \\
& \leq E(\xi_{1\tilde{\rho}t-1}^{1-\tilde{\iota}} + \xi_{2\tilde{\rho}t-1}^{1-\tilde{\iota}})^{1+\delta} \left\| \frac{1}{h_{1t}} \frac{\partial h_{1t}(\theta_{10})}{\partial \gamma} - \frac{1}{h_{2t}} \frac{\partial h_{2t}(\theta_{20})}{\partial \gamma} \right\|^{1-\delta} \\
& \leq \left[E(\xi_{1\tilde{\rho}t-1}^{1-\tilde{\iota}} + \xi_{2\tilde{\rho}t-1}^{1-\tilde{\iota}})^{2(1+\delta)} E \left\| \frac{1}{h_{1t}} \frac{\partial h_{1t}(\theta_{10})}{\partial \gamma} - \frac{1}{h_{2t}} \frac{\partial h_{2t}(\theta_{20})}{\partial \gamma} \right\|^{2(1-\delta)} \right]^{\frac{1}{2}} = o(1).
\end{aligned}$$

This completes the proof. \square

References

1. LING, S. and LI, W.K. (1997) Fractional autoregressive integrated moving-average time series conditional heteroskedasticity. *J. Amer. Statist. Assoc.* **92**, 1184-1194.
2. BOUGEROL, P. and PICARD, N.M. (1992) Stationarity of GARCH processes and of some nonnegative timeseries. *J. Econometrics* **52**, 115-127.