A FRONT-FIXING FINITE DIFFERENCE METHOD FOR THE VALUATION OF AMERICAN OPTIONS

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Abstract. The difficulty for the accurate valuation of American type financial options lies on the unknown free boundaries associated with the early exercise feature. A front-fixing transformation is used in this paper to transform the unknown free boundary into a known and fixed one. An efficient finite difference method is then developed, which produces the optimal exercise boundary and multiple option values at once. Numerical results show that the front-fixing finite difference method has accuracy comparable to that of the binomail method, and it is computationally competitive when multiple option positions need to be priced.

1. Introduction

The valuation of American options has long been an intriguing problem. It is widely acknowledged that analytical formula may not exist for an American option when early exercise may be optimal. As a result, the valuation of American options routinely resorts to numerical or or quasi-analytical methods. Since most traded options are American options, considerable interests exist in new valuation techniques.

The numerical methods are symbolized by the finite difference method (Brennan and Schwartz, 1977), and particularly the binomial method (Cox, Ross and Rubinstein, 1979). These methods are pedagogically appealing, easy to implement, and adaptive to options with nonstandard features or exotic options. Rigorous justification has also been established for these methods (Jaillet, Lamberton, and Shastri, 1990; Amin and Khanna, 1994). Nevertheless, numerical methods are considered too

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slow for accurate valuation. Richardson extrapolation, first used by Geske and Johnson(1984) for option pricing, was employed to achieve higher accuracy with small number of time steps (Breen, 1991).

The quasi-analytical solutions were introduced by Geske and Johnson(1984), MacMillan(1986), and Barone-Adesi and Whaley(1987). These methods generate approximate solutions of an American option by either restricting early exercise at discrete dates, or solving some modified Black-Scholes equation. Notable recent developments of quasi-analytical methods include the *analytical method of lines* by Carr and Faguet (1994), the *integral equation approach* by Huang, Subrahmanyam and Yu (1996) and the *capped option approximation* by Broadie and Detemple (1996). Both *integral equation approach* and *capped option approach* require Newton's iteration for the early exercise boundary. Richardson extrapolation is a critical component of *analytical method of line* and the *integral equation approach*. It is reasonable to believe that these methods can be generalized to many other options. To some exotic options, such as Asian option, which don't have analytical formula when early exercise is not allowed, the prospect of generalization is not clear.

Recently, Wilmott, Dewyenn and Harrison (1993) have developed a new framework to price exotic options, such as barrier, Asian and lookback options. They model these exotic options by a linear complementary problem of partial differential equation, which can be solved effectively by project SOR method (Elliot and Ockendon, 1982). The projection requires an embedded iteration at each time step. The method is more accurate but slower than the finite difference method by Brennan and Schwartz(1977).

In this papers we introduce an old technique for free boundary problems into option pricing. By the so-call front-fixing transformation (Landau, 1950) we let the unknown boundary get into the equation in exchange for a fixed boundary. Such transformation has also been considered by Carr (1995). The fixed boundary facilitates effective discretization of a partial differential equation. We then propose a linear difference scheme for the transformed equation. Our scheme doesn't need embedded iteration at each time step of evolution. In addition to option values, our method captures the whole optimal exercise boundary. The procedure works for an option as long as a front-fixing transformation exists, which is true at least for standard American options, barrier options, Asian options and lookback options. In subsequent sections we will present the procedure and test results with the prototype American put options.

The paper is organized as follow. In §2 we introduce a front-fixing transformation. In §3 we propose a finite difference discretization to the transformed equation and describe the solution procedure. Numerical comparisons with binomail method are given in §4. We conclude the paper in §5.

2. The Front-Fixing Transformation

Let $P(S,\tau;X)$ denote the value of an American put option. Here, S is the price of the underlying asset price, τ the time to maturity, and X the strike price. We assume that S follows the risk-neutral process

$$dS = rSdt + \sigma S dz, \tag{1}$$

where r is the risk-free interest rate, and σ is the volatility of the asset price. Both r and σ are assumed constants. It has been well-known that at any moment, there exists optimal exercise boundary $B(\tau)$ such that it is optimal to exercise the put option when S is at or below $B(\tau)$. Hence, when $S \leq B(\tau)$ the put option is of value

$$P(S,\tau;X) = X - S. \tag{2}$$

For asset price above $S > B(\tau)$, instead, $P(S,\tau)$ satisfies the celebrated Black-Scholes equation (Black and Scholes, 1973; Merton, 1973)

$$P_\tau - \frac{1}{2}\sigma^2 S^2 P_{SS} - rSP_S + rP = 0, \quad S \in (B(\tau,\infty), \tag{3}$$

the "smooth pasting " condition

$$P(B(\tau),\tau) = X - B(\tau), \quad P_S(B(\tau),\tau) = -1, \tag{4}$$

at $B(\tau)$, and the upper boundary condition

$$\lim_{s\to\infty} P(S,\tau) = 0. \tag{5}$$

The subindices in (3) represent partial derivatives with respect to respective variables. The terminal payoff gives rise to the initial condition

$$P(S,0) = 0, \quad S \in (B(0),\infty) \quad \text{with} \quad \mathrm{B}(0) = X. \tag{6}$$

Since $P(S,\tau)$ is linearly homogeneous in S and X, and S is linearly homogeneous in X, the equation and boundary conditions for normalized functions $\tilde{P} = \frac{P}{X}$ and $\tilde{B}(\tau) = \frac{B(\tau)}{X}$ on normalized variable $\tilde{S} = \frac{S}{X}$ are the same as (3)-(6), except that strike price be replaced by 1. Assume no confusion is caused, we let P, B and S stand for the normalized variables in the subsequent discussions.

The difficulty for accurate valuation of the American put option lies on the unknown boundary $B(\tau)$. If we apply finite difference and finite element method directly to (3)-(6), we will have trouble managing the computational mesh points or elements. It was first suggested by Landau (1950) that such difficulty can be removed by transforming the unknown and varying boundary into a known and fixed one. The following transformation of state variable serves this purpose:

$$y = \ln(S/B(\tau)). \tag{7}$$

The process for y is

$$dy = \left(r - \frac{\sigma^2}{2} - \frac{B'(\tau)}{B(\tau)}\right) dt + \sigma dz. \tag{8}$$

By either forming a riskless portfolio or direct substitution, we can derive the equation and boundary conditions under the new variable y:

$$\frac{\partial P}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial y^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\partial P}{\partial y} + r\tilde{P} = \frac{B'(\tau)}{B(\tau)}\frac{\partial P}{\partial y}, \tag{9}$$

$$P(y,0) = 0, \quad y \in (0,\infty), \tag{10}$$

$$P(0,\tau) = 1 - B(\tau), \quad \frac{\partial P(0,\tau)}{\partial y} = -B(\tau), \tag{11}$$

$$P(\infty,\tau) = 0. \tag{12}$$

Stemmed from the term $\frac{B'(\tau)}{B(\tau)}\frac{\partial P}{\partial y}$ the nonlinear nature of valuation problem is exposed by the transformation. Note that transformation (7) is valid only if $B(\tau) > 0$ for all $\tau \geq 0$. This is indeed true as it has been known already (Samulson, 1979) that $B(\tau)$ is a monotonically decreasing function of τ with a nontrivial asymptotic limit:

$$B(\infty) = \frac{1}{1+\gamma}, \quad \gamma = \frac{\sigma^2}{2r}.$$

Unlike many other free boundary problems, there is no separate equation exists for $B(\tau)$. At $y = 0$, equation (9) becomes

$$-\frac{\sigma^2}{2}\frac{\partial^2 P(0,\tau)}{\partial y^2} - \frac{\sigma^2}{2}B(\tau) + r = 0, \tag{13}$$

due to some cancellations. Since the left boundary value $P(0,\tau)$ is an unknown, equation (13) will be needed for numerical solution.

3. Finite difference approximation

The finite difference discretization of the equations is to substitute all derivatives by the appropriate difference quotients. For this purpose, we introduce a two-dimensional mesh of the size (h,k) in the first quadrant of the y-τ plane, as is shown in Figure 1.

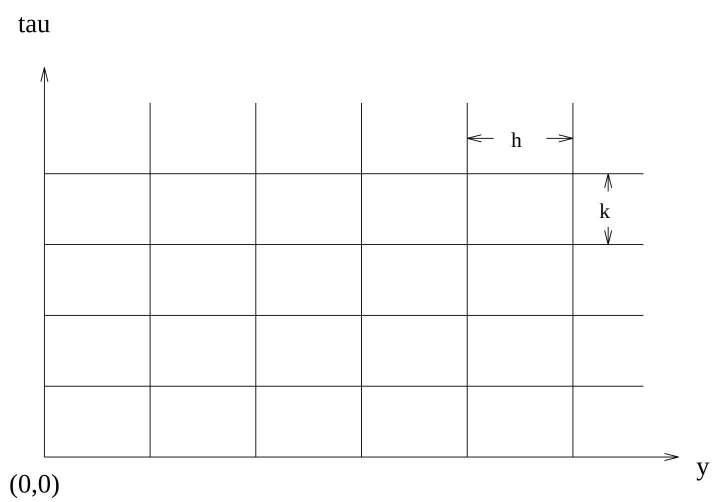

Figure 1. Computational mesh

To present our finite difference scheme in a compact form, we define the following difference operators:

$$D_+ = \frac{E - I}{h}, \quad D_- = \frac{I - E^{-1}}{h}, \quad D_0 = \frac{E - E^{-1}}{2h}, \tag{14}$$

where E is the spatial shifting operator such that for any discrete function P_j,

$$E^i P_j = P_{j+i}. \tag{15}$$

In order to avoid nonlinearity and achieve high accuracy, we adopt the following three-level discretization to equation (9):

$$\frac{P_j^{n+1} - P_j^{n-1}}{2k} - \left\{ \frac{\sigma^2}{2} D_+ D_- + (r - \frac{\sigma^2}{2}) D_0 - r \right\} \frac{(P_j^{n+1} + P_j^{n-1})}{2} = g^n D_0 P_j^n, \tag{16}$$
$$j = 1, 2, \ldots, M.$$

Here, P_j^n is the numerical approximation to $P(jh, nk)$, and

$$g^n = \frac{B^{n+1} - B^{n-1}}{2kB^n}, \tag{17}$$

which approximates $\frac{B'(nk)}{B(nk)}$. We choose M large enough so that we can comfortably put $P_{M+1}^n = 0$ for all n. The discretized version of equation (13) is

$$-\frac{\sigma^2}{2} D_+ D_- P_0^n - \frac{\sigma^2}{2} B^n + r = 0, \tag{18}$$

which involves a ghost value P_{-1}^n. The discretization of the "smooth pasting condition" (11) by central differencing gives rise to

$$P_0^n = 1 - B^n, \quad \text{and} \tag{19}$$

$$\frac{P_1^n - P_{-1}^n}{2h} = -B^n, \quad \text{for all } n \geq 1. \tag{20}$$

From (18), (19) and (20) we can eliminate P_{-1}^n and obtain

$$P_1^n = \alpha - \beta B^n, \quad n \geq 1, \tag{21}$$

where

$$\alpha = 1 + h^2 \sigma^{-2} r, \quad \beta = [1 + (1 + h)^2]/2. \tag{22}$$

Note that the numerical discretization is not unique. We adopt (16) based on the following considerations. First, when $g^n = 0$, (16) reduces to the Crank-Nicholson scheme used by Courtadon (1982) on European call option. If we look at our finite difference scheme from the viewpoint of approximate general jump process, then the underlying jump process has no biad variance. Second, the three-level discretization permits the explicit treatment of nonlinear term, without sacrificing the accuracy of the Crank-Nicholson discretization, which is of order $O(k^2 + h^2)$.

We now explain how to advance from P_j^{n-1} and P_j^n to get $P_j^{n+1}, j = 0, 1, \ldots, M$. We first rewrite (16) using matrix notations. Denote

$$a = \mu\sigma^2 + kr, \quad b = \frac{\mu}{2}\left[\sigma^2 - h(r - \frac{\sigma^2}{2})\right], \quad c = \frac{\mu}{2}\left[\sigma^2 + h(r - \frac{\sigma^2}{2})\right],$$

where $\mu = k/h^2$, and define matrix

$$A = \begin{pmatrix} a & -c & 0\ldots & \ldots & \ldots & 0 \\ -b & a & -c & 0 & \ldots & 0 \\ 0 & -b & a & -c & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \ldots & \ldots & -b & a & -c \\ 0 & 0 & \ldots & 0 & -b & a \end{pmatrix}.$$

Then in terms of A, equation (16) can be rewritten as

$$\begin{aligned} (I + A)\mathbf{P}^{n+1} &= (I - A)\mathbf{P}^{n-1} + bP_0^{n+1}\mathbf{e}_1 + 2kg^n D_0\mathbf{P}^n, \\ &= (I - A)\mathbf{P}^{n-1} + bP_0^{n+1}\mathbf{e}_1 + \lambda g^n(2hD_0\mathbf{P}^n), \quad n > 1, \end{aligned} \tag{23}$$

where $\lambda = k/h$ and

$$\begin{aligned} \mathbf{P} &= (P_1, P_2, \ldots, P_M)^T, \\ \mathbf{e}_1 &= (1, 0, \ldots, 0)^T. \end{aligned} \tag{24}$$

The solution to (23) can be expressed as

$$\mathbf{P}^{n+1} = \mathbf{f}_1 + bP_0^{n+1}\mathbf{f}_2 + \lambda g^n\mathbf{f}_3, \tag{25}$$

where

$$\begin{aligned}\mathbf{f}_1 &= (I+A)^{-1}(I-A)\mathbf{P}^{n-1},\\ \mathbf{f}_2 &= (I+A)^{-1}\mathbf{e}_1,\\ \mathbf{f}_3 &= (I+A)^{-1}(2hD_0\mathbf{P}^n).\end{aligned} \tag{26}$$

Substituting P_1^{n+1} into (21) and using (17) we can solve for B^{n+1}:

$$B^{n+1} = \frac{\alpha - f_{1,1} - bf_{1,2} + \frac{\lambda f_{1,3} B^{n-1}}{2kB^n}}{\beta - bf_{1,2} + \frac{\lambda f_{1,3}}{2kB^n}}. \tag{27}$$

The solutions for g^n and P_0^{n+1} then follow. The pseudo-code for the method is

$[L, U] = LU\text{-}decompose\ I + A$

$\mathbf{f}_2 = U^{-1}L^{-1}\mathbf{e}_1$

for $n = 1, 2, \ldots, N-1$ *do*

  $\mathbf{f}_1 \leftarrow U^{-1}L^{-1}(I-A)\mathbf{P}^{n-1}$

  $\mathbf{f}_3 \leftarrow U^{-1}L^{-1}(2hD_0\mathbf{P}^n)$

  Solve for B^{n+1}, g^n *and* P_0^{n+1}

  $\mathbf{P}^{n+1} \leftarrow \mathbf{f}_1 + bP_0^{n+1}\mathbf{f}_2 + \lambda g^n \mathbf{f}_3$

end

It takes $11M$ multiplications (divisions) and $9M$ additions (subtractions) to compute each $\mathbf{P}^n$.

Since equation (16) is a three-level scheme, we need $\mathbf{P}^1$ in addition to $\mathbf{P}^0$ to initialize the computation. To maintain the overall second order of accuracy we use the two-step predictor-corrector technique to obtain $\mathbf{P}^1$:

$$\begin{aligned}(I + \frac{A}{2})\tilde{\mathbf{P}} &= (I - \frac{A}{2})\mathbf{P}^0 + \frac{b}{2}\tilde{P}_0 + k\tilde{g}D_0\mathbf{P}^0,\\ (I + \frac{A}{2})\mathbf{P}^1 &= (I - \frac{A}{2})\mathbf{P}^0 + \frac{b}{2}P_0^1 + kg^1 D_0(\frac{\tilde{\mathbf{P}} + \mathbf{P}^0}{2}),\end{aligned} \tag{28}$$

where

$$\tilde{g} = \frac{\tilde{B} - B^0}{kB^0}, \qquad g^1 = \frac{B^1 - B^0}{k\frac{\tilde{B}+B^0}{2}}. \tag{29}$$

The code for $\mathbf{P}^n$ can be used to realize this predictor-corrector procedure after slight modification.

The specification of grid size (k, h) and the integer M is an important issue to be addressed. Following the convention of the numerical methods we let k be one of the input parameters defined according to the number of time steps N, i.e, $k = T/N$. For h, it is well-known that the convergence of the finite difference solution requires $k/h \to 0$ as $k \to 0$. From the viewpoint of approximate general jump process we want to have nonnegative $1 - a, b$ and c, as they then can be interpreted as probabilities (multiplied by $1 - kr$, the time discount factor). The nonnegativity requirement leads to $h \geq \sigma\sqrt{k}$. From experiences we recommend $h = 1.5\sigma\sqrt{k}$. This selection implies that our finite difference method is **first order accurate in** k. When penny accuracy is demanded, M should be chosen according to $P(Mh, T) < (100X)^{-1}$. Clearly, M is a function of all input parameters. At this point we don't have a general formula of M that guarantees penny accuracy in all situations. We have instead chosen M in a rather simple way. For $0 \leq T \leq O(1)$, we observe the magnitude of the solution in the far field ($y \gg 1$) depends on $\sigma\sqrt{T}$. We thus consider $Mh = c\sigma\sqrt{T}$, or $M = [c\sigma\sqrt{T}/h]$, here c is a constant insensitive to the input parameters. When $T \leq 3$, we have uniformly taken $c = 8$. This selection is supported by our numerical results. For bigger σ, T or X we may need bigger c.

Given M chosen above we can calculate the number of arithmetic operations needed for the entire iterations. The total numbers of multiplications (divisions) and additions (subtractions) are

$$\text{No. of } \times/\div = 11MN = [\frac{22c}{3}N^{\frac{3}{2}}] \tag{30}$$

and

$$\text{No. of } +/- = 9MN = [6cN^{\frac{3}{2}}]. \tag{31}$$

The power over N is $\frac{3}{2}$. When the number of time steps doubles, the CPU time for front-fixing method will increase by a factor $2^{\frac{3}{2}} = 2.8$. Meanwhile, binomial method takes $N(N+1)$ multiplications and the same number of additions. When the number of time steps doubles, the CPU time for the binomial method will ncrease by the factor 4. If the CPU time for one multiplication (division) significantly dominates the CPU time for one addition (subtraction), then the front-fixing method will take less CPU time than binomial method for *each run* when the number of time steps $N \geq [(\frac{22c}{3})^2]$. Hence, if there are p option positions with the same maturity to be valuated, we should consider the front-fixing method when the number of time steps $N \geq [(\frac{22c}{3p})^2]$. Take $p = c = 8$ for example, $N \geq 54$.

Finally we remark that interpolation treatment is generally part of the front-fixing method. The finite difference method on the transformed equation produces option values at

$$S_j = XB(T)e^{jh}, \quad j = 0, 1, \ldots, M. \tag{32}$$

For option values at any designated asset prices other than these S_j's, we adopt the cubic spline interpolation(Press et. al., 1992) with $P(S_j, T)$. One can prove that interpolated option values over the interval $[B(T), B(T)e^{c\sigma\sqrt{T}}]$ will have the same accuracy as that of $P(S_j, T)$. However, if the *delta* is obtained by differentiating the cubic spline polynomial, then theoretically we can only guarantee the accuracy of order $O(h)$.

4. Numerical Results

In this section we show the performance of the front-fixing method with three test cases. The test cases cover short term, medium term and long term options with various parameters. For the same number of time steps, front-fixing method is tested against the standard binomial method. Throughout these test cases we take $h = \frac{3}{2}\sigma\sqrt{k}$ and $M = [8\sigma\sqrt{T}/h]$ for the front-fixing method. For various numbers of time step, we tabulate the option values, deltas, root-mean-square-errors(RMSE) and CPU times of both the standard binomial method and the front-fixing method. In Example 1 and 2, we generate the "exact" solutions for the computation of RMSE by the binomial method with $1,000$ time steps. The "exact" solutions in Example 3 are taken directly from Huang, Subrahmanyam and Yu (Huang et al., 1996), which were obtained by $10,000$ binomail iterations. We would like to emphasize here that the CPU times given in these examples are the CPU time for each run of either method.

Example 1: The first test case is the prototype (Carr and Faguet, 1994) with the following characteristics:

- Strike price X =\$100;
- Risk-free interest rate $r = 0.1$;
- Volatility $\sigma = 0.3$;
- Time to maturity $T = 1$ (year).

Table 1A lists the option values and deltas obtained by binomial and front-fixing methods for two sets of asset prices, where "F-F-F" stands for front-fixing finite difference method. The asset prices in first set are near the optimal exercise boundary $B(T) = 76.25$ and the asset prices in second set lie within 20% range of the strike price. The *delta* for the front-fixing method is obtained by differentiating the cubic spline interpolant. The RMSE indicates that the two methods have close accuracy, and both are well within the truncation error $O(k)$. Near the optimal exercise boundary, the front-fixing method is slightly more accurate. However, the *deltas* calculated for the front-fixing method have much bigger error than that of the *deltas* obtained by the binomial method.

In Table 1B we display the changes of RMSE and CPU time with respect to N. We define

$$\text{Factor of RMSE decrease} = \frac{\text{RMSE(N)}}{\text{RMSE(N/2)}}, \tag{33}$$

and

$$\text{Factor of CPU time decrease} = \frac{\text{CPU(N)}}{\text{CPU(N/2)}}, \tag{34}$$

and RMSE(N) and CPU(N) denote the RMSE and CPU time of either method with N time steps. These two factors measure the order of the accuracy and rate of increase of CPU times. It can be seen that when time step doubles, the RMSE of the front-fixing method decreases by a factor around 0.5, and the CPU time increases by factors approaching $\sqrt{8}$. This factor of decrease confirms the first order temporal accuracy of the front-fixing method. Note that for $N = 512$, the run time of front-fixing method becomes less than that of binomial method. Figure 2 offers the early exercise boundary obtained by the front-fixing method for $0 \leq \tau \leq T$.

$r = 0.1, \sigma = 0.3, T = 1, X = 100, k = 0.01$

	Option Values			Delta		
Stock Price	Binomial n=1000	Binomial n=100	F-F-F n=100	Binomial n=1000	Binomial n=100	F-F-F n=100
77	23.0131	23.0000	23.0128	-0.9686	-0.9619	-0.9718
78	22.0615	22.0567	22.0621	-0.9318	-0.9342	-0.9353
79	21.1483	21.1442	21.1469	-0.8971	-0.8987	-0.9001
80	20.2687	20.2576	20.2662	-0.8632	-0.8634	-0.8661
RMSE		0.0092	0.0015		0.0036	0.0032
CPU(sec)		1.89	5.9500			
80	20.2689	20.2576	20.2662	-0.8631	-0.8634	-0.8661
85	16.3467	16.3412	16.3396	-0.7109	-0.7107	-0.7133
90	13.1228	13.1208	13.1124	-0.5829	-0.5832	-0.5848
95	10.4847	10.4798	10.4733	-0.4755	-0.4761	-0.4769
100	8.3348	8.3255	8.3277	-0.3856	-0.3860	-0.3866
105	6.6071	6.6108	6.5936	-0.3108	-0.3110	-0.3116
110	5.2091	5.2250	5.2004	-0.2491	-0.2493	-0.2497
115	4.0976	4.1034	4.0872	-0.1986	-0.1988	-0.1990
120	3.2059	3.1964	3.2023	-0.1575	-0.1574	-0.1578
RMSE		0.0086	0.0090		0.0003	0.0016
CPU(sec)		1.89	6.08			

Table 1a: Comparison of speed and accuracy, I

$r = 0.1, \sigma = 0.3, T = 1, X = 100$

	Binomial		F-F-F		Binomial		F-F-F	
Time Step N	RMSE	Factor of decrease	RMSE	Factor of decrease	CPU time	Factor of increase	CPU time	Factor of increase
16	5.1E-02		7.4E-02		6.7E-02		7.2E-01	
32	1.0E-02	0.20	3.4E-02	0.46	2.3E-01	3.50	1.2E+00	1.67
64	1.2E-02	1.22	1.5E-02	0.44	8.3E-01	3.57	2.7E+00	2.26
128	3.3E-03	0.27	6.9E-03	0.46	3.2E+00	3.90	6.8E+00	2.48
256	5.8E-03	1.79	3.8E-03	0.55	1.3E+01	3.91	1.8E+01	2.70
512	1.3E-03	0.22	2.8E-03	0.73	5.1E+01	4.03	5.0E+01	2.74

Table 1b: RMSE and CPU time vs number of time steps

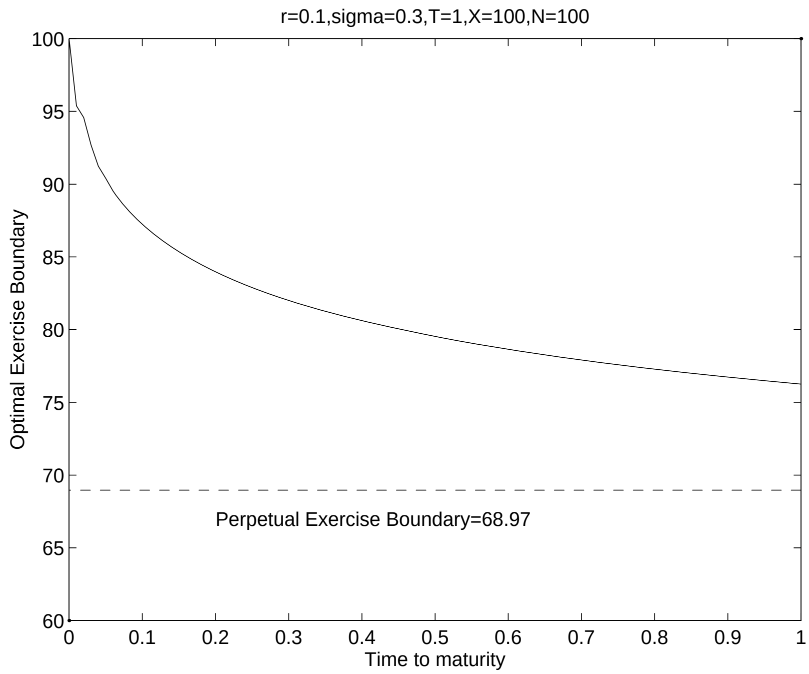

FIGURE 2 Optimal exercise boundary

Example 2: The second example (Carr and Faguet, 1994) is a long term option with the following characteristics:

- Strike price X =\$100;
- Risk-free interest rate $r = 0.06$;
- Volatility $\sigma = 0.4$;
- Time to maturity $T = 3$ (year).

As is shown in Table 2, the accuracy of option values by the front-fixing method is slightly better than that by binomail method.

$r = 0.1, \sigma = 0.3, T = 1, X = 100, k = 0.01$

	Option Values			Delta		
Stock Price	Binomial n=1000	Binomial n=100	F-F-F n=100	Binomial n=1000	Binomial n=100	F-F-F n=100
80	28.0708	28.0886	28.0607	-0.5040	-0.5039	-0.5060
85	25.6850	25.6895	25.6698	-0.4524	-0.4526	-0.4542
90	23.5395	23.5411	23.5203	-0.4073	-0.4073	-0.4088
95	21.6027	21.6320	21.5832	-0.3675	-0.3672	-0.3688
100	19.8513	19.8251	19.8335	-0.3323	-0.3328	-0.3334
105	18.2716	18.2967	18.2500	-0.3011	-0.3010	-0.3020
110	16.8385	16.8700	16.8141	-0.2732	-0.2731	-0.2741
115	15.5359	15.5039	15.5099	-0.2484	-0.2486	-0.2492
120	14.3500	14.3714	14.3233	-0.2262	-0.2261	-0.2269
RMSE		0.0235	0.0207		0.0002	0.0013
CPU(sec)		1.89	6.08			

TABLE 2: Option values and deltas

Example 3: The last test case is used by Huang, Subrahmanyam and Yu (Huang et al., 1996). With fixed interest rate and stock price, options of different strike prices, volatilities and maturities are valuated. The details of the characteristic are listed in Table 3. Again we witness the comparable accuracy of the two methods. We would comment that the accuracy of the option values by front-fixing is very close to that of recursive method by Huang, Subrahmanyam and Yu (Huang et al., 1996).

5. Conclusion

From the approach of numerical solution of Black-Scholes equation, we have proposed and tested a new finite difference method. The main gradient of this method is the front-fixing transformation. The new method has several advantages. First, it can valuate option positions with the same maturity for essentially all possible asset prices at once. It becomes increasing economical when the number of option position increases. Second, it offers the optimal exercise boundary together with option prices without extra cost. Third, the accuracy of the method is comparable to that of the binomial method, which is significantly better than the well-known finite difference method by Brennan and Schwartz(1977). Fourth and perhaps the most practical advantage is that the method is adaptive to other options as long as a front-fixing transformation exists. This includes barrier option and Asian options. Our method has some disadvantages as well. It doesn't possess natural mean to accurately evaluate *deltas*. Also, the front-fixing transformation may not work for American options on multiple assets.

$r = 0.0488, S = 40$

Strike Price	σ	T	Binomial N=10,000	Binomial N=150	F-F-F N=150
35	0.2	0.0833	0.0062	0.0061	0.0065
35	0.2	0.3333	0.2004	0.1995	0.2007
35	0.2	0.5833	0.4328	0.4340	0.4325
35	0.3	0.0833	0.8522	0.8513	0.8498
35	0.3	0.3333	1.5798	1.5784	1.5766
35	0.3	0.5833	1.9904	1.9887	1.9872
35	0.4	0.0833	5.0000	5.0000	5.0194
35	0.4	0.3333	5.0883	5.0886	5.0865
35	0.4	0.5833	5.2670	5.2677	5.2645
40	0.2	0.0833	0.0774	0.0776	0.0783
40	0.2	0.3333	0.6975	0.6994	0.6961
40	0.2	0.5833	1.2198	1.2239	1.2169
40	0.3	0.0833	1.3099	1.3085	1.3059
40	0.3	0.3333	2.4825	2.4800	2.4764
40	0.3	0.5833	3.1696	3.1666	3.1630
40	0.4	0.0833	5.0597	5.0600	5.0573
40	0.4	0.3333	5.7056	5.7066	5.6990
40	0.4	0.5833	6.2436	6.2448	6.2363
45	0.2	0.0833	0.2466	0.2456	0.2472
45	0.2	0.3333	1.3460	1.3506	1.3419
45	0.2	0.5833	2.1549	2.1603	2.1484
45	0.3	0.0833	1.7681	1.7661	1.7624
45	0.3	0.3333	3.3874	3.3837	3.3783
45	0.3	0.5833	4.3526	4.3481	4.3424
45	0.4	0.0833	5.2868	5.2877	5.2808
45	0.4	0.3333	6.5099	6.5104	6.4990
45	0.4	0.5833	7.3830	7.3898	7.3712
RMSE				2.6292e-03	6.6574e-03
CPU(sec)				4.1500	10.78

TABLE 3: Option values and deltas

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