# LOCAL SPACINGS ALONG CURVES 

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#### Abstract

We investigate local spacing problems along curves via smooth maps. Moreover we provide explicit formulas for the nearest neighbor spacing distribution function of torsion points on elliptic curves over $\mathbb{R}$, and of rational points on the unit circle.


## 1. Introduction And statement of RESULTS

The statistics of local spacings measure the fine structure of sequences of real numbers in a more subtle way than the classical uniform distribution. Their study was initiated by physicists (see for example [7] and [24]), in order to understand the spectra of high energies. This notion has received a great deal of attention in areas such as mathematical physics, analysis, probability and number theory (see [1]-[6],[8]-[10],[12]-[20]). For most cases considered so far, the problem can be interpreted in terms of the distribution of a given sequence of points which lie on a straight line. There are, however, many important sequences of points which lie on a curve rather than a straight line. Interesting examples arising naturally, which will be considered below, are torsion points on elliptic curves over $\mathbb{R}$ and rational points on the unit circle. In [25] the last author raised the problem of spacing distribution along curves, and showed under certain conditions how spacing distribution functions along a curve can be obtained from local data. In this paper, motivated by the two examples mentioned above, we first study how the spacing distribution function deforms via smooth maps between curves. In particular this explains how the spacing distribution function of a sequence of points on a curve can be obtained from a known distribution on a segment via a parametrization. Next, we provide explicit formulas for the nearest neighbor spacing distribution function of torsion points on elliptic curves over $\mathbb{R}$ and of rational points on the unit circle.

Let $\mathbf{I}=[a, b]$ be an interval with length $l(\mathbf{I})=b-a$ and $\mathcal{C} \subset \mathbb{R}^{k}$ a curve with parametrization $f: \mathbf{I} \longrightarrow \mathcal{C}$, where $f$ is continuous, piecewise

[^0]continuously differentiable, and $f^{\prime}$ does not vanish in I. Suppose $\mathscr{F}=$ $\left(\mathscr{F}_{Q}\right)_{Q \in \mathbb{N}}$ is a sequence of sets on $\mathbf{I}, \mathscr{F}_{Q}=\left\{t_{j}^{Q}: 1 \leq j \leq N_{Q}\right\} \subset \mathbf{I}$, $a \leq t_{1}^{Q}<t_{2}^{Q}<\cdots<t_{N_{Q}}^{Q} \leq b$. Denoting $x_{j}^{Q}=f\left(t_{j}^{Q}\right)$, we form a sequence on $\mathcal{C}$ by letting $\mathscr{M}=\left(\mathscr{M}_{Q}\right)_{Q \in \mathbb{N}}$ where
$$
\mathscr{M}_{Q}=\left\{x_{j}^{Q}: 1 \leq j \leq N_{Q}\right\} .
$$

Each set $\mathscr{M}_{Q}$ cuts the curve $\mathcal{C}$ in finitely many arcs. For any positive real number $\lambda$ we consider the proportion $G_{\mathcal{C}, \mathscr{M}_{Q}}(\lambda)$ of such arcs whose length is at least $\lambda$ times the average. The nearest neighbor spacing distribution function $G_{\mathcal{C}, \mathscr{M}}:[0, \infty) \longrightarrow[0,1]$ is defined, if it exists, by

$$
G_{\mathcal{C}, \mathscr{M}}(\lambda)=\lim _{Q \rightarrow \infty} G_{\mathcal{C}, \mathscr{M}_{Q}}(\lambda),
$$

for any $\lambda \geq 0$. For more notation and terminology the reader is referred to the beginning of Section 2 below.
Theorem 1. Let $\mathcal{C}, \mathbf{I}, f, \mathscr{M}, \mathscr{F}$ be as above. Suppose that $\mathscr{F}$ is uniformly distributed on $\mathbf{I}$ and for any subinterval $\mathbf{J}$ of $\mathbf{I}$, the functions $G_{\mathbf{J}, \mathscr{F}_{Q}}$ converge pointwise as $Q \rightarrow \infty$ to a continuous function $H_{\mathscr{F}}$ which is independent of J. Then the nearest neighbor spacing distribution function $G_{\mathcal{C}, \mathscr{M}}$ of $\mathscr{M}$ on $\mathcal{C}$ exists and is continuous. More specifically, for any $\lambda \geq 0$,

$$
\begin{equation*}
G_{\mathcal{C}, \mathscr{M}}(\lambda)=\frac{1}{l(\mathbf{I})} \cdot \int_{a}^{b} H_{\mathscr{F}}\left(\frac{l(\mathcal{C})}{l(\mathbf{I})\left|f^{\prime}(t)\right|} \cdot \lambda\right) d t \tag{1}
\end{equation*}
$$

A more general theorem will be proved in Section 2. In order to apply Theorem 1 in practice, if a curve and a particular sequence of sets of points on it are given, one needs to find a convenient parametrization of the curve in such a way that the corresponding sequence of points has a known local spacing distribution function on the interval, or the distribution function can be found by known methods.

One such example is provided by torsion points on an elliptic curve over $\mathbb{R}$. Elliptic curves have been studied for a long time. For a presentation of various aspects of the theory see [21]. Let $E$ be an elliptic curve defined over $\mathbb{R}$, given by the equation

$$
E: y^{2}=4 x^{3}-g_{2} x-g_{3},
$$

where $g_{2}, g_{3} \in \mathbb{R}$ and $g_{2}^{3}-27 g_{3}^{2} \neq 0$. The set $E(\mathbb{R})$ of real points of $E$ has a natural group structure which makes $E(\mathbb{R})$ an abelian group. Recall that there is a complex analytic isomorphism of complex Lie groups (see [21])

$$
\begin{aligned}
\exp : \mathcal{C} / \Lambda & \longrightarrow E(\mathbb{C}) \subseteq P^{2}(\mathbb{C}) \\
z & \longmapsto\left(\wp(z), \wp^{\prime}(z)\right),
\end{aligned}
$$

where $\Lambda$ is a lattice in $\mathbb{C}$ associated to $E(\mathbb{C})$, and $\wp(z)$ is the Weierstrass $\wp$-function associated to $\Lambda$. Here $E(\mathbb{R})$ may have one or two connected components. The unbounded one $E_{U}(\mathbb{R})$ is isomorphic under $\exp$ to $\mathcal{S}^{1} \simeq$ $\mathbb{R} / \mathbb{Z}$ as real Lie groups. Identifying $\mathbb{R} / \mathbb{Z}$ with $[0,1)$, this gives us in turn a $C^{\infty} \operatorname{map} \phi:[0,1) \longrightarrow E_{U}(\mathbb{R})$ such that $\phi(0)=\mathcal{O}$ is the point at infinity and for any $t \in[0,1)$, any integer $n$,

$$
\phi(n t(\bmod 1))=[n] \phi(t) \in E_{U}(\mathbb{R})
$$

Theorem 2. Let $E$ be the elliptic curve defined over $\mathbb{R}$. For any $Q \geq 1$, let $\mathscr{M}_{Q}$ be the set of torsion points of order less than or equal to $Q$ on $E_{U}(\mathbb{R})$ and $\mathscr{M}=\left(\mathscr{M}_{Q}\right)_{Q \in \mathbb{N}}$. Then for any finite connected subarc $\mathcal{C}$ of $E_{U}(\mathbb{R})$, the nearest neighbor spacing distribution function of $\mathscr{M}$ on $\mathcal{C}$ exists and is given by

$$
\begin{equation*}
G_{\mathcal{C}, \mathscr{M}}(\lambda)=1-\frac{2}{l(\mathbf{I})} \cdot \int_{a}^{b} A_{1}\left(\frac{3 l(\mathbf{I})\left|\phi^{\prime}(t)\right|}{\pi^{2} l(\mathcal{C}) \lambda}\right) d t \tag{2}
\end{equation*}
$$

for any $\lambda>0$, where $\mathbf{I}=[a, b]=\phi^{-1}(\mathcal{C})$ and $A_{1}$ is defined by

$$
A_{1}(a)= \begin{cases}1-a-\sqrt{1-4 a} / 2+2 a \log ((1+\sqrt{1-4 a}) / 2) & : 0<a \leq \frac{1}{4} \\ 1-a+a \log a & : \frac{1}{4}<a<1 \\ 0 & : \quad a \geq 1\end{cases}
$$

Corollary 1. For any elliptic curve $E$ over $\mathbb{R}$ and any point $P \in E_{U}(\mathbb{R}), P \neq$ $\mathcal{O}$, the limit

$$
G_{E, P}(\lambda)=\lim _{\substack{P \in \mathcal{C} \subset E_{U}(\mathbb{R}) \\ l(\mathcal{C}) \rightarrow 0}} G_{\mathcal{C}, \mathscr{M}}(\lambda)
$$

exists and is given by

$$
G_{E, P}(\lambda)=\left\{\begin{array}{cc}
1 & : 0<\lambda \leq \frac{3}{\pi^{2}} \\
\frac{6}{\pi^{2} \lambda} \cdot\left(1-\log \left(\frac{3}{\pi^{2} \lambda}\right)\right)-1 & : \frac{3}{\pi^{2}}<\lambda<\frac{12}{\pi^{2}} \\
\frac{6}{\pi^{2} \lambda}+\sqrt{1-\frac{12}{\pi^{2} \lambda}}-\frac{12}{\pi^{2} \lambda} & \\
\times \log \left(\left(1+\sqrt{1-\frac{12}{\pi^{2} \lambda}}\right) / 2\right)-1 & : \quad \lambda \geq \frac{12}{\pi^{2}} .
\end{array}\right.
$$

This corollary says that the nearest neighbor distribution function of torsion points around any point $P$ on any elliptic curve $E$ defined over $\mathbb{R}$, is independent of the point $P$. Moreover, the fact that $G_{E, P}(\lambda)=1$ on the entire interval $\left[0, \frac{3}{\pi^{2}}\right]$ shows a very strong repulsion between torsion points. This distribution coincides locally with the distribution of Farey fractions. It is closer to, from this point of view, for instance, the distribution of zeros
of the Riemann Zeta function, where one also has a well known repulsion phenomenon, than to the distribution of a randomly chosen sequence of points, where such repulsion phenomenon is not present.

These comments also apply to our second example, concerning rational points on the unit circle. Pythagorean triangles and their connection to rational points on the unit circle are of course well understood. Our aim here is to see how these points are distributed along the circle. The unit circle $\mathcal{S}^{1}=\left\{(x, y) \in \mathbb{R}^{2}: x^{2}+y^{2}=1\right\}$ has a parametrization

$$
\begin{aligned}
f_{1}: \mathbb{R} & \longrightarrow \mathcal{S}^{1} /\{(-1,0)\} \\
t & \mapsto\left(\frac{1-t^{2}}{1+t^{2}}, \frac{2 t}{1+t^{2}}\right),
\end{aligned}
$$

under which the rational points of $\mathcal{S}^{1}$ (omitting $\left.(-1,0)\right)$ are exactly those of the form (see [11])

$$
\left(\frac{1-t^{2}}{1+t^{2}}, \frac{2 t}{1+t^{2}}\right), \quad t \in \mathbb{Q} .
$$

Write any non-zero rational number $t=a / q$ in reduced form, i.e., $a, q \in$ $\mathbb{Z}, q>0$ and $\operatorname{gcd}(a, q)=1$. We take the corresponding rational point $P=\left(\frac{q^{2}-a^{2}}{q^{2}+a^{2}}, \frac{2 q a}{q^{2}+a^{2}}\right)$ on the unit circle and denote $H(P)=q^{2}+a^{2}$. This is closely related to the height of $P$, which equals to $H(P) / 2$ when $a, q$ are both odd and $H(P)$ otherwise (see [21]). Consider the sequence $\mathscr{M}=\left(\mathscr{M}_{Q}\right)_{Q \in \mathbb{N}}$ defined by

$$
\mathscr{M}_{Q}=\left\{\text { rational points } P \in \mathcal{S}^{1}: H(P) \leq Q, P \neq( \pm 1,0)\right\},
$$

The union of the sets $\mathscr{M}_{Q}$ consists of all the rational points on $\mathcal{S}^{1}-\{(-1,0)\}$. The nearest neighbor spacing distribution function of $\mathscr{M}$ on $\mathcal{S}^{1}$ is explicitly determined in the next theorem.

Theorem 3. For any arc $\mathcal{C}$ of the unit circle $\mathcal{S}^{1}-\{( \pm 1,0)\}$, the nearest neighbor spacing distribution function of $\mathscr{M}$ exists, is continuous and is independent of the arc $\mathcal{C}$. Specifically,

$$
G_{\mathcal{C}, \mu}(\lambda)=\left\{\begin{array}{cc}
1 & : \quad 0<\lambda \leq \frac{3}{\pi^{2}}  \tag{3}\\
\frac{6}{\pi^{2} \lambda} \cdot\left(1-\log \left(\frac{3}{\pi^{2} \lambda}\right)\right)-1 & : \frac{3}{\pi^{2}}<\lambda<\frac{12}{\pi^{2}} \\
\frac{6}{\pi^{2} \lambda}+\sqrt{1-\frac{12}{\pi^{2} \lambda}}-\frac{12}{\pi^{2} \lambda} & \\
\times \log \left(\left(1+\sqrt{1-\frac{12}{\pi^{2} \lambda}}\right) / 2\right)-1 & : \quad \lambda \geq \frac{12}{\pi^{2}} .
\end{array}\right.
$$

One sees from Corollary 1 and Theorem 3 that torsion points on an elliptic curve are locally distributed like the rational points on the unit circle.

The situation changes dramatically if one considers rational points on the elliptic curve. By the Mordell-Weil theorem the group of rational points on an elliptic curve defined over $\mathbb{Q}$ is finitely generated. Combining this with classical results on the fractional parts of linear forms (see [22] [23]), it follows that rational points on an elliptic curve do not have a limiting nearest neighbor spacing distribution function.

## 2. Proof of theorem 1

We will derive Theorem 1 from a more general result. For the sake of completeness we first recall some notation and terminology from [25].

Let $\mathcal{C}$ be a piecewise smooth, compact curve in $\mathbb{R}^{k}$. For any arc $\mathcal{J}$ on $\mathcal{C}$ we denote by $l(\mathcal{J})$ the length of $\mathcal{J}$. Given a finite sequence $\mathscr{U}$ of points on $\mathcal{C}, \mathscr{U}=\left\{u_{n} \in \mathcal{C}: 1 \leq n \leq N\right\}$, and any arc $\mathcal{J}$ on $\mathcal{C}$, let $\mu_{\mathcal{C}, \mathscr{U}}(\mathcal{J})$ denote the proportion of points from $\mathscr{U}$ which belong to $\mathcal{J}$,

$$
\mu_{\mathcal{C}, \mathscr{U}}(\mathcal{J})=\frac{\#\left\{1 \leq n \leq N: u_{n} \in \mathcal{J}\right\}}{N} .
$$

Let now $\mathscr{M}=\left(\mathscr{M}_{Q}\right)_{Q \in \mathbb{N}}$ be a sequence of finite sequences $\mathscr{M}_{Q}=\left\{P_{Q, n} \in\right.$ $\left.\mathcal{C}: 1 \leq n \leq N_{Q}\right\}$ of points on the curve $\mathcal{C}$, such that $N_{Q} \rightarrow \infty$ as $Q \rightarrow \infty$. Given an arc $\mathcal{J}$ on $\mathcal{C}$, if the sequence $\left(\mu_{C, \mathscr{M}_{Q}}(\mathcal{J})\right)_{Q \in \mathbb{N}}$ is convergent, we denote

$$
\mu_{\mathcal{C}, \mathscr{M}}(\mathcal{J})=\lim _{Q \rightarrow \infty} \mu_{\mathcal{C}, \mathscr{M}_{Q}}(\mathcal{J}) .
$$

We say that $\mathscr{M}$ is uniformly distributed along $\mathcal{C}$ provided that

$$
\mu_{\mathcal{C}, \mathscr{M}}(\mathcal{J})=\frac{l(\mathcal{J})}{l(\mathcal{C})}
$$

for any $\operatorname{arc} \mathcal{J}$ on $\mathcal{C}$. If $\mu_{\mathcal{C}, \mathscr{M}}(\mathcal{J})$ is defined, then we set

$$
\rho_{\mathcal{C}, \mathscr{M}}(\mathcal{J})=\frac{\mu_{\mathcal{C}, \mathscr{M}}(\mathcal{J}) l(\mathcal{C})}{l(\mathcal{J})}
$$

and call $\rho_{\mathcal{C}, \mathscr{M}}(\mathcal{J})$ the density of $\mathscr{M}$ on $\mathcal{J}$. If $\rho_{\mathcal{C}, \mathscr{M}}(\mathcal{J})$ is defined for any arc $\mathcal{J}$ on $\mathcal{C}$, then we say that $\mathscr{M}$ has a density along $\mathcal{C}$.

Our definition of local spacing distribution differs slightly from that of [25], but they are the same once the limit $Q \rightarrow \infty$ is taken. Let $\mathcal{C}$ be a connected, piecewise smooth, compact curve in $\mathbb{R}^{k}$ and let $\mathscr{U}=\left\{u_{n} \in \mathcal{C}\right.$ : $1 \leq n \leq N\}$ be a finite sequence of consecutive points $u_{1}, u_{2}, \ldots, u_{N}$ on $\mathcal{C}$. Let $\mathcal{J}$ be a connected $\operatorname{arc}$ of $\mathcal{C}$. Denote $\mathscr{U}(\mathcal{J})=\left\{1 \leq j \leq N: u_{j} \in \mathcal{J}\right\}$.

Let $h \geq 1$ be an integer. For any real numbers $\lambda_{1}, \ldots, \lambda_{h} \geq 0$, let $\lambda=$ $\left(\lambda_{1}, \ldots, \lambda_{h}\right)$ and define

$$
\begin{aligned}
G_{\mathcal{J}, \mathscr{U}}(\lambda) & =\frac{1}{\# \mathscr{U}(\mathcal{J})} \\
& \times \# \bigcap_{s=1}^{h}\left\{j: l\left({\widetilde{u_{j+s-1} u_{j+s}}}\right) \geq \frac{\lambda_{s} l(\mathcal{J})}{\# \mathscr{U}(\mathcal{J})}, u_{j+s-1}, u_{j+s} \in \mathscr{U}(\mathcal{J})\right\} .
\end{aligned}
$$

Let now $\mathscr{M}=\left(\mathscr{M}_{Q}\right)_{Q \in \mathbb{N}}$ be a sequence of finite sequences $\mathscr{M}_{Q}=\left\{P_{Q, n} \in\right.$ $\left.\mathcal{C}: 1 \leq n \leq N_{Q}\right\}$ of consecutive points on the curve $\mathcal{C}$, such that $N_{Q} \rightarrow \infty$ as $Q \rightarrow \infty$. We say that $\mathscr{M}$ has an $h$-spacing distribution function $G_{\mathcal{J}, \mathscr{M}}$ on a subarc $\mathcal{J} \subset \mathcal{C}$ provided that the sequence of functions $\left(G_{\mathcal{J}, \mathscr{M}_{Q}}\right)_{Q \in \mathbb{N}}$ is pointwise convergent to $G_{\mathcal{J}, \mathscr{M}}$. We need one more definition:

Definition 1. Suppose $\mathcal{C}$ is a connected compact curve, $\mathscr{M}=\left(\mathscr{M}_{Q}\right)_{Q \in \mathbb{N}}$ is a sequence of sequences of points on $\mathcal{C}$ such that $\# \mathscr{M}_{Q} \rightarrow \infty$ as $Q \rightarrow \infty$. Suppose $h \geq 1$ and the $h$-spacing distribution function $G$ of $\mathscr{M}$ along $\mathcal{C}$ exists and is continuous. $G$ is called uniformly continuous along $\mathcal{C}$ if for any $\epsilon>0$, there is a $\delta>0$ such that for any $\lambda=\left(\lambda_{1}, \ldots, \lambda_{h}\right), \mu=\left(\mu_{1}, \ldots, \mu_{h}\right), \lambda_{i}, \mu_{i} \geq$ 0 and any subarc $\mathcal{J} \subset \mathcal{C}$, we have

$$
\left|G_{\mathcal{J}, M}(\lambda)-G_{\mathcal{J}, M}(\mu)\right|<\epsilon
$$

whenever $\|\lambda-\mu\|_{\mathbb{R}^{h}}<\delta$.
Let $\mathbf{I}=[a, b]$ be an interval with length $l(\mathbf{I})=b-a$ and let $\mathcal{C} \subset \mathbb{R}^{m}, \mathcal{D} \subset$ $\mathbb{R}^{n}$ be two curves of finite length $l(\mathcal{C})$ and $l(\mathcal{D})$ with parameterizations $\phi: \mathbf{I} \longrightarrow \mathcal{C}$ and $\psi: \mathbf{I} \longrightarrow \mathcal{D}$ given by

$$
\phi(t)=\left(\phi_{1}(t), \ldots, \phi_{m}(t)\right), \quad \psi(t)=\left(\psi_{1}(t), \ldots, \psi_{n}(t)\right) .
$$

Suppose further that both $\phi, \psi$ are continuous, piecewise continuously differentiable and the functions

$$
\phi^{\prime}(t)=\left(\phi_{1}^{\prime}(t), \ldots, \phi_{m}^{\prime}(t)\right), \quad \psi^{\prime}(t)=\left(\psi_{1}^{\prime}(t), \ldots, \psi_{m}^{\prime}(t)\right)
$$

do not vanish in $\mathbf{I}$.
Let $\mathscr{F}=\left(\mathscr{F}_{Q}\right)_{Q \in \mathbb{N}}$ be a sequence of sets where $\mathscr{F}_{Q}=\left\{t_{j}^{Q}: 1 \leq j \leq\right.$ $\left.N_{Q}\right\} \subset \mathbf{I}$ with increased order $a \leq t_{1}^{Q}<t_{2}^{Q}<\cdots<t_{N_{Q}}^{Q} \leq b$ such that $N_{Q} \rightarrow \infty$ as $Q \rightarrow \infty$. Denoting $x_{j}^{Q}=\phi\left(t_{j}^{Q}\right)$ and $y_{j}^{Q}=\psi\left(t_{j}^{Q}\right)$ we form two sequences $\mathscr{M}=\left(\mathscr{M}_{Q}\right)_{Q \in \mathbb{N}}$ and $\mathscr{N}=\left(\mathscr{N}_{Q}\right)_{Q \in \mathbb{N}}$ by

$$
\mathscr{M}_{Q}=\left\{x_{j}^{Q}: 1 \leq j \leq N_{Q}\right\}, \quad \mathscr{N}_{Q}=\left\{y_{j}^{Q}: 1 \leq j \leq N_{Q}\right\} .
$$

We have the following result:

Theorem 4. Suppose that on $\mathcal{C}$ the sequence $\mathscr{M}=\left(\mathscr{M}_{Q}\right)_{Q \in \mathscr{N}}$ has bounded density. Let $h$ be a positive integer. If the $h$-spacing distribution function $G_{\mathcal{C}, \mathscr{M}}$ of $\mathscr{M}$ is uniformly continuous along $\mathcal{C}$, then the $h$-spacing distribution function $G_{\mathcal{D}, \mathscr{N}}$ of $\mathscr{N}$ on $\mathcal{D}$ exists and is continuous. More precisely, for any $\lambda=\left(\lambda_{1}, \ldots, \lambda_{h}\right), \lambda_{i} \geq 0$,
$G_{\mathcal{D}, \mathscr{N}}(\lambda)=\lim _{\delta(\pi) \rightarrow 0} \frac{1}{l(\mathcal{C})} \cdot \sum_{i=0}^{L-1} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) l\left(\mathcal{J}_{i}\right) \cdot G_{\mathcal{J}_{i}, \mathscr{M}}\left(\frac{l(\mathcal{D})\left|\phi^{\prime}\left(a_{i}\right)\right|}{l(\mathcal{C})\left|\psi^{\prime}\left(a_{i}\right)\right|} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) \cdot \lambda\right)$
where the limit is taken over partitions $\pi: a=a_{0}<a_{1}<a_{2}<\cdots<a_{L}=b$ of $\mathbf{I}, \delta(\pi)=\max \left\{a_{i+1}-a_{i}: 0 \leq i \leq L-1\right\}, \mathcal{J}_{i}=\phi\left(a_{i}\right) \phi\left(a_{i+1}\right)$ is the $i$-th arc of $\mathcal{C}, \rho$ is the density function and $l\left(\mathcal{J}_{i}\right)$ is the length of the arc $\mathcal{J}_{i}$.

Proof of Theorem 4. Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{h}\right), \lambda_{i} \geq 0$. We need to study the behavior of the quantity

$$
G_{\mathcal{D}, \mathcal{N}_{Q}}(\lambda)=\frac{1}{N_{Q}} \# \bigcap_{s=1}^{h}\left\{1 \leq j \leq N_{Q}+1-h: l\left(\widetilde{y_{j+s-1}^{Q} y_{j+s}^{Q}}\right) \geq \frac{\lambda_{s} l(\mathcal{D})}{N_{Q}}\right\}
$$

as $Q \rightarrow \infty$. For this purpose, we make a partition of $\mathbf{I}$,

$$
\pi: a=a_{0}<a_{1}<a_{2}<\cdots<a_{L}=b,
$$

and denote $A_{i}=\phi\left(a_{i}\right), B_{i}=\psi\left(a_{i}\right)$ for $i=0,1, \ldots, L, \mathbf{I}_{i}=\left[a_{i}, a_{i+1}\right], \mathcal{J}_{i}=$ $\widetilde{A_{i} A_{i+1}}, \mathcal{J}_{i}^{\prime}=\widetilde{B_{i} B_{i+1}}$ as subarcs of $\mathcal{J}, \mathcal{C}, \mathcal{D}$ respectively for $i=0, \ldots, L-1$. Then

$$
\mathbf{I}=\bigcup_{i=0}^{L-1} \mathbf{I}_{i}, \quad \mathcal{C}=\bigcup_{i=0}^{L-1} \mathcal{J}_{i}, \quad \mathcal{D}=\bigcup_{i=0}^{L-1} \mathcal{J}_{i}^{\prime}
$$

Moreover, for $i=0,1, \ldots, L-1$ denote

$$
\begin{aligned}
\mathscr{F}_{Q}\left(\mathbf{I}_{i}\right) & =\left\{t_{j}^{Q} \in \mathbf{I}_{i}: 1 \leq j \leq N_{Q}\right\} \\
\mathscr{M}_{Q}\left(\mathcal{J}_{i}\right) & =\left\{x_{j}^{Q} \in \mathcal{J}_{i}: 1 \leq j \leq N_{Q}\right\} \\
\mathscr{N}_{Q}\left(\mathcal{J}_{i}\right) & =\left\{y_{j}^{Q} \in \mathcal{J}_{i}^{\prime}: 1 \leq j \leq N_{Q}\right\}
\end{aligned}
$$

and for simplicity

$$
N_{Q, i}=\# \mathscr{F}_{Q}\left(\mathbf{I}_{i}\right)=\# \mathscr{M}_{Q}\left(\mathcal{J}_{i}\right)=\# \mathscr{N}_{Q}\left(\mathcal{J}_{i}^{\prime}\right)
$$

Let

$$
H_{\mathcal{J}_{i}^{\prime}, \mathcal{N}_{Q}}(\lambda)=\frac{1}{N_{Q, i}} \# \bigcap_{s=1}^{h}\left\{j: l\left(y_{j+s-1}^{Q} y_{j+s}^{Q}\right) \geq \frac{\lambda_{s} l(\mathcal{D})}{N_{Q}}, y_{j+s-1}^{Q}, y_{j+s}^{Q} \in \mathcal{J}_{i}^{\prime}\right\}
$$

It is easy to see that

$$
\begin{equation*}
N_{Q} G_{\mathcal{D}, \mathscr{N}_{Q}}(\lambda)-h(L-1) \leq \sum_{i=0}^{L-1} N_{Q, i} H_{\mathcal{J}_{i}^{\prime}, \mathscr{N}_{Q}}(\lambda) \leq N_{Q} G_{\mathcal{D}, \mathscr{N}_{Q}}(\lambda) \tag{4}
\end{equation*}
$$

Denote

$$
\begin{array}{ll}
M_{i}=\max _{t \in \mathbf{I}_{i}}\left(\left|\phi^{\prime}(t)\right|\right), & m_{i}=\min _{t \in \mathbf{I}_{i}}\left(\left|\phi^{\prime}(t)\right|\right), \\
M_{i}^{\prime}=\max _{t \in \mathbf{I}_{i}}\left(\left|\psi^{\prime}(t)\right|\right), & m_{i}^{\prime}=\min _{t \in \mathbf{I}_{i}}\left(\left|\psi^{\prime}(t)\right|\right) .
\end{array}
$$

When $t_{j+s-1}^{Q}, t_{j+s}^{Q} \in \mathbf{I}_{i}$, we have

$$
\begin{aligned}
m_{i}\left(t_{j+s}^{Q}-t_{j+s-1}^{Q}\right) & \leq l\left(\widetilde{x_{j+s-1}^{Q} x_{j+s}^{Q}}\right)=\int_{t_{j+s-1}^{Q}}^{t_{j+s}^{Q}}\left|\phi^{\prime}(t)\right| d t \\
& \leq M_{i}\left(t_{j+s}^{Q}-t_{j+s-1}^{Q}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
m_{i}^{\prime}\left(t_{j+s}^{Q}-t_{j+s-1}^{Q}\right) & \leq l\left(y_{j+s-1}^{Q} x_{j+s}^{Q}\right)=\int_{t_{j+s-1}^{Q}}^{t_{j+s}^{Q}}\left|\psi^{\prime}(t)\right| d t \\
& \leq M_{i}^{\prime}\left(t_{j+s}^{Q}-t_{j+s-1}^{Q}\right)
\end{aligned}
$$

Define

$$
L_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda)=\frac{1}{N_{Q, i}} \# \bigcap_{s=1}^{h}\left\{j: l\left(x_{j+s-1}^{Q} x_{j+s}^{Q}\right) \geq \frac{\lambda_{s} l(\mathcal{D}) M_{i}}{N_{Q} m_{i}^{\prime}}, x_{j+s-1}^{Q}, x_{j+s}^{Q} \in \mathcal{J}_{i}\right\}
$$

and

$$
U_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda)=\frac{1}{N_{Q, i}} \# \bigcap_{s=1}^{h}\left\{j: l\left(x_{j+s-1}^{Q} x_{j+s}^{Q}\right) \geq \frac{\lambda_{s} l(\mathcal{D}) m_{i}}{N_{Q} M_{i}^{\prime}}, x_{j+s-1}^{Q}, x_{j+s}^{Q} \in \mathcal{J}_{i}\right\} .
$$

By the inequalities above and the definition, we have

$$
L_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda) \leq H_{\mathcal{J}_{i}^{\prime}, \mathcal{N}_{Q}}(\lambda) \leq U_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda) .
$$

Taking (4) into account we have

$$
\sum_{i=0}^{L-1} \frac{N_{Q, i}}{N_{Q}} \cdot L_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda) \leq G_{\mathcal{D}, \mathscr{N}_{Q}}(\lambda) \leq \sum_{i=0}^{L-1} \frac{N_{Q, i}}{N_{Q}} \cdot U_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda)+\frac{h(L-1)}{N_{Q}}
$$

By our assumptions,

$$
\lim _{Q \rightarrow \infty} \frac{N_{Q, i}}{N_{Q}}=\mu_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right)=\rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) \cdot \frac{l\left(\mathcal{J}_{i}\right)}{l(\mathcal{C})} .
$$

Denote

$$
\delta_{i}=\frac{l(\mathcal{D}) M_{i}}{l(\mathcal{C}) m_{i}^{\prime}} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right), \quad \delta_{i}^{\prime}=\frac{l(\mathcal{D}) m_{i}}{l(\mathcal{C}) M_{i}^{\prime}} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right)
$$

Since the $h$-spacing distribution function $G_{\mathcal{J}_{i}, \mathscr{M}_{Q}}(\lambda)$ exists and is continuous, we have

$$
L_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda) \longrightarrow G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i} \cdot \lambda\right) \quad(Q \rightarrow \infty)
$$

and

$$
U_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda) \longrightarrow G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i}^{\prime} \cdot \lambda\right) \quad(Q \rightarrow \infty)
$$

Therefore

$$
\begin{aligned}
\lim _{Q \rightarrow \infty} \sum_{i=0}^{L-1} \frac{N_{Q, i}}{N_{Q}} \cdot L_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda) & =\frac{1}{l(\mathcal{C})} \cdot \sum_{i=0}^{L-1} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) l\left(\mathcal{J}_{i}\right) G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i} \lambda\right) \\
& =L_{\pi}(\lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
\lim _{Q \rightarrow \infty} \sum_{i=0}^{L-1} \frac{N_{Q, i}}{N_{Q}} \cdot U_{\mathcal{J}_{i}, \mathscr{F}_{Q}}(\lambda) & =\frac{1}{l(\mathcal{C})} \cdot \sum_{i=0}^{L-1} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) l\left(\mathcal{J}_{i}\right) G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i}^{\prime} \lambda\right) \\
& =U_{\pi}(\lambda)
\end{aligned}
$$

Hence

$$
\begin{equation*}
L_{\pi}(\lambda) \leq \liminf _{Q \rightarrow \infty} G_{\mathcal{D}, \mathscr{N}_{Q}}(\lambda) \leq \limsup _{Q \rightarrow \infty} G_{\mathcal{D}, \mathscr{N}_{Q}}(\lambda) \leq U_{\pi}(\lambda) \tag{5}
\end{equation*}
$$

for any partition $\pi$ of $\mathbf{I}$. Our next goal is to prove that $\lim _{Q \rightarrow \infty} G_{\mathcal{D}, \mathcal{N}_{Q}}(\lambda)$ exists by manipulating the lower bound $L_{\pi}(\lambda)$ and upper bound $U_{\pi}(\lambda)$ for different partitions $\pi$. Since the $h$-spacing distribution function $G_{\mathcal{C}, \mathscr{M}}$ of $\mathscr{M}$ is uniformly continuous along $\mathcal{C}$, for any $\epsilon>0$, there is a $\delta>0$, such that when $x=\left(x_{1}, \ldots, x_{h}\right), y=\left(y_{1}, \ldots, y_{h}\right), x_{i}, y_{i} \geq 0$ and $\|x-y\|_{\mathbb{R}^{h}}<\delta$, then

$$
\left|G_{\mathcal{C}^{\prime}, \mathscr{M}}(x)-G_{\mathcal{C}^{\prime}, \mathscr{M}}(y)\right|<\epsilon
$$

for any subarc $\mathcal{C}^{\prime} \subset \mathcal{C}$. Because $\rho_{\mathbf{I}, \mathscr{F}}$ is bounded, there is an $\eta>0$ such that if $\delta(\pi)=\max _{i}\left\{l\left(\mathcal{J}_{i}\right)\right\}<\eta$, then $\left\|\left(\delta_{i}-\delta_{i}^{\prime}\right) \lambda\right\|_{R^{h}}<\delta$ and we have

$$
\begin{aligned}
0 \leq U_{\pi}(\lambda)-L_{\pi}(\lambda) & =\frac{1}{l(\mathcal{C})} \cdot \sum_{i=0}^{L-1} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) l\left(\mathcal{J}_{i}\right) \cdot\left[G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i} \lambda\right)-G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i}^{\prime} \lambda\right)\right] \\
& \leq \frac{1}{l(\mathcal{C})} \cdot \sum_{i=0}^{L-1} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) l\left(\mathcal{J}_{i}\right) \cdot \epsilon \\
& =\epsilon \cdot \sum_{i=0}^{L-1} \mu_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right)=\epsilon
\end{aligned}
$$

Therefore

$$
\begin{aligned}
G_{\mathcal{D}, \mathscr{N}}(\lambda) & =\lim _{Q \rightarrow \infty} G_{\mathcal{D}, \mathcal{N}_{Q}}(\lambda) \\
& =\lim _{\delta(\pi) \rightarrow 0} L_{\pi}(\lambda)=\lim _{\delta(\pi) \rightarrow 0} U_{\pi}(\lambda),
\end{aligned}
$$

the limit exists and $\mathscr{N}$ has an $h$-spacing distribution function on $\mathcal{D}$.
In order to prove that this function is continuous, observe that

$$
\begin{aligned}
\left|G_{\mathcal{D}, \mathscr{N}}(\lambda)-G_{\mathcal{D}, \mathscr{N}}(\mu)\right| \leq & \left|G_{\mathcal{D}, \mathscr{N}}(\lambda)-U_{\pi}(\lambda)\right|+\left|G_{\mathcal{D}, \mathscr{N}}(\mu)-U_{\pi}(\mu)\right| \\
& +\left|U_{\pi}(\lambda)-U_{\pi}(\mu)\right| .
\end{aligned}
$$

The first two terms on the right side tend to 0 as $\delta(\pi) \rightarrow 0$, and

$$
\left|U_{\pi}(\lambda)-U_{\pi}(\mu)\right| \leq \frac{1}{l(\mathcal{C})} \cdot \sum_{i=0}^{L-1} \rho_{\mathcal{C},, \mathscr{M}}\left(\mathcal{J}_{i}\right) l\left(\mathcal{J}_{i}\right) \cdot\left|G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i}^{\prime} \lambda\right)-G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i}^{\prime} \mu\right)\right|
$$

By similar argument we see that $\left|U_{\pi}(\lambda)-U_{\pi}(\mu)\right| \longrightarrow 0$ as $\lambda \rightarrow \mu$. Therefore $G_{\mathcal{D}, \mathscr{N}}(\lambda)$ is continuous as a function of $\lambda$.

Next denote

$$
G_{\mathcal{D}, \pi}(\lambda)=\frac{1}{l(\mathcal{C})} \cdot \sum_{i=0}^{L-1} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) l\left(\mathcal{J}_{i}\right) \cdot G_{\mathcal{J}_{i}, \mathscr{M}}\left(\frac{l(\mathcal{D})\left|\phi^{\prime}\left(a_{i}\right)\right|}{l(\mathcal{C})\left|\psi^{\prime}\left(a_{i}\right)\right|} \rho_{\mathcal{C}, \mathscr{M}}\left(\mathcal{J}_{i}\right) \cdot \lambda\right) .
$$

Since

$$
\frac{m_{i}}{M_{i}^{\prime}} \leq \frac{\left|\phi^{\prime}\left(a_{i}\right)\right|}{\left|\psi^{\prime}\left(a_{i}\right)\right|} \leq \frac{M_{i}}{m_{i}^{\prime}},
$$

one has

$$
L_{\pi}(\lambda) \leq G_{\mathcal{D}, \pi}(\lambda) \leq U_{\pi}(\lambda)
$$

Hence

$$
G_{\mathcal{D}, \mathscr{M}}(\lambda)=\lim _{Q \rightarrow \infty} G_{\mathcal{D}, \mathscr{M}_{Q}}(\lambda)=\lim _{\delta(\pi) \rightarrow 0} G_{\mathcal{D}, \pi}(\lambda),
$$

and Theorem 4 is proved.
Under the assumption that the sequence is uniformly distributed, i.e., the density function $\rho$ identically equals to 1 , we have a stronger result.
Corollary 2. Assume the same conditions and notations as in Theorem 4. Suppose further that $\mathscr{M}$ is uniformly distributed on $\mathcal{C}$. Then the h-spacing distribution function $G_{\mathcal{D}, \mathcal{N}}$ of $\mathscr{N}$ on $\mathcal{D}$ exits, is uniformly continuous along $\mathcal{D}$ and is given by,

$$
\begin{equation*}
G_{\mathcal{D}, \mathscr{N}}(\lambda)=\lim _{\delta(\pi) \rightarrow 0} \frac{1}{l(\mathcal{C})} \cdot \sum_{i=0}^{L-1} l\left(\mathcal{J}_{i}\right) \cdot G_{\mathcal{J}_{i}, \mathscr{M}}\left(\frac{l(\mathcal{D})\left|J \phi\left(a_{i}\right)\right|}{l(\mathcal{C})\left|J \psi\left(a_{i}\right)\right|} \cdot \lambda\right) \tag{6}
\end{equation*}
$$

where the limit is taken over partitions $\pi: a=a_{0}<a_{1}<a_{2}<\cdots<a_{L}=b$ of $\mathbf{I}$.

Proof of Corollary 2. For any subinterval $\mathbf{I}^{\prime}=\left[a^{\prime}, b^{\prime}\right] \subset \mathbf{I}=[a, b]$, denote $\mathcal{C}^{\prime}=\phi\left(\mathbf{I}^{\prime}\right) \subset \mathcal{C}, \mathcal{D}^{\prime}=\psi\left(\mathbf{I}^{\prime}\right) \subset \mathcal{D}$. Consider a partition $\pi: a^{\prime}=a_{0}<a_{1}<$ $\cdots<a_{L}=b^{\prime}$ of $\mathbf{I}^{\prime}$. We use same notations as in the proof of Theorem 1. It is easy to see that, since $\rho_{\mathcal{C}, \mathscr{M}} \equiv 1$, we still have the inequality

$$
L_{\pi}(\lambda) \leq \liminf _{Q \rightarrow \infty} G_{\mathcal{D}^{\prime}, \mathscr{N}_{Q}}(\lambda) \leq \limsup _{Q \rightarrow \infty} G_{\mathcal{D}^{\prime}, \mathcal{N}_{Q}}(\lambda) \leq U_{\pi}(\lambda)
$$

for any partition $\pi$ of $\mathbf{I}^{\prime}$, where

$$
\begin{aligned}
L_{\pi}(\lambda) & =\frac{1}{l\left(\mathcal{C}^{\prime}\right)} \cdot \sum_{i=0}^{L-1} l\left(\mathcal{J}_{i}\right) G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i} \lambda\right), \\
U_{\pi}(\lambda) & =\frac{1}{l\left(\mathcal{C}^{\prime}\right)} \cdot \sum_{i=0}^{L-1} l\left(\mathcal{J}_{i}\right) G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i}^{\prime} \lambda\right),
\end{aligned}
$$

and

$$
\delta_{i}=\frac{l\left(\mathcal{D}^{\prime}\right) M_{i}}{l\left(\mathcal{C}^{\prime}\right) m_{i}^{\prime}}, \quad \delta_{i}^{\prime}=\frac{l\left(\mathcal{D}^{\prime}\right) m_{i}}{l\left(\mathcal{C}^{\prime}\right) M_{i}^{\prime}} .
$$

Following the same argument as in Theorem 1, we see that

$$
G_{\mathcal{D}^{\prime}, \mathscr{N}}(\lambda)=\lim _{Q \rightarrow \infty} G_{\mathcal{D}^{\prime}, \mathcal{N}_{Q}}(\lambda)
$$

exists, is continuous and $G_{\mathcal{D}, \mathscr{N}}(\lambda)$ can be written explicitly as in (6).
To prove that this function is uniformly continuous along $\mathcal{D}$, first note that the $h$-spacing distribution function $G_{\mathcal{C}, \mathscr{M}}$ of $\mathscr{M}$ is uniformly continuous along $\mathcal{C}$, for any $\epsilon>0$, there is a $\delta>0$, such that when $x=\left(x_{1}, \ldots, x_{h}\right), y=$ $\left(y_{1}, \ldots, y_{h}\right), x_{i}, y_{i} \geq 0$ and $\|x-y\|_{\mathbb{R}^{h}}<\delta$, then

$$
\left|G_{\mathcal{C}^{\prime}, \mathscr{M}}(x)-G_{\mathcal{C}^{\prime}, \mathscr{M}}(y)\right|<\epsilon
$$

for any subarc $\mathcal{C}^{\prime} \subset \mathcal{C}$.
Denote

$$
\begin{aligned}
& M=\max _{t \in \mathbf{I}}\left(\left|\phi^{\prime}(t)\right|\right), m=\min _{t \in \mathbf{I}}\left(\left|\phi^{\prime}(t)\right|\right), \\
& M^{\prime}=\max _{t \in \mathbf{I}}\left(\left|\psi^{\prime}(t)\right|\right), \quad m^{\prime}=\min _{t \in \mathbf{I}}\left(\left|\psi^{\prime}(t)\right|\right) .
\end{aligned}
$$

Notice that

$$
0 \leq \delta_{i}^{\prime} \leq \delta_{i} \leq \frac{M M^{\prime}}{m m^{\prime}}
$$

For any $\lambda=\left(\lambda_{1}, \ldots, \lambda_{h}\right), \mu=\left(\mu_{1}, \ldots, \mu_{h}\right), \lambda_{i}, \mu_{i} \geq 0$ with $\|(\lambda-\mu)\|_{\mathbb{R}^{h}}<$ $\delta \cdot \frac{m m^{\prime}}{M M^{\prime}}$, then $\left\|(\lambda-\mu) \delta_{i}^{\prime}\right\|_{\mathbb{R}^{h}}<\delta$, we have

$$
\begin{aligned}
\left|U_{\pi}(\lambda)-U_{\pi}(\mu)\right| & \leq \frac{1}{l\left(\mathcal{C}^{\prime}\right)} \cdot \sum_{i=0}^{L-1} l\left(\mathcal{J}_{i}\right) \cdot\left|G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i}^{\prime} \lambda\right)-G_{\mathcal{J}_{i}, \mathscr{M}}\left(\delta_{i}^{\prime} \mu\right)\right| \\
& \leq \frac{1}{l\left(\mathcal{C}^{\prime}\right)} \cdot \sum_{i=0}^{L-1} l\left(\mathcal{J}_{i}\right) \cdot \epsilon=\epsilon
\end{aligned}
$$

for any subinterval $\mathbf{I}^{\prime} \subset \mathbf{I}$ and any partition $\pi$ of $\mathbf{I}^{\prime}$. Also,

$$
\begin{aligned}
\left|G_{\mathcal{D}^{\prime}, \mathscr{N}}(\lambda)-G_{\mathcal{D}^{\prime}, \mathscr{N}}(\mu)\right| \leq & \left|G_{\mathcal{D}^{\prime}, \mathscr{N}}(\lambda)-U_{\pi}(\lambda)\right|+\left|G_{\mathcal{D}^{\prime}, \mathscr{N}}(\mu)-U_{\pi}(\mu)\right| \\
& +\left|U_{\pi}(\lambda)-U_{\pi}(\mu)\right| .
\end{aligned}
$$

Here the first two terms on the right side tend to 0 as $\delta(\pi) \rightarrow 0$, we have

$$
\left|G_{\mathcal{D}^{\prime}, \mathscr{N}}(\lambda)-G_{\mathcal{D}^{\prime}, \mathscr{N}}(\mu)\right| \leq \lim _{\delta(\pi) \rightarrow 0}\left|U_{\pi}(\lambda)-U_{\pi}(\mu)\right| \leq \epsilon
$$

Therefore the $h$-spacing distribution function $G_{\mathcal{D}, \mathscr{N}}(\lambda)$ is uniformly continuous along $\mathcal{D}$ and Corollary 2 is proved.

Proof of Theorem 1. For the nearest neighbor spacing distribution function we have $h=1$. Since $\mathscr{F}$ is uniformly distributed on $\mathbf{I}$ and obvious by the assumption, the nearest neighbor spacing distribution function $G$ of $\mathscr{F}$ is uniformly continuous along I, by Corollary 2 , the function $G_{\mathcal{C}, \mathscr{M}}$ of $\mathscr{M}$ on $\mathcal{C}$ is also uniformly continuous along $\mathcal{C}$ and can be written explicitly as

$$
\begin{aligned}
G_{\mathcal{C}, \mathscr{M}}(\lambda) & =\lim _{\delta(\pi) \rightarrow 0} \frac{1}{l(\mathbf{I})} \cdot \sum_{i=0}^{L-1} l\left(\mathbf{I}_{i}\right) \cdot G_{\mathbf{I}_{i}, \mathscr{F}}\left(\frac{l(\mathcal{C})}{l(\mathbf{I})\left|f^{\prime}\left(a_{i}\right)\right|} \cdot \lambda\right) \\
& =\lim _{\delta(\pi) \rightarrow 0} \frac{1}{l(\mathbf{I})} \cdot \sum_{i=0}^{L-1} l\left(\mathbf{I}_{i}\right) \cdot H\left(\frac{l(\mathcal{C})}{l(\mathbf{I})\left|f^{\prime}\left(a_{i}\right)\right|} \cdot \lambda\right) \\
& =\frac{1}{l(\mathbf{I})} \cdot \int_{a}^{b} H\left(\frac{l(\mathcal{C})}{l(\mathbf{I})\left|f^{\prime}(t)\right|} \cdot \lambda\right) d t,
\end{aligned}
$$

for any $\lambda \geq 0$. This completes the proof of Theorem 1 .

## 3. TORSION POINTS ON ELLIPTIC CURVES

We will compute for any $h \geq 1$ the $h$-spacing distribution function associated to the set of torsion points on the given elliptic curve $E$ defined over
$\mathbb{R}$. For any $Q \in \mathbb{N}$, denote by $\mathscr{F}_{Q}(\mathbf{I})$ the set of Farey fractions of order $Q$ from $\mathbf{I}$, that is

$$
\mathscr{F}_{Q}(\mathbf{I})=\left\{\gamma=\frac{p}{q} \in \mathbf{I}: 1 \leq q \leq Q, \operatorname{gcd}(p, q)=1, a, q \in \mathbb{N}\right\}
$$

and order increasingly its elements $\gamma_{j}^{Q}=p_{j} / q_{j}$ as

$$
a \leq \gamma_{1}^{Q}<\gamma_{2}^{Q}<\cdots<\gamma_{N_{Q}(\mathbf{I})}^{Q} \leq b
$$

The number $N_{Q}(\mathbf{I})$ of elements of $\mathscr{F}_{Q}(\mathbf{I})$ satisfies (see [1])

$$
N_{Q}(\mathbf{I})=3 l(\mathbf{I}) Q^{2} / \pi^{2}+O(Q \log Q) .
$$

The sequence $\mathscr{F}(\mathbf{I})=\left(\mathscr{F}_{Q}(\mathbf{I})\right)_{Q \in \mathbb{N}}$ is uniformly distributed on $\mathbf{I}$. Denote $x_{j}^{Q}=\phi\left(\gamma_{j}^{Q}\right)$ and

$$
\mathscr{M}_{Q}(\mathcal{C})=\left\{x_{j}^{Q}: 1 \leq j \leq N_{Q}(\mathbf{I})\right\}
$$

where $x_{1}^{Q}, x_{2}^{Q}, \ldots, x_{N_{Q}(\mathbf{I})}^{Q}$ are consecutive points on $\mathcal{C}$. Then $\mathscr{M}_{Q}(\mathcal{C})$ is exactly the set of torsion points on $\mathcal{C}$ with order less than or equal to $Q$. Let $\mathscr{M}(\mathcal{C})=\left(\mathscr{M}_{Q}(\mathcal{C})\right)_{Q \in \mathbb{N}}$. For $h$ any positive integer, $\lambda=\left(\lambda_{1}, \ldots, \lambda_{h}\right), \lambda_{i} \geq 0$, consider the quantity

$$
G_{\mathcal{C}, \mu_{Q}}(\lambda)=\frac{1}{N_{Q}(\mathbf{I})} \# \bigcap_{s=1}^{h}\left\{1 \leq j \leq N_{Q}(\mathbf{I})+1-h: l\left(x_{j+s-1}^{Q} x_{j+s}^{Q}\right) \geq \frac{\lambda_{s} l(\mathcal{C})}{N_{Q}(\mathbf{I})}\right\}
$$

According to [1], $G_{\mathcal{C}, M_{Q}}$ converges as $Q \rightarrow \infty$ to a continuous function, hence by Theorem 1, the $h$-spacing distribution function

$$
G_{\mathcal{C}, \mathscr{M}}(\lambda)=\lim _{Q \rightarrow \infty} G_{\mathcal{C}, \mathscr{M}_{Q}}(\lambda)
$$

exists and is continuous.
To make things more precise, let $\mathscr{T}$ be the Farey triangle

$$
\{(x, y): 0<x \leq 1,0<y \leq 1, x+y>1\}
$$

and consider, for each $(x, y) \in \mathbb{R}^{2}$, the sequence $\left(L_{i}(x, y)\right)_{i \geq 0}$ defined by $L_{0}(x, y)=x, L_{1}(x, y)=y$ and then recursively, for $i \geq 2$,

$$
L_{i}(x, y)=\left[\frac{1+L_{i-2}(x, y)}{L_{i-1}(x, y)}\right] \cdot L_{i-1}(x, y)-L_{i-2}(x, y)
$$

Consider as well the map $\Phi_{h}: \mathscr{T} \longrightarrow(0, \infty)^{h}$ given by

$$
\Phi_{h}(x, y)=\frac{3}{\pi^{2}}\left(\frac{1}{L_{0}(x, y) L_{1}(x, y)}, \frac{1}{L_{1}(x, y) L_{2}(x, y)}, \ldots, \frac{1}{L_{h-1}(x, y) L_{h}(x, y)}\right)
$$

To each $\lambda=\left(\lambda_{i}, \ldots, \lambda_{h}\right), \lambda_{i} \geq 0$, we associate a set $\mathscr{B}_{\lambda}=\prod_{i=1}^{h}\left(\lambda_{i},+\infty\right) \subset$ $(0, \infty)^{h}$, and define the following subset of $\mathscr{T}$ :

$$
\widetilde{\Omega}_{\mathscr{B}_{\lambda}}=\bigcap_{i=1}^{h}\left\{(x, y) \in \mathscr{T}: L_{i-1}(x, y) L_{i}(x, y) \leq \frac{3}{\pi^{2} \lambda_{i}}\right\}=\Phi_{h}^{-1}\left(\mathscr{B}_{\lambda}\right)
$$

Theorem 1 of [1] states that the $h$-spacing distribution function of $\mathscr{F}_{Q}(\mathbf{I})$ on $\mathbf{I}$ is given by

$$
G_{\mathbf{I}, \mathscr{F}}(\lambda)=2 \cdot \operatorname{Area}\left(\widetilde{\Omega}_{\mathscr{B}_{\lambda}}\right) .
$$

For any $a=\left(a_{1}, \ldots, a_{h}\right), a_{i} \geq 0$, denote

$$
\Omega_{a}=\bigcup_{i=1}^{h}\left\{(x, y) \in \mathscr{T}: L_{i-1}(x, y) L_{i}(x, y)>a_{i}\right\}
$$

and define a continuous function $A_{h}:(0,+\infty)^{h} \longrightarrow[0,1]$ by

$$
A_{h}(a)=\operatorname{Area}\left(\Omega_{a}\right)
$$

Then

$$
G_{\mathbf{I}, \mathscr{F}}(\lambda)=1-2 A_{h}\left(\frac{3}{\pi^{2} \lambda}\right) .
$$

By Theorem 1,

$$
\begin{aligned}
G_{\mathcal{C}, \mathscr{M}}(\lambda) & =\frac{1}{l(\mathbf{I})} \cdot \int_{a}^{b}\left(1-2 A_{h}\left(\frac{3 l(\mathbf{I})\left|\phi^{\prime}(t)\right|}{\pi^{2} l(\mathcal{C}) \lambda}\right)\right) d t \\
& =1-\frac{2}{l(\mathbf{I})} \cdot \int_{a}^{b} A_{h}\left(\frac{3 l(\mathbf{I})\left|\phi^{\prime}(t)\right|}{\pi^{2} l(\mathcal{C}) \lambda}\right) d t
\end{aligned}
$$

As for the case $h=1$, it is known (see [1]) that

$$
A_{1}(a)= \begin{cases}1-a-\sqrt{1-4 a} / 2+2 a \log ((1+\sqrt{1-4 a}) / 2) & : 0<a \leq \frac{1}{4} \\ 1-a+a \log a & : \frac{1}{4}<a<1 \\ 0 & : a \geq 1\end{cases}
$$

Here $A_{1}$ is a piecewise smooth function and

$$
A_{1}^{\prime}(a)= \begin{cases}2 \log ((1+\sqrt{1-4 a}) / 2) & : 0<a<\frac{1}{4} \\ \log a & : \frac{1}{4}<a<1 \\ 0 & : a>1\end{cases}
$$

The nearest neighbor spacing distribution function of torsion points on $\mathcal{C}$ is then given explicitly by

$$
G_{\mathcal{C}, \mathfrak{M}, h=1}(\lambda)=1-\frac{2}{l(\mathbf{I})} \cdot \int_{a}^{b} A_{1}\left(\frac{3 l(\mathbf{I})\left|\phi^{\prime}(t)\right|}{\pi^{2} l(\mathcal{C}) \lambda}\right) d t
$$

Its density function is

$$
\begin{aligned}
g_{\mathcal{C}, \mathscr{M}, h=1}(\lambda) & =\lim _{\mu \rightarrow \lambda} \frac{G_{\mathcal{C}, \mathscr{M}, h=1}(\lambda)-G_{\mathcal{C}, \mathscr{M}, h=1}(\mu)}{\lambda-\mu} \\
& =\frac{6}{\pi^{2} l(\mathcal{C}) \lambda^{2}} \cdot \int_{a}^{b} A_{1}^{\prime}\left(\frac{3 l(\mathbf{I})\left|\phi^{\prime}(t)\right|}{\pi^{2} l(\mathcal{C}) \lambda}\right) \cdot\left|\phi^{\prime}(t)\right| d t .
\end{aligned}
$$

Clearly, for any point $P \in E_{U}(\mathbb{R}), P \neq \mathcal{O}$, the local density around $P$ is

$$
G_{E, P}(\lambda)=\lim _{\substack{P \in \mathcal{C} \subset E_{U}(\mathbb{R}) \\ l(\mathcal{C}) \rightarrow 0}} G_{\mathcal{C}, \mathscr{M}, h=1}(\lambda)=1-2 A_{1}\left(\frac{3}{\pi^{2} \lambda}\right)
$$

Theorem 2 and Corollary 1 are now proved.

## 4. Local Spacings of Rational points on Unit Circle

We need to calculate the nearest neighbor spacing distribution function of rational points on the Unit Circle $\mathcal{S}^{1}$. We use the same notation as in the statement of Theorem 3. Consider the map

$$
\begin{aligned}
& f_{2}:\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \\
& \theta \mapsto \operatorname{R} \\
& \tan \theta,
\end{aligned}
$$

so that we have $\phi=f_{2} \circ f_{1}:\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathcal{S}^{1}-\{(-1,0)\}$ given by

$$
\phi(\theta)=(\cos 2 \theta, \sin 2 \theta) .
$$

Here $\left|\phi^{\prime}(\theta)\right|=|2(-\cos (2 \theta), \sin (2 \theta))|=2$. Let $\mathscr{F}=\left(\mathscr{F}_{Q}\right)_{Q \in \mathbb{N}}$ be the sequence in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ given by $\mathscr{F}_{Q}=\phi^{-1}\left(\mathscr{M}_{Q}\right)$. For $\theta \in \mathscr{F}_{Q}, \theta \neq 0$, write $\tan \theta=a / q$ in reduced form. This $\theta$ equals to the angle between the $q$ axis and the straight line passing through the origin and the "visible" point $(a, q) \in \mathbb{Z}^{2}$ inside the disk

$$
D_{Q}=\left\{(a, q) \in \mathbb{R}^{2}: a^{2}+q^{2} \leq Q\right\}
$$

Without any loss of generality, suppose $\mathcal{C} \subset \mathcal{S}^{1}$ is a connected arc not containing the point $(-1,0)$. Then $\phi^{-1}(\mathcal{C})$ is a subinterval of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Denote this interval $\mathbf{I}=[a, b]$. We have $l(\mathcal{C})=2 \cdot l(\mathbf{I})$. According to Theorem 0.1 of [5], $\mathscr{F}$ is uniformly distributed on I and the nearest neighbor spacing distribution function of $\mathscr{F}$ on $\mathbf{I}$ is given by

$$
G_{\mathbf{I}, \mathscr{F}}(\lambda)=2 \eta(\lambda),
$$

where

$$
\eta(\lambda)=\left\{\begin{array}{cc}
\frac{1}{2} & : \quad 0<\lambda \leq \frac{3}{\pi^{2}} \\
\frac{3}{\pi^{2} \lambda} \cdot\left(1-\log \left(\frac{3}{\pi^{2} \lambda}\right)\right)-\frac{1}{2} & : \frac{3}{\pi^{2}}<\lambda<\frac{12}{\pi^{2}} \\
\frac{3}{\pi^{2} \lambda}+\frac{1}{2} \sqrt{1-\frac{12}{\pi^{2} \lambda}}-\frac{6}{\pi^{2} \lambda} & \\
\times \log \left(\left(1+\sqrt{1-\frac{12}{\pi^{2} \lambda}}\right) / 2\right)-\frac{1}{2} & : \quad \lambda \geq \frac{12}{\pi^{2}} .
\end{array}\right.
$$

By Theorem 1, then we have

$$
G_{\mathcal{C}, \mathscr{M}}(\lambda)=2 \eta(\lambda),
$$

and this completes the proof of Theorem 3.

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