Pricing multi-asset options with an external barrier

by

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Abstract

An external barrier of an option contract is a variable which determines whether the option is knocked in or out when the value of the variable is above or below some predetermined level, but itself is not the price of an asset which underlies the option. In this paper, we present analytic formulation for the valuation of European options on one or multiple assets with single external barrier, where the barrier level can be exponential. As the domain of the problem becomes semi-infinite due to the presence of the external barrier, we employ the method of images to find the Green function of the governing differential equation. An efficient and accurate implicit finite difference scheme is proposed for the numerical valuation of these barrier options.

Keywords: Multi-asset options, external barrier, Green's function, implicit finite difference schemes

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I. Introduction

Heynen and Kat (1994) have presented analytic valuation formulas for European style barrier options with single external barrier, where the external variable is not the price of the underlying asset. For example, the underlying asset may be a foreign stock and the external barrier may be the exchange rate. Such barrier options become nullified (out-options) or activated (in-options) when the value of the barrier variable hits some level either from above (down-options) or below (up-options). In this paper, we go beyond the scope of Heynen and Kat's formulas by allowing more than one underlying asset and the barrier level can be exponential. In the valuation of these options, we take the usual Black-Scholes assumptions. The diffusion processes for the prices of the assets which underly the option and the barrier variable are taken to be lognormal, and the correlation coefficients between these diffusion processes are all constant. Also, the volatility values and the riskless interest rate are assumed to be constant. Besides the presentation of analytic formulations, efficient implicit finite difference schemes are also proposed for the numerical valuation of these barrier options.

This paper is organized as follows. In the next section, we present the valuation formulas for European style options on one or multiple assets with single external barrier where the barrier level can be exponential. The transition density function of the underlying processes for the asset prices associated with the barrier feature is derived using the partial differential equation approach rather than the usual statistical approach. We solve for the Green function of the governing differential equation argumented with the homogeneous Dirichlet boundary condition corresponding to the absorbing barrier. The Green function can be recognized to be the discounted value of the transition density function. Since the domain of the problem becomes semi-infinite with the presence of the barrier, we derive the Green function using the method of images. The method of images resembles the reflection principle in statistics (Rich, 1994), but the analytic procedures involved using the method of images are more routine and direct. It is known that the binomial method cannot provide sufficient accuracy for the numerical valuation of barrier options (Ritchken, 1995). In Section III, we propose a class of fractional step implicit finite difference schemes for the accurate valuation of these barrier options, including those with American early exercise feature. The paper is ended with conclusive remarks in Section IV.

II. European options with single external barrier

Suppose there is an external barrier variable which determines whether the option is nullified ("out") or activated ("in") when the value of the barrier variable hits some prescribed level. For European style options, an in-option plus an out-option with the same expiration date, strike price and barrier will have the same terminal payoff as the corresponding ordinary option without the barrier. Therefore, it becomes necessary only to find values for out-options since those for in-option can be obtained without much added efforts.

Let B denote the external barrier variable, and $b(\tau) = b_0 e^{-\beta\tau}, \beta > 0$, be the exponential barrier level for B. The barrier variable is said to be an *up* (or *down*) barrier if the out-option becomes nullified when the value of the barrier variable hits the corresponding barrier level from *below* (or *above*), where the barrier level is *upstream* (or *downstream*) of the movement of the value taken by the barrier variable.

Formulation of the Green function

Assume that there are m underlying assets and let $S_i, i = 1, 2, \cdots, m$, denote the price of the underlying asset i. In the risk neutral world, the underlying asset prices are assumed to follow the lognormal diffusion processes

$$\frac{dS_i}{S_i} = (r - q_i)dt + \sigma_i dZ_i, \quad i = 1, 2, \cdots, m, \tag{1}$$

where r is the constant riskless interest rate and σ_i is the constant volatility of the stochastic process for S_i, q_i is the constant dividend yield of asset i. We let $n = m + 1$. The value of the barrier variable is also assumed to follow the lognormal process

$$\frac{dB}{B} = r dt + \sigma_n dZ_n, \tag{2}$$

where σ_n is the constant volatility of the stochastic process for B. Here, $dZ_i, i = 1, 2, \cdots, n$, are standardized Weiner processes, and let ρ_{ij} denote the correlation between dZ_i and dZ_j. It is reasonable to assume $|\rho_{ij}| < 1, i, j = 1, 2, \cdots, n, i \neq j$.

Suppose we apply the following transformation of variables

$$x_i = \frac{1}{\sigma_i} \ln S_i, i = 1, 2, \cdots, m, \quad \text{and} \quad x_n = \frac{1}{\sigma_n} \ln \frac{B}{b(\tau)}, \tag{3}$$

the governing partial differential equation for the value of an option on the m underlying assets and with downstream external barrier is given by

$$\frac{\partial V}{\partial \tau} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{m} \frac{r - q_i - \frac{\sigma_i^2}{2}}{\sigma_i} \frac{\partial V}{\partial x_i} + \frac{r - \beta - \frac{\sigma_n^2}{2}}{\sigma_n} \frac{\partial V}{\partial x_n} - rV,$$
$$-\infty < x_i < \infty, i = 1, 2, \cdots, m, \quad 0 < x_n < \infty, \tau > 0. \tag{4}$$

(The interval of definition for x_n becomes $-\infty < x_n < 0$ when the barrier is upstream.) We would like to find the Green function for the above differential equation which satisfies the homogeneous Dirichlet boundary condition at $x_n = 0$ and vanishing far field boundary conditions at $x_i \to \infty, i = 1, 2, \cdots, n$ and $x_i \to -\infty, i = 1, 2, \cdots, m$. First, we find the fundamental solution for the above differential equation in the infinite domain: $-\infty < x_1, x_2, \cdots, x_n < \infty$; then use the method of images to find the corresponding Green function in the semi-infinite domain: $-\infty < x_1, \cdots, x_m < \infty, 0 < x_n < \infty$. The fundamental solution in the infinite domain is found to be (see Appendix A)

$$g(x_1, x_2, \cdots x_n) = \frac{e^{-r\tau}}{(2\pi\tau)^{n/2}} \frac{1}{\sqrt{\det R}} \exp\left(-\frac{1}{2\tau} \mathbf{y}^T R^{-1} \mathbf{y}\right), -\infty < x_1, x_2, \cdots, x_n < \infty. \tag{5}$$

Here, R is a $n \times n$ symmetric matrix whose entries are $R_{ij} = \rho_{ij}, i, j = 1, 2, \cdots, n$, and $\mathbf{y} = (y_1 \ y_2 \cdots y_n)^T$ where

$$y_i = x_i + \frac{\left(r - q_i - \frac{\sigma_i^2}{2}\right)\tau}{\sigma_i}, i = 1, 2, \cdots, m, \quad y_n = x_n + \frac{\left(r - \beta - \frac{\sigma_n^2}{2}\right)\tau}{\sigma_n}. \tag{6}$$

Also, we write $\mathbf{x} = (x_1\ x_2 \cdots x_n)^T, \boldsymbol{\xi} = (\xi_1\ \xi_2 \cdots \xi_n)^T$ and $\boldsymbol{\eta} = (\eta_1\ \eta_2 \cdots \eta_n)^T$. Corresponding to the homogeneous boundary condition at $x_n = 0$, the Green function for equation (4) defined in the semi-infinite domain can be expressed as

$$\begin{aligned} G_n(\mathbf{x}, \tau; \boldsymbol{\xi}, u) =& \frac{e^{-r(\tau-u)}}{[2\pi(\tau-u)]^{n/2}} \frac{1}{\sqrt{\det R}} \left[\exp\left(-\frac{1}{2\tau} (\mathbf{y}-\boldsymbol{\xi})^T R^{-1} (\mathbf{y}-\boldsymbol{\xi}) \right) \right. \\ & \left. -H(\boldsymbol{\xi}) \exp\left(-\frac{1}{2\tau} (\mathbf{y}-\boldsymbol{\eta})^T R^{-1} (\mathbf{y}-\boldsymbol{\eta}) \right) \right]. \end{aligned} \tag{7}$$

Here, $\boldsymbol{\eta}$ is to be determined in term of $\boldsymbol{\xi}$ such that the homogeneous boundary condition at $x_n = 0$ is satisfied. The second term in equation (7) can be considered as the contribution from the negative source at $\boldsymbol{\eta}$ which is the image of $\boldsymbol{\xi}$ with respect to the boundary mirror $x_n = 0$. The present approach of finding the Green function is called the *method of images*. The above Green function is guaranteed to satisfy the differential equation and all required boundary conditions. It can be shown that $\eta_1, \eta_2, \cdots, \eta_n$ are obtained by solving the following system of algebraic equations (see Appendix B)

$$\begin{cases} \boldsymbol{\xi}^T R^{-1}\ \boldsymbol{\xi} = \boldsymbol{\eta}^T R^{-1}\ \boldsymbol{\eta} \\ \mathbf{e}_i^T R^{-1}\ \boldsymbol{\xi} = \mathbf{e}_i^T R^{-1}\ \boldsymbol{\eta}, \quad i = 1, 2, \cdots, m, \end{cases} \tag{8}$$

where $\mathbf{e}_i$ is the ith coordinate vector. The solution to the above algebraic system is found to be

$$\boldsymbol{\eta} = \boldsymbol{\xi} - 2\xi_n R \mathbf{e}_n. \tag{9}$$

From equation (B.4), the corresponding value for $H(\boldsymbol{\xi})$ then becomes

$$H(\boldsymbol{\xi}) = \exp\left(\frac{r - \beta - \frac{\sigma_n^2}{2}}{\sigma_n} \mathbf{e}_n^T R^{-1} (\boldsymbol{\xi} - \boldsymbol{\eta}) \right) = \exp\left(2 \frac{r - \beta - \frac{\sigma_n^2}{2}}{\sigma_n} \xi_n \right). \tag{10}$$

The Green function can be expressed as

$$\begin{aligned} G_n(\mathbf{x}, \tau; \boldsymbol{\xi}, u) \ =& \frac{e^{-r(\tau-u)}}{[2\pi(\tau-u)^{n/2}} \frac{1}{\sqrt{\det R}} \left[\exp\left(-\frac{1}{2\tau} (\mathbf{y}-\boldsymbol{\xi})^T R^{-1} (\mathbf{y}-\boldsymbol{\xi}) \right) \right. \\ & \left. - \exp\left(2 \frac{r - \beta - \frac{\sigma_n^2}{2}}{\sigma_n} \xi_n \right) \exp\left(-\frac{1}{2\tau} (\mathbf{y} + 2\xi_n R \mathbf{e}_n - \boldsymbol{\xi})^T R^{-1} (\mathbf{y} + 2\xi_n R \mathbf{e}_n - \boldsymbol{\xi}) \right) \right]. \end{aligned} \tag{11}$$

The above Green function is valid for external barrier which is either downstream or upstream. To facilitate the subsequent derivation of the option formulas, it is more convenient to use an alternative form of the Green function as given by equation (C.3) in Appendix C.

We restrict the later discussion to options where the number of underlying assets is either one or two. The generalization to arbitrary number of underlying assets is quite straightforward.

Barrier options on one underlying asset

We consider the valuation of European out-options on one underlying asset and with single external barrier. Let θ denote a binary variable which equals 1 or -1 depending on respectively whether the barrier is downstream or upstream. The interval of definition for ξ_2 is $(0, \infty)$ or $(-\infty, 0)$

corresponding to $\theta = 1$ or -1 respectively. Also, let η be a binary variable which equals 1 or -1 corresponding to the option being a call or a put respectively. The terminal payoff for any type of option, either a call or a put, can be represented by $\max(\eta(S_{1,T} - X), 0)$. Here X is the strike price and $S_{1,T}$ is the asset price at expiry. The price of an out-option with an external barrier is given by

$$\begin{aligned} V(S,B,\tau) &= \int_0^\infty \int_{\frac{\eta}{\sigma_1}\ln X}^\infty \eta(S_{1,T} - X) G_2(x_1, x_2, \tau; \xi_1, \xi_2, 0) d(\eta\xi_1) d(\theta\xi_2) \\ &= \eta S_1 e^{q_1\tau}\left[N_2(\eta d_1, -\theta e_1; \eta\theta\rho_{12}) - \exp(-2(\alpha + \rho_{12}\sigma_1)x_2) N_2(\eta d_1', -\theta e_1'; \eta\theta\rho_{12})\right] \\ &\quad - \eta X e^{-r\tau}\left[N_2(\eta d_2, -\theta e_2; \eta\theta\rho_{12}) - \exp(-2\alpha x_2) N_2(\eta d_2', -\theta e_2'; \eta\theta\rho_{12})\right], \end{aligned} \tag{12}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_1}{X} + \left(r - q_1 + \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, d_2 = d_1 - \sigma_1\sqrt{\tau}, d_1' = d_1 - \frac{2\rho_{12}x_2}{\sqrt{\tau}}, d_2' = d_2 - \frac{2\rho_{12}x_2}{\sqrt{\tau}}, \\ e_1 &= -\frac{x_2 + (\alpha + \rho_{12}\sigma_1)\tau}{\sqrt{\tau}}, e_2 = -\frac{x_2 + \alpha\tau}{\sqrt{\tau}}, e_1' = e_1 + \frac{2x_2}{\sqrt{\tau}}, e_2' = e_2 + \frac{2x_2}{\sqrt{\tau}}, \\ \alpha &= \frac{r - \beta - \frac{\sigma_2^2}{2}}{\sigma_2}, x_2 = \frac{1}{\sigma_2}\ln\frac{B}{b(\tau)}. \end{aligned} \tag{13}$$

A few corrections have been made to the formula given by Heynen and Kat (1994).

Options on the extremum of two risky assets with an external barrier

As an illustration, we consider the valuation of a European call option on the maximum of two risky assets and with a down-and-out external barrier. The payoff function of the present European option is given by $c^2_{\max}(S_1, S_2, B, 0) = \max(\max(S_{1,T}, S_{2,T}) - X, 0)$, where X is the strike price and $S_{i,T}$ is the price of asset i at expiry, $i = 1, 2$. The option remains alive only when $B > b(\tau) = b_0 e^{-B\tau}$ throughout the whole life of the option. The background on the use of options on the extremum of several risky assets can be found in Stulz's paper (1982). In what follows, we adopt similar derivation procedure proposed by Johnson (1987).

Formally speaking, the price of the present European call is given by

$$c^2_{\max}(S_1, S_2, B, \tau) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty c^2_{\max}(S_1, S_2, B, 0) G_3(x_1, x_2, x_3, \tau; \xi_1\xi_2\xi_3, 0) d\xi_1 d\xi_2 d\xi_3. \tag{14}$$

Like other all price formulas, the above call price contains the term which represents the discounted expectation of the cash paid out conditional on the exercising of the call. This cash payment is given by

$$-Xe^{-r\tau}[1 - P_r(S_{1,T}, S_{2,T} < X, B > b(\tau) \text{ for all } \tau)],$$

where $1 - P_r(S_{1,T}, S_{2,T} < X, B > b(\tau) \text{ for all } \tau)$ gives the probability that the call will be exercised at expiry. The probability term can be expressed as

$$\begin{aligned} & e^{-r\tau} P_r(S_{1,T}, S_{2,T} < X, B > b(\tau) \text{ for all } \tau) \\ &= \int_0^\infty \int_{-\infty}^{\frac{1}{\sigma_2}\ln X} \int_{-\infty}^{\frac{1}{\sigma_1}\ln X} G_3(x_1, x_2, x_3, \tau; \xi_1, \xi_2, \xi_3, 0) d\xi_1 d\xi_2 d\xi_3 \end{aligned} \tag{15}$$

where $\xi_i = \dfrac{1}{\sigma_i}\ln S_{i,T},\ i=1,2,$ and $\xi_3 = \dfrac{1}{\sigma_2}\ln\dfrac{B}{b(\tau)}$. By defining

$$u_i = \frac{\xi_i - y_i}{\sqrt{\tau}},\ \ v_i = \frac{\xi_i - y_i + 2\rho_{i3}x_3}{\sqrt{\tau}}, i = 1,2,3, \tag{16}$$

the above probability term can be expressed as

$$\begin{aligned}
e^{-r\tau}&\left[\int_{-\frac{y_3}{\sqrt{\tau}}}^{\infty}\int_{-\infty}^{\left(\frac{\ln X}{\sigma_2}-y_2\right)/\sqrt{\tau}}\int_{-\infty}^{\left(\frac{\ln X}{\sigma_1}-y_1\right)/\sqrt{\tau}} n_3(u_1,u_2,u_3;\rho_{12},\rho_{13},\rho_{23})du_1du_2du_3\right.\\
&\left. - e^{-2\alpha x_3}\int_{\frac{-y_3+2x_3}{\sqrt{\tau}}}^{\infty}\int_{-\infty}^{\left(\frac{\ln X}{\sigma_2}-y_2+2\rho_{23}x_3\right)/\sqrt{\tau}}\int_{-\infty}^{\left(\frac{\ln X}{\sigma_1}-y_1+2\rho_{13}x_3\right)/\sqrt{\tau}} n_3(v_1,v_2,v_3;\rho_{12},\rho_{13},\rho_{23})dv_1dv_2dv_3\right]\\
= e^{-r\tau}&\left[N_3\left(-d,-e,f;\rho_{12},-\rho_{13},-\rho_{23}\right) - \exp(-2\alpha x_3)N_3\left(-d',-e',f';\rho_{12},-\rho_{13},-\rho_{23}\right)\right]
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
d =& \frac{\ln\frac{S_1}{X} + \left(r - q_1 - \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, e = \frac{\ln\frac{S_2}{X} + \left(r - q_2 - \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}}, f = \frac{\ln\frac{B}{b(\tau)} + \left(r - \beta - \frac{\sigma_3^2}{2}\right)\tau}{\sigma_3\sqrt{\tau}},\\
d' =& d - \frac{2\rho_{13}x_3}{\sqrt{\tau}}, e' = e - \frac{2\rho_{23}x_3}{\sqrt{\tau}}, f' = f - \frac{2x_3}{\sqrt{\tau}},\\
\alpha =& \frac{r-\beta-\frac{\sigma_3^2}{2}}{\sigma_3}, x_3 = \frac{1}{\sigma_3}\ln\frac{B}{b(\tau)}.
\end{aligned} \tag{18}$$

Suppose $S_{1,T}$ is the larger of the two asset prices at expiry. Following the procedure proposed by Johnson, we define

$$\tilde{S}_1 = \frac{Xe^{-r\tau}}{S_1e^{q_1\tau}},\quad \tilde{S}_2 = \frac{S_2e^{q_2\tau}}{S_1e^{q_1\tau}},\quad \tilde{S}_3 = \frac{B}{b(\tau)}, \tag{19}$$

and let $\tilde{\sigma}_i^2$ to denote the variance rate of the normal diffusion process for $\ln\tilde{S}_i, i = 1,2,3$. These variance rates are found to be

$$\tilde{\sigma}_1^2 = \sigma_1^2,\quad \tilde{\sigma}_2^2 = \sigma_1^2 - \rho_{12}\sigma_1\sigma_2 + \sigma_2^2,\quad \tilde{\sigma}_3^2 = \sigma_3^2. \tag{20}$$

Let $\tilde{\rho}_{ij}$ denote the correlation coefficient between the Wiener processes associated with the logorithms of $\tilde{S}_i$ and $\tilde{S}_j, i \neq j$. It can be shown that

$$\tilde{\rho}_{12} = \frac{\sigma_1 - \rho_{12}\sigma_2}{\tilde{\sigma}_2},\quad \tilde{\rho}_{13} = -\rho_{13},\quad \tilde{\rho}_{23} = \frac{\sigma_2\rho_{23} - \sigma_1\rho_{13}}{\tilde{\sigma}_2}. \tag{21}$$

Similar to the probability calculation given in equation (17), the expectation of S_1 in the risk neutral world, given that $\tilde{S}_1 < 1, \tilde{S}_2 < 1$ at expiry and $\tilde{S}_3 < 1$ for all τ, is found to be

$$S_1e^{q_1\tau}\left[N_3(d_1,e_1,f_1;\tilde{\rho}_{12},-\tilde{\rho}_{13},-\tilde{\rho}_{23}) - \exp(-2(\alpha+\tilde{\rho}_{13}\sigma_1)x_3)N_3(d_1',e_1',f_1';\tilde{\rho}_{12},-\tilde{\rho}_{13},-\tilde{\rho}_{23})\right]$$

where

$$d_1 = \frac{\ln \frac{S_1}{X} + \left(r - q_1 + \frac{\sigma_1^2}{2}\right)\tau}{\sigma_1\sqrt{\tau}}, e_1 = \frac{\ln \frac{S_1}{S_2} + \left(q_2 - q_1 + \frac{\widetilde{\sigma_2^2}}{2}\right)\tau}{\tilde{\sigma}_2\sqrt{\tau}}, f_1 = \frac{\ln \frac{B}{b(\tau)} + \left(r - \beta - \frac{\sigma_3^2}{2} + \tilde{\rho}_{13}\sigma_1\sigma_3\right)\tau}{\sigma_3\sqrt{\tau}},$$
$$d_1' = d_1 - \frac{2\tilde{\rho}_{13}x_3}{\sqrt{\tau}}, \quad e_1' = e_1 - \frac{2\tilde{\rho}_{23}x_3}{\sqrt{\tau}}, \quad f_1' = f_1 - \frac{2x_3}{\sqrt{\tau}}. \tag{22}$$

Next, suppose $S_{2,T}$ is the larger of the two asset prices at expiry. Accordingly, we define

$$\hat{S}_1 = \frac{S_1 e^{q_1\tau}}{S_2 e^{q_2\tau}}, \quad \hat{S}_2 = \frac{Xe^{-r\tau}}{S_2 e^{q_2\tau}}, \quad \hat{S}_3 = \frac{B}{b(\tau)}. \tag{23}$$

The variance rates for $\ln \hat{S}_i$, $i = 1, 2, 3$, are found to be

$$\hat{\sigma}_1^2 = \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2, \quad \hat{\sigma}_2^2 = \sigma_2^2, \quad \hat{\sigma}_3^2 = \sigma_3^2, \tag{24}$$

and the corresponding correlation coefficients between $\ln \hat{S}_i$ and $\ln \hat{S}_j$, $i \neq j$, are given by

$$\hat{\rho}_{12} = \frac{\sigma_2 - \rho_{12}\sigma_1}{\hat{\sigma}_1}, \quad \hat{\rho}_{13} = \frac{\sigma_1\rho_{13} - \sigma_2\rho_{23}}{\hat{\sigma}_1}, \quad \hat{\rho}_{23} = -\rho_{23}. \tag{25}$$

The expectation of S_2 in the risk neutral world, given that $\hat{S}_1 < 1$, $\hat{S}_2 < 1$ at expiry and $\hat{S}_3 < 1$ for all τ, is found to be

$$S_2 e^{q_2\tau}\left[N_3(d_2, e_2, f_2; \hat{\rho}_{12}, -\hat{\rho}_{13}, -\hat{\rho}_{23}) - \exp(-2(\alpha + \hat{\rho}_{23}\sigma_2)x_3)N_3(d_2', e_2', f_2'; \hat{\rho}_{12}, -\hat{\rho}_{13}, -\hat{\rho}_{23})\right]$$

where

$$d_2 = \frac{\ln \frac{S_2}{S_1} + \left(q_1 - q_2 + \hat{\sigma}_1^2\right)\tau}{\hat{\sigma}_1\sqrt{\tau}}, \quad e_2 = \frac{\ln \frac{S_2}{X} + \left(r - q_2 + \frac{\sigma_2^2}{2}\right)\tau}{\sigma_2\sqrt{\tau}},$$
$$f_2 = \frac{\ln \frac{B}{b(\tau)} + \left(r - \beta - \frac{\sigma_3^2}{2} + \hat{\rho}_{23}\sigma_2\sigma_3\right)\tau}{\sigma_3\sqrt{\tau}}, \quad d_2' = d_2 - \frac{2\hat{\rho}_{13}x_3}{\sqrt{\tau}}, \quad e_2' = e_2 - \frac{2\hat{\rho}_{23}x_3}{\sqrt{\tau}}, \quad f_2' = f_2 - \frac{2x_3}{\sqrt{\tau}}. \tag{26}$$

Combining the results, the price of the European call option on the maximum of two risky assets is given by

$$\begin{aligned}
&c_{\max}^2(S_1, S_2, B, \tau) \\
&= S_1 e^{q_1\tau}[N_3(d_1, e_1, f_1; \tilde{\rho}_{12}, -\tilde{\rho}_{13}, -\tilde{\rho}_{23}) - \exp(-2(\alpha + \tilde{\rho}_{13}\sigma_1)x_3)N_3(d_1', e_1', f_1'; \tilde{\rho}_{12}, -\tilde{\rho}_{13}, -\tilde{\rho}_{23}) \\
&\quad + S_2 e^{q_2\tau}[N_3(d_2, e_2, f_2; \hat{\rho}_{12}, -\hat{\rho}_{13}, -\hat{\rho}_{23}) - \exp(-2(\alpha + \hat{\rho}_{23}\sigma_2)x_3)N_3(d_2', e_2', f_2'; \hat{\rho}_{12}, -\hat{\rho}_{13}, -\hat{\rho}_{23})] \\
&\quad + Xe^{-r\tau}\{1 - [N_3(-d, -e, f; \rho_{12}, -\rho_{13}, -\rho_{23}) - \exp(-2\alpha x_3)N_3(-d', -e', f'; \rho_{12}, -\rho_{13}, -\rho_{23})]\}.
\end{aligned} \tag{27}$$

The prices of other European options on the extremum of several risky assets and with an external barrier can be derived in similar manner.

III. Numerical algorithms

It has been fortunate that analytic price formulas are available for the above European options with an external barrier. However, in most other option models with similar nature, the corresponding price formula may appear to be too cumbersome or unavailable. A good example is a single-asset European option with both internal (dependent on the underlying asset price) and external barriers, whose analytic price formula involves an infinite series since infinite number of images will be generated based on the present framework of method of images. Also, no analytic price formula would be available for multi-asset options when the terminal payoff function is a linear function of the underlying asset prices. In these cases, one has to resort to numerical method for valuation of the option models.

The popular approaches for numerical schemes are the binomial method and finite difference method. The binomial method may not be a good choice for the present external barrier option models since the binomial method is well known to have difficulty to cope with the presence of barriers (Ritchken, 1995). In this paper, we propose a fractional step finite difference schemes for the numerical valuation of European options with an external barrier. The proposed finite difference scheme can be considered as fractional splitting of the second order time accurate Crank-Nichoson scheme. The advancement in each time step is split into n stages, where n is the spatial dimension of the governing equation. In every stage, only the implicit differencing of spatial derivative terms in one of the spatial variables (one by one in turn) is involved; and so only the solution of a tridiagonal system of equations is required at each stage.

We illustrate the splitting procedure through the valuation of the following option model: a European call option on a single asset with an external down-and-out barrier and no rebate. The governing equation of the call is given by

$$\frac{\partial c}{\partial \tau} = \frac{1}{2}\left(\frac{\partial^2 c}{\partial x_1^2} + 2\rho_{12}\frac{\partial c^2}{\partial x_1 \partial x_2} + \frac{\partial^2 c}{\partial x_2^2}\right) + \frac{r - q_1 - \frac{\sigma_1^2}{2}}{\sigma_1}\frac{\partial c}{\partial x_1} + \frac{r - \beta - \frac{\sigma_2^2}{2}}{\sigma_2}\frac{\partial c}{\partial x_2} - rc, \tag{28}$$
$$-\infty < x_1 < \infty, \quad 0 < x_2 < \infty, \quad \tau > 0,$$

where

$$x_1 = \frac{1}{\sigma_1}\ln S, \quad x_2 = \frac{1}{\sigma_2}\ln\frac{B}{b(\tau)}, \quad b(\tau) = b_0 e^{-\beta\tau}, \quad \beta > 0. \tag{29}$$

The boundary conditions at $S = 0$ and $B = b(\tau)$ can be deduced to be

$$\lim_{x_1 \to -\infty} c(x_1, x_2, \tau) = 0 \quad \text{and} \quad c(x_1, 0, \tau) = 0. \tag{30}$$

When B attains sufficiently large value, it becomes much unlikely that the down-and-out barrier is breached. Hence, the far field boundary condition at large value of B is

$$\lim_{x_2 \to \infty} c(x_1, x_2, \tau) = c_E(x_1, \tau), \tag{31}$$

where $c_E(x_1, \tau)$ is the price formula for the corresponding European option without the barrier. It is slightly more trickly to deduce the far field boundary condition at large value of S. We choose the following boundary condition

$$\begin{aligned}\lim_{x_1 \to \infty} c(x_1, x_2, \tau) &= (e^{\sigma_1 x_1 - q_1\tau} - Xe^{-r\tau})p_r(x_2 > 0) \\ &= (e^{\sigma_1 x_1 - q_1\tau} - Xe^{-r\tau})\left[N(\bar{d}_1) - e^{-\frac{2\alpha x_2}{\sigma_2}}N(\bar{d}_2)\right]\end{aligned} \tag{32}$$

where

$$\bar{d}_1 = \frac{\ln \frac{B}{b(\tau)} + \alpha\tau}{\sigma_2\sqrt{\tau}}, \quad \bar{d}_2 = \bar{d}_1 - \frac{2x_2}{\sqrt{\tau}}. \tag{33}$$

The initial condition is simply the terminal payoff function of the call, namely,

$$c(x_1, x_2, 0) = \max(e^{\sigma_1 x_1} - X, 0). \tag{34}$$

The computational procedures of the fractional step finite difference scheme is outlined in Appendix D.

We test the proposed fractional step finite difference algorithm and compare its performance with the binomial method proposed by Kamrad and Ritchken (1991). The accuracy of the numerical schemes is assessed based on the exact value obtained by the direct evaluation of the analytic formula (see equation (12)). In addition, we examine the order of convergence of the numerical schemes.

Case (i): barrier variable value is far from the down-and-out barrier

The parameters of the option model are: $B = 1.0$, $b = 0.5$, $r = 0.05$, $q_1 = 0.02$, $\beta = 0$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, $\rho_{12} = 0.5$, $X = 1.0$, $\tau = 1.0$. We find the option value at varying asset values using the numerical schemes and compare the computed values with the exact values. Summing the square of these errors at different asset prices and taking the square root gives the root mean squared error (RMSE). We attempt to deduce the level of accuracy of the numerical schemes and their rate of convergence by performing the calculations at varying time steps and comparing the RMSE values. The numerical results are tabulated in Table 1. Roughly speaking, we observe that when the number of time steps is doubled, the RMSE is halved in binomial calculations but reduced by a factor of one fourth in finite difference calculations. These observations reveal that the finite difference scheme is second order time accurate while the binomial method is only first order time accurate.

Case (ii): barrier variable value is close to the down-and-out barrier

The parameters of the option model are the same as that in case (i), except that now $b = 0.9$, which is closer to the barrier variable value $B = 1.0$. The numerical results for the RMSE values are also tabulated in Table 1. We observe that the accuracy of the binomial method cannot be improved with increasing number of time steps when the barrier variable value is close to the down-and-out barrier, but the finite difference scheme is shown to remain to be second order time accurate.

IV Conclusions

We have developed a general approach for deriving analytic formulas for the pricing of European multi-asset options with an external barrier. The method of image is shown to be an effective technique for finding the transition density function of the asset price movement with the presence of an external barrier. A second order time accurate fractional step finite difference scheme is proposed. The computational efficiency of the difference scheme is maintained through the splitting procedure. The finite difference scheme is shown to outperform the binomial method, in particular, when the value of the barrier variable is close to the knock out barrier.

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Appendix A

We consider the fundamental solution of the following n-dimensional parabolic equation defined on the infinite domain

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1}\frac{r-q_i-\frac{\sigma_i^2}{2}}{\sigma_i}\frac{\partial V}{\partial x_i} + \frac{r-\beta-\frac{\sigma_n^2}{2}}{\sigma_n}\frac{\partial V}{\partial x_n} - rV, \tag{A.1}$$
$$-\infty < x_i < \infty, \quad i = 1, 2, \cdots, n.$$

Define a new set of variables

$$y_i = x_i + \frac{1}{\sigma_i}\left(r - q_i - \frac{\sigma_i^2}{2}\right)\tau, \quad i = 1, 2, \cdots, n-1, \tag{A.2a}$$

$$y_n = x_n + \frac{1}{\sigma_n}\left(r - \beta - \frac{\sigma_n^2}{2}\right)\tau \tag{A.2b}$$

and

$$\phi = e^{r\tau}V, \tag{A.2c}$$

then the above equation can be transformed into

$$\frac{\partial \phi}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 \phi}{\partial y_i \partial y_j} = \frac{1}{2}\begin{pmatrix}\frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \cdots & \frac{\partial}{\partial y_n}\end{pmatrix} R \begin{pmatrix}\frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \\ \vdots \\ \frac{\partial}{\partial y_n}\end{pmatrix}\phi, \quad -\infty < y_i < \infty, \tag{A.3}$$

where R is a $n \times n$ symmetric matrix whose entries are $R_{ij} = \rho_{ij}, i, j = 1, 2, \cdots, n$. Provided $|\rho_{ij}| < 1, i \neq j$, then R is positive definite. As a consequence, there exists an orthonormal matrix Q such that

$$Q^T R Q = \Lambda \tag{A.4}$$

where Λ is a diagonal matrix whose entries are the eignevalues of R. Since R is positive definite, all eigenvalues of R are positive. We apply the following linear transformation of variables

$$(z_1\ z_2 \cdots z_n)^T = \Lambda^{-\frac{1}{2}}Q^T(y_1\ y_2 \cdots y_n)^T \tag{A.5a}$$

so that

$$\left(\frac{\partial}{\partial z_1}\ \frac{\partial}{\partial z_2}\ \cdots \frac{\partial}{\partial z_n}\right)^T = \Lambda^{\frac{1}{2}}Q^T\left(\frac{\partial}{\partial y_1}\ \frac{\partial}{\partial y_2}\ \cdots \frac{\partial}{\partial y_n}\right)^T, \tag{A.5b}$$

where $\Lambda^{-\frac{1}{2}}$ denotes the inverse of the positive square root of the diagonal matrix Λ. Substituting the transformation (A.5b) into equation (A.3), we obtain the prototype n-dimensional diffusion equation

$$\frac{\partial \phi}{\partial \tau} = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \phi}{\partial z_i^2}. \tag{A.6}$$

The fundamental solution to equation (A.6) in the infinite domain is known to be

$$\phi(z_1, z_2, \cdots, z_n, \tau) = \frac{1}{(2\pi\tau)^{\frac{n}{2}}} \exp\left(-\frac{z_1^2 + z_2^2 + \cdots + z_n^2}{2\tau}\right), \quad -\infty < z_i < \infty, i = 1, 2, \cdots, n. \tag{A.7}$$

Let $\mathbf{y} = (y_1 \; y_2 \cdots y_n)^T$ and observe that $z_1^2 + z_2^2 + \cdots + z_n^2 = \mathbf{y}^T R^{-1} \mathbf{y}$ and the Jacobian of the transformation (A.5a) is $\frac{1}{\sqrt{\det R}}$. The above fundamental solution can be rewritten as

$$\phi(y_1, y_2, \cdots, y_n, \tau) = \frac{1}{(2\pi\tau)^{\frac{n}{2}}} \frac{1}{\sqrt{\det\ R}} \exp\left(-\frac{1}{2\tau}\, \mathbf{y}^T R^{-1} \mathbf{y}\right), \quad -\infty < y_1, y_2, \cdots, y_n < \infty. \tag{A.8}$$

Appendix B

We would like to find $\boldsymbol{\eta}$ in term $\boldsymbol{\xi}$ such that $G(\mathbf{x}, \tau; \boldsymbol{\xi}, u) = 0$ at $x_n = 0$. Observe that in general the following quantity

$$F = \left.\frac{\exp\left(-\frac{1}{2\tau}(\mathbf{y} - \boldsymbol{\xi})^T R^{-1}(\mathbf{y} - \boldsymbol{\xi})\right)}{\exp\left(-\frac{1}{2\tau}(\mathbf{y} - \boldsymbol{\eta})^T R^{-1}(\mathbf{y} - \boldsymbol{\eta})\right)}\right|_{x_n = 0} \tag{B.1}$$

is a function of $\boldsymbol{\xi}$ and τ since $\mathbf{y}$ contains τ. We determine $\boldsymbol{\eta}$ such that F is a function of $\boldsymbol{\xi}$ only; otherwise, the Green function cannot be expressed in the form given in equation (7). Now consider

$$\begin{aligned} &-\frac{1}{2\tau}\left[(\mathbf{y} - \boldsymbol{\xi})^T R^{-1}(\mathbf{y} - \boldsymbol{\xi}) - (\mathbf{y} - \boldsymbol{\eta})^T R^{-1}(\mathbf{y} - \boldsymbol{\eta})\right] \\ =& -\frac{1}{2\tau}\left\{\left(\boldsymbol{\xi}^T R^{-1}\boldsymbol{\xi} - \boldsymbol{\eta}^T R^{-1}\boldsymbol{\eta}\right) - 2\left[y_1 \mathbf{e}_1^T R^{-1}(\boldsymbol{\xi} - \boldsymbol{\eta}) + \cdots + y_{n-1}\mathbf{e}_{n-1}^T R^{-1}(\boldsymbol{\xi} - \boldsymbol{\eta}) + y_n \mathbf{e}_n^T R^{-1}(\boldsymbol{\xi} - \boldsymbol{\eta})\right]\right\}, \end{aligned} \tag{B.2}$$

and observe that at $x_n = 0$, the last term $\frac{1}{\tau} y_n \mathbf{e}_n^T R^{-1}(\boldsymbol{\xi} - \boldsymbol{\eta})$ becomes $\frac{r - \beta - \frac{\sigma_n^2}{2}}{\sigma_n}\mathbf{e}_n^T R^{-1}(\boldsymbol{\xi} - \boldsymbol{\eta})$, which is independent of τ. Hence, it becomes possible to make F to be independent of τ by choosing $\eta_1, \eta_2, \cdots, \eta_n$ such that

$$\begin{cases} \boldsymbol{\xi}^T R^{-1}\boldsymbol{\xi} = \boldsymbol{\eta}^T R^{-1}\boldsymbol{\eta} \\ \mathbf{e}_i^T R^{-1}(\boldsymbol{\xi} - \boldsymbol{\eta}) = 0, \quad i = 1, 2, \cdots, n-1. \end{cases} \tag{B.3}$$

Appendix C

Consider the last exponential terms in equation (11), we have

$$
\begin{aligned}
&(\mathbf{y} + 2\xi_n R\mathbf{e}_n - \boldsymbol{\xi})^T R^{-1} (\mathbf{y} + 2\xi_n R\mathbf{e}_n - \boldsymbol{\xi}),\\
=&[\mathbf{y} - (I - 2R\mathbf{e}_n\mathbf{e}_n^T)\boldsymbol{\xi}]^T R^{-1} [\mathbf{y} - (I - 2R\mathbf{e}_n\mathbf{e}_n^T)\boldsymbol{\xi}]\\
=&(A^{-1}\mathbf{y} - \boldsymbol{\xi})^T \widehat{R}^{-1} (A^{-1}\mathbf{y} - \boldsymbol{\xi}),
\end{aligned}
\tag{C.1}
$$

where $A = I - 2R\mathbf{e}_n\mathbf{e}_n^T$ and $\widehat{R}^{-1} = A^T R^{-1} A$. Further, let $\mathbf{u} = A^{-1}\mathbf{y}$, $\alpha = \dfrac{r - \beta - \frac{\sigma_n^2}{2}}{\sigma_n}$, and consider

$$
\begin{aligned}
&\exp(2\alpha\xi_n)\exp(-\frac{1}{2\tau}(\mathbf{u} - \boldsymbol{\xi})^T \widehat{R}^{-1} (\mathbf{u} - \boldsymbol{\xi}))\\
=&\exp\left(-\frac{1}{2\tau}\left(\mathbf{u} + 2\alpha\tau\widehat{R}\mathbf{e}_n - \boldsymbol{\xi}\right)^T \widehat{R}^{-1}\left(\mathbf{u} + 2\alpha\tau\widehat{R}\mathbf{e}_n - \boldsymbol{\xi}\right)\right)\exp\left(2\alpha\mathbf{e}_n^T\mathbf{u}\right)\exp\left(2\alpha^2\tau\left(\widehat{R}\mathbf{e}_n\right)^T\mathbf{e}_n\right)\\
=&\exp\left(-\frac{1}{2\tau}(\mathbf{v} - \boldsymbol{\xi})^T \widehat{R}^{-1} (\mathbf{v} - \boldsymbol{\xi})\right)\exp(-2\alpha x_n),
\end{aligned}
\tag{C.2}
$$

where $\mathbf{v} = \mathbf{u} + 2\alpha\tau\widehat{R}\mathbf{e}_n$. It can be shown easily that $A^{-1} = A$ and $\widehat{R}^{-1} = R^{-1}$. Hence, the Green function can be simplified to the form

$$
\begin{aligned}
G_n(\mathbf{x}, \tau; \boldsymbol{\xi}, u) =&\frac{e^{-r(\tau - u)}}{[2\pi(\tau - u)]^{n/2}}\frac{1}{\sqrt{\det R}}\left[\exp\left(-\frac{1}{2\tau}(\mathbf{y} - \boldsymbol{\xi})^T R^{-1} (\mathbf{y} - \boldsymbol{\xi})\right)\right.\\
&\left. - \exp(-2\alpha x_n)\exp\left(-\frac{1}{2\tau}(\mathbf{y} - 2x_n R\mathbf{e}_n - \boldsymbol{\xi})^T R^{-1} (\mathbf{y} - 2x_n R\mathbf{e}_n - \boldsymbol{\xi})\right)\right].
\end{aligned}
\tag{C.3}
$$

Appendix D

Let $c_{i,j}^n$ denote the approximate value of $c(i\triangle x_1, j\triangle x_2, n\triangle\tau)$, where $\triangle x_1$ and $\triangle x_2$ are the step-widths in x_1- and x_2-directions, and $\triangle\tau$ is the time step. Let $\tilde{c}_{i,j}^{n+1}$ denote the provisional value of $c_{i,j}^{n+1}$ immediate between nth and $(n+1)$th time step. Define the centered difference operators as

$$
\delta_{x_1}^2 c_{i,j} = \frac{c_{i+1,j} - 2c_{i,j} + c_{i-1,j}}{\triangle x_1^2},\ D_{x_1} c_{i,j} = \frac{c_{i+1,j} - c_{i-1,j}}{2\triangle x_1},
\tag{D.1}
$$

and similar definitions for $\delta_{x_2}^2$ and D_{x_2}. The fractional step algorithm is depicted as follows:

(i) provisional stage - implicit differencing of terms involving x_1-derivatives

$$
\begin{aligned}
\frac{\tilde{c}_{i,j}^{n+1} - c_{i,j}^n}{\triangle\tau} =&\frac{1}{2}\delta_{x_1}^2\frac{\tilde{c}_{i,j}^{n+1} + c_{i,j}^n}{2} + \frac{1}{2}\delta_{x_2}^2 c_{i,j}^n + \rho_{12}D_{x_1}D_{x_2}\frac{\tilde{c}_{i,j}^{n+1} + c_{i,j}^n}{2} - \rho_{12}D_{x_1}\frac{\tilde{c}_{i,j+1}^{n+1} - c_{i,j+1}^n}{2}\\
&+ \frac{r - q_1 - \frac{\sigma_1^2}{2}}{\sigma_1}D_{x_1}\frac{\tilde{c}_{i,j}^{n+1} + c_{i,j}^n}{2} + \frac{r - \beta - \frac{\sigma_2^2}{2}}{\sigma_2}D_{x_2}c_{i,j}^n - r\left(\frac{\tilde{c}_{i,j}^{n+1} + c_{i,j}^n}{2}\right),
\end{aligned}
\tag{D.2a}
$$

(ii) final stage - implicit differencing of terms involving x_2-derivatives

$$\frac{c_{i,j}^{n+1}-c_{i,j}^{n}}{\triangle\tau} = \frac{1}{2}\delta_{x_1}^2\frac{\tilde{c}_{i,j}^{n+1}+c_{ij}^{n}}{2}+\frac{1}{2}\delta_{x_2}^2\frac{c_{i,j}^{n+1}+c_{i,j}^{n}}{2}+\rho_{12}D_{x_1}D_{x_2}\frac{c_{i,j}^{n+1}+c_{i,j}^{n}}{2}-\rho_{12}D_{x_2}\frac{c_{i+1,j}^{n+1}-\tilde{c}_{i+1,j}^{n+1}}{2}$$
$$+\frac{r-q_1-\frac{\sigma_1^2}{2}}{\sigma_1}D_{x_1}\frac{\tilde{c}_{i,j}^{n+1}+c_{i,j}^{n}}{2}+\frac{r-\beta-\frac{\sigma_2^2}{2}}{\sigma_2}D_{x_2}\frac{c_{i,j}^{n+1}+c_{i,j}^{n}}{2}-r\left(\frac{c_{i,j}^{n+1}+c_{i,j}^{n}}{2}\right), \tag{D.2b}$$

The solution of the tridiagonal system at each stage in the computational procedure can be effected efficiently by the Thomas algorithm.

	$b_0 = 0.5$		$b_0 = 0.9$	
time steps	binomial	fin. diff.	binomial	fin. diff.
4	$1.16e-03$	$9.99e-03$	$2.44e-02$	$9.49e-03$
8	$5.00e-04$	$3.21e-03$	$2.51e-02$	$3.19e-03$
16	$2.34e-04$	$6.98e-04$	$2.93e-02$	$7.12e-04$
32	$1.19e-04$	$1.69e-04$	$2.70e-02$	$1.71e-04$
64	$9.68e-05$	$4.18e-05$	$2.80e-02$	$4.91e-05$
128	$4.29e-05$	$1.03e-05$	$2.47e-02$	$3.42e-05$

Table 1. Comparison of the root-mean-squared error values with regard to the binomial scheme and the fractional step finite difference scheme at varying number of time stpes.