PAIR CORRELATION OF SUMS OF RATIONALS WITH BOUNDED HEIGHT

EMRE ALKAN, MAOSHENG XIONG, AND ALEXANDRU ZAHARESCU

Abstract. For each positive integer $Q$, let $\mathcal{F}_Q$ denote the Farey sequence of order $Q$. We prove the existence of the pair correlation measure associated to the sum $\mathcal{F}_Q + \mathcal{F}_Q$ modulo 1, as $Q$ tends to infinity, and compute the corresponding limiting pair correlation function.

1. Introduction

The study of local spacings, which measure the distribution of a sequence in a more subtle way than the classical Weyl uniform distribution ([46]), was initiated by physicists (see Wigner [47] and Dyson [11], [12], [13]), in order to understand the spectra of high energies. These notions have received a great deal of attention in many areas of mathematical physics, analysis, probability theory and number theory. In most cases of interest in number theory it is very challenging to prove the existence of the limiting spacing measures. Many such sequences are predicted to have a Poisson distribution, and some important results of this type are due to Hooley [19], [20] on residue classes relatively prime with a large modulus $q$, Gallagher [17] on gaps between primes, Sarnak [39] on values at integers of binary quadratic forms, and Rudnick and Sarnak on pair correlation of fractional parts of polynomials [34]. Further results have been obtained by a number of authors. Primitive roots modulo $p$ were studied in [9], and the distribution of visible points from the origin in dilations of a region $\Omega$ was established in [4]. The spacing distribution of fractional parts of lacunary sequences has been obtained by Rudnick and one of the authors in [36] and [38] (see also [7]), and the distribution of small powers of a primitive root was studied in [37]. Boca and one of the authors [6] investigated the pair correlation of values of rational functions modulo $p$. Kurlberg and Rudnick [27] (see also [26]) established the distribution of squares modulo highly composite integers. The spacings between the energy levels of a two-dimensional harmonic oscillator (see Pandey, Bohigas and Giannoni [32] and Bleher [2], [3]) are essentially those between the numbers $\alpha n \pmod{1}$, where the gaps take at most three values (see Sós [40] and Świerczkowski [41]). The distribution of energy levels of a boxed oscillator reduces to that of $\alpha n^2 \pmod{1}$, which is conjectured to be Poissonian (see Berry and Tabor [1]). Rudnick, Sarnak and one of the authors [35] (see also [50]) proved that this conjecture holds true for a large class of numbers $\alpha$ satisfying certain Diophantine conditions. Eigenvalues on

2000 Mathematics Subject Classification. 11K06, 11L07.

Key words and phrases. Pair correlation, Farey fractions.

First author is supported in part by TUBITAK Career Award and Distinguished Young Scholar Award, TUBA-GEBIP of Turkish Academy of Sciences. Third author is supported in part by National Science Foundation Grant DMS-0456615.
multidimensional flat tori, and values at integers of homogeneous polynomials, were studied by Vanderkam [42], [43], [44]. Correlation densities of inhomogeneous quadratic forms were investigated by Marklof [28], [29]. The distribution of fractional part of $\sqrt{n}$ was established by Elkies and McMullen [14]. The distribution of imaginary parts of zeros of primitive L-functions is believed to be the same as the GUE distribution studied by Random Matrix Theory. Important work in this area was done by Montgomery [30], Rudnick and Sarnak [33], and Katz and Sarnak [24], see also [25]. One striking difference between the GUE model and the Poissonian model is that the density function vanishes at the origin in the GUE case but not in the Poissonian case. For this reason, it is said that in the Poissonian case one has “level clustering” while in the GUE case one has “level repulsion”. Here the word “level”, coined by physicists, refers to the possibly infinitely many stages of a process. One has an even stronger repulsion in the context described below.

Here we investigate a new type of question, which concerns two different notions: the pair correlation of the given sequence and the natural additive structure of the ambient space. More specifically, for each positive integer $Q$ let $\mathcal{F}_Q$ denote the Farey sequence of order $Q$ (for basic properties of the Farey sequence see [18]), as the $Q$th level of our process, that is, the set of all rationals in $[0, 1]$ of height bounded by $Q$ (the height of a rational number, in irreducible form, is defined to be the maximum of the absolute values of its numerator and denominator). The pair correlation measure associated to $\mathcal{F}_Q$ was proved to converge, as $Q \to \infty$, by Boca and one of the authors [8]. They showed that the limiting measure is absolutely continuous with respect to the Lebesgue measure, and provided an explicit formula for the corresponding limiting pair correlation function $g(\lambda)$,

$$g(\lambda) = \frac{6}{\pi^2 \lambda^2} \sum_{1 \leq k \leq \frac{\pi^2 \lambda}{2}} \varphi(k) \log \frac{\pi^2 \lambda}{3k},$$

for any $\lambda > 0$, where $\varphi$ is Euler’s totient function.

Let $\mathcal{F}_Q = \{ \frac{a}{b} : 1 \leq a \leq b \leq Q, \ (a, b) = 1 \}$ be the set of Farey fractions of order $Q$ and also let $\mathcal{F}_Q + \mathcal{F}_Q \subset [0, 1)$ denote the set of all sums of pairs of fractions in $\mathcal{F}_Q$ written modulo 1. Our goal is to understand whether addition of Farey fractions influences the pair correlation measure. For this purpose, we compare the pair correlation of $\mathcal{F}_Q + \mathcal{F}_Q$ (mod 1) with that of $\mathcal{F}_Q$, as $Q \to \infty$. From a technical point of view the pair correlation measure of the sum $\mathcal{F}_Q + \mathcal{F}_Q$ is more difficult to handle than that of $\mathcal{F}_Q$. The Weil bounds [45], [15] for Kloosterman sums, which played a decisive role in [8], fail to solve the problem. A natural strategy would be to employ Deligne bounds [10] for two dimensional hyper-Kloosterman sums, but the range of the sums turns out to be too short for this method to succeed either. Karatsuba [21], [22], [23] devised a method for bounding certain exponential sums, and Friedlander and Iwaniec applied it successfully in [16], but our short ranges are outside the scope of this method either. As pointed out in [48], [49], one sometimes obtains more cancellation by averaging the pair correlations themselves rather than by averaging their expressions in terms of exponential sums. Inspired by this idea, we adjust our use of exponential sums, and barely obtain enough cancellation to complete the proof. In order to state our main result, we introduce a multiplicative arithmetic function $\psi$, which plays a
similar role for \( \mathcal{F}_Q + \mathcal{F}_Q \) to the one played by Euler’s function for \( \mathcal{F}_Q \). We define \( \psi \) in terms of its associated Dirichlet series,

\[
\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta^4(s)} \prod_{p \text{ prime}} H_p(s),
\]

where \( \zeta(s) \) is the Riemann Zeta function, and \( H_p(s) \) is given by

\[
H_p(s) = 1 + \frac{(p-1)(p+4)}{p(p+3)} \left\{ \frac{4}{p-1} + \sum_{k=1}^{\infty} \frac{(1-p^{(k+1)}(1-s))^4 - (1-p^{k(1-s)})^4}{(1-p^{1-s})p^{k(2-s)}} \right\}.
\]

\section*{Theorem 1.}
The limiting pair correlation function of \( \mathcal{F}_Q + \mathcal{F}_Q \) modulo 1 exists, as \( Q \to \infty \), on any subinterval \( I \subset [0,1] \), and is given by

\[
g_2(\lambda) = \frac{c}{\pi^2 \lambda^2} \sum_{1 \leq k \leq \frac{\pi^4 \lambda}{9}} \psi(k) \log \frac{\pi^4 \lambda}{9k},
\]

for any \( \lambda > 0 \), where

\[
c = \prod_{p \text{ prime}} \left( 1 - \frac{2}{p(p+1)} \right) \left( 1 - \frac{3}{p(p+2)} \right).
\]

The above functions \( g(\lambda) \) and \( g_2(\lambda) \) being distinct, we see that addition of Farey fractions does influence, in this sense, the pair correlation. Their graphs are shown in Figure 1, together with \( g_{\text{GUE}}(\lambda) = 1 - \left( \frac{\sin \pi \lambda}{\pi \lambda} \right)^2 \) and \( g_{\text{Poisson}} = \text{constant equal to 1} \).

\section*{Acknowledgments.}
The author is grateful to the referee for many valuable suggestions.

\section*{2. A Uniform Distribution Result}

Let \( \mathcal{F}_Q = \{\gamma_1, \ldots, \gamma_{N(Q)}\} \) denote the Farey sequence of order \( Q \) with \( 1/Q = \gamma_1 < \gamma_2 < \cdots < \gamma_{N(Q)} = 1 \) and \( \mathcal{F} = (\mathcal{F}_Q)_Q \). Let \( x_{ij} \equiv \gamma_i + \gamma_j \pmod{1} \) and denote by the set \( \mathcal{G}_Q := \{x_{ij} : 1 \leq i,j \leq N(Q)\} = \mathcal{F}_Q + \mathcal{F}_Q \pmod{1} \) counting multiplicities. The sequence of sequences \( \mathcal{G} = (\mathcal{G}_Q)_{Q \in \mathbb{N}} \) is uniformly distributed along the unit interval. More precisely,

\section*{Lemma 1.}

For any subinterval \( I \subset [0,1] \), denote \( \mathcal{G}_I(Q) := \mathcal{G}_Q \cap I \). Then

\[
\#\mathcal{G}_I(Q) = \frac{9|I|}{\pi^4} Q^4 + O(Q^3(\log Q)^{3/2}).
\]

Our method actually gives a more general counting result. For any continuously differentiable function \( f : R^k \to R \) with compact support, we define

\[
Df = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_k} \right\rangle
\]

and

\[
\|Df\|_\infty = \sum_{j=1}^{k} \left\| \frac{\partial f}{\partial x_j} \right\|_\infty.
\]
Lemma 2. Suppose $I$ is a finite interval, $G \in C^1(\mathbb{R})$ with $\text{Supp} (G) \subset I$. Define
\[
g(y) = \sum_{n \in \mathbb{Z}} G(y + n), \quad S_{Q,G} = \sum_{\gamma, \gamma' \in \mathcal{F}_Q} g(\gamma + \gamma').
\]

Then
\[
S_{Q,G} = \left( \int_I G(x) dx \right) \frac{9Q^4}{\pi^4} + E_{Q,G},
\]

where
\[
E_{Q,G} \ll Q^2 (\log Q)^3 ||DG||_\infty |I| + Q^3 \log Q \left| \int_I G(x) dx \right|.
\]

Proof of Lemma 2: For $y \in \mathbb{R}$, let
\[
g(y) = \sum_{n \in \mathbb{Z}} a_n e(ny)
\]
be the Fourier series expansion of \( g \). Then
\[
S_{Q,G} = \sum_{\gamma, \gamma' \in \mathcal{F}_Q} \sum_{n \in \mathbb{Z}} a_n e(n(\gamma + \gamma')) = \sum_{n \in \mathbb{Z}} a_n \sum_{\gamma \in \mathcal{F}_Q} e(n\gamma) \sum_{\gamma' \in \mathcal{F}_Q} e(n\gamma')
\]
\[
= \sum_{n \in \mathbb{Z}} a_n \left( \sum_{1 \leq d \leq Q \atop d \nmid n} d M \left( \frac{Q}{d} \right) \right)^2
\]
\[
= \sum_{1 \leq d_1, d_2 \leq Q} d_1 d_2 M \left( \frac{Q}{d_1} \right) M \left( \frac{Q}{d_2} \right) \sum_{l \in \mathbb{Z}} a_{[d_1, d_2]} l.
\]

Consider for each \( y > 0 \) the function
\[
G_y(x) = \frac{1}{y} G \left( \frac{x}{y} \right), \quad x \in \mathbb{R}.
\]

By properties of the Fourier transform,
\[
\widehat{G}_y(x) = \frac{x}{y} G(x),
\]

and using Poisson summation formula,
\[
\sum_{l \in \mathbb{Z}} a_{[d_1, d_2]} l = \sum_{l \in \mathbb{Z}} \widehat{G}_y([d_1, d_2] l) = \sum_{l \in \mathbb{Z}} \widehat{G}_y([d_1, d_2]) (l) = \sum_{l \in \mathbb{Z}} G([d_1, d_2]) (l).
\]

Thus
\[
S_{Q,G} = \sum_{1 \leq d_1, d_2 \leq Q} d_1 d_2 M \left( \frac{Q}{d_1} \right) M \left( \frac{Q}{d_2} \right) \sum_{l \in \mathbb{Z}} \frac{1}{[d_1, d_2]} G \left( \frac{l}{[d_1, d_2]} \right).
\]

Applying Lemma 8 of [5], we obtain
\[
\sum_{l \in \mathbb{Z}} \frac{1}{[d_1, d_2]} G \left( \frac{l}{[d_1, d_2]} \right) = \int_{I} G(x) dx + O \left( \|DG\|_\infty \left( \frac{|I|}{[d_1, d_2]} + \frac{2}{[d_1, d_2]^2} \right) \right).
\]

Therefore,
\[
S_{Q,G} = \left( \int_{I} G(x) dx \right) \left( \sum_{1 \leq d_1, d_2 \leq Q} d_1 d_2 M \left( \frac{Q}{d_1} \right) M \left( \frac{Q}{d_2} \right) \right) + E_{G,1}
\]
\[
= \left( \int_{I} G(x) dx \right) \left( \sum_{1 \leq d \leq Q} d M \left( \frac{Q}{d} \right) \right)^2 + E_{G,1},
\]

where
\[
E_{G,1} \ll \sum_{1 \leq d_1, d_2 \leq Q} Q^2 \left( \|DG\|_\infty \left( \frac{|I|}{[d_1, d_2]} + \frac{2}{[d_1, d_2]^2} \right) \right).
\]
Since
\[ \sum_{1 \leq d_1, d_2 \leq Q} \frac{1}{[d_1, d_2]} = \sum_{1 \leq \delta \leq Q} \sum_{[d_1, d_2] = \delta} \frac{1}{[d_1, d_2]} = \sum_{1 \leq \delta \leq Q} \sum_{1 \leq q_1, q_2 \leq Q} \frac{1}{\delta q_1 q_2} \]
\[ \ll (\log Q)^3, \]
and
\[ \sum_{1 \leq d_1, d_2 \leq Q} \frac{1}{[d_1, d_2]^2} \leq \sum_{1 \leq \delta \leq Q} \sum_{1 \leq q_1, q_2 \leq Q} \frac{1}{\delta^2 q_1 q_2} \leq \left( \sum_{1 \leq \delta \leq Q} \frac{1}{\delta^2} \right)^3 = O(1), \]
it follows that,
\[ E_{G,1} \ll Q^2 (\log Q)^3 \|DG\|_\infty \|I\|. \]
Moreover we observe that
\[ \sum_{d \leq Q} dM \left( \frac{Q}{d} \right) = \sum_{d \leq Q} d \sum_{r \leq Q/d} \mu(r) = \sum_{r \leq Q} \mu(r) \sum_{d \leq Q/r} d \]
\[ = \sum_{r \leq Q} \mu(r) \left( \frac{Q}{r} + O(1) \right)^2 \]
\[ = \frac{Q^2}{2} \sum_{r \leq Q} \frac{\mu(r)}{r^2} + O(Q \log Q), \]
and therefore
\[ \sum_{d \leq Q} dM \left( \frac{Q}{d} \right) = \frac{Q^2}{2} \left( \frac{6}{\pi^2} + O \left( \frac{1}{Q} \right) \right) + O(Q \log Q) \]
\[ = \frac{3Q^2}{\pi^2} + O(Q \log Q), \]
which finally gives
\[ \left( \sum_{1 \leq d \leq Q} dM \left( \frac{Q}{d} \right) \right)^2 = \frac{9Q^4}{\pi^4} + O(Q^3 \log Q). \]
Combining all these estimates completes the proof of Lemma 2.

**Proof of Lemma 1.** We will approximate the characteristic function \( \chi_I \) of \( I \) by a \( C^1 \) function. To this end, consider the function \( f(x) = 3x^2 - 2x^3 \) for \( x \in [0, 1] \). First note the following properties:
- \( f'(x) = 6x(1-x) \geq 0 \) and \( |f'(x)| \leq 3/2 \) for \( x \in [0, 1] \);
- \( f'(0) = f'(1) = 0, f(0) = 0, f(1) = 1 \);
- \( \int_0^1 f(x)dx = 1/2 \).
For real numbers $a < b < c < d$, we define the function $g_{a,b,c,d} : \mathbb{R} \rightarrow [0, 1]$ by

$$g_{a,b,c,d}(t) = \begin{cases} 
0 & : t \leq a; \\
\frac{t-a}{b-a} & : a < t \leq b; \\
1 & : b < t \leq c; \\
\frac{1}{d-c} & : c < t \leq d; \\
0 & : d < t.
\end{cases}$$

It is easy to see that $g_{a,b,c,d} \in C^1(\mathbb{R})$ with $\text{Supp} (g_{a,b,c,d}) \subset [a, d]$, and

$$||Dg_{a,b,c,d}||_\infty \leq \frac{3}{2} \max \left( \frac{1}{b-a}, \frac{1}{d-c} \right),$$

$$\int_{\mathbb{R}} g_{a,b,c,d}(x)dx = c - b + \frac{b-a}{2} + \frac{d-c}{2}.$$

Now let $G = \chi_{[a,b]}$, the characteristic function of interval $I = [a, b] \subset [0, 1]$. Putting $a_1 = a - \epsilon, a_2 = a + \epsilon, b_1 = b + \epsilon, b_2 = b - \epsilon$ and $G_1 = g_{a_1,a,b,b_1}, G_2 = g_{a,a_2,b_2,b}$, we may denote by

$$f(y) = \sum_{n \in \mathbb{Z}} G(y+n), \quad f_1(y) = \sum_{n \in \mathbb{Z}} G_1(y+n)$$

and $f_2(y) = \sum_{n \in \mathbb{Z}} G_2(y+n)$, to obtain that

$$S_{Q,G} = \sum_{\gamma, \gamma' \in \mathcal{F}_Q} f(\gamma + \gamma'), \quad S_{Q,G_1} = \sum_{\gamma, \gamma' \in \mathcal{F}_Q} f_1(\gamma + \gamma'), \quad S_{Q,G_2} = \sum_{\gamma, \gamma' \in \mathcal{F}_Q} f_2(\gamma + \gamma').$$

Since $G_2 \leq G = \chi_{[a,b]} \leq G_1$, we have

$$S_{Q,G_2} \leq S_{Q,G} \leq S_{Q,G_1}.$$

Noticing that

$$\int_{\mathbb{R}} G_1(x)dx = b - a + \epsilon = |I| + \epsilon, \quad \int_{\mathbb{R}} G_2(x)dx = b - a - \epsilon = |I| - \epsilon,$$

and

$$||DG_1||_\infty \leq \frac{3}{\epsilon}, \quad ||DG_2||_\infty \leq \frac{3}{\epsilon},$$

we may use lemma 2 to obtain

$$S_{Q,G_1} = (|I| + \epsilon)\frac{9}{\pi^4}Q^4 + E_{Q,G_1},$$

$$S_{Q,G_2} = (|I| - \epsilon)\frac{9}{\pi^4}Q^4 + E_{Q,G_2},$$

where

$$E_{Q,G_1} \ll Q^2 (\log Q)^3 \frac{3}{\epsilon}(|I| + 2\epsilon) + Q^3 \log Q(|I| + \epsilon),$$

and

$$E_{Q,G_2} \ll Q^2 (\log Q)^3 \frac{3}{\epsilon}(|I| + Q^3 \log Q(|I| - \epsilon).$$
Choosing 
\[ \epsilon = \frac{(\log Q)^{3/2}}{Q}, \]
we have 
\[ S_{Q,G_1} = S_{Q,G_2} = \frac{9|I|}{\pi^4} Q^4 + E, \]
where 
\[ E \ll Q^3(\log Q)^{3/2}, \]
Therefore 
\[ \#G_I(Q) = S_{Q,G} = \frac{9|I|}{\pi^4} Q^4 + O(Q^3(\log Q)^{3/2}), \]
which completes the proof of Lemma 1.

3. Pair correlation of sums of Farey fractions

For each positive integer \( Q \), let \( \mathcal{F}_Q = \{\gamma_1, \ldots, \gamma_{N(Q)}\} \) denote the Farey sequence of order \( Q \) with \( 1/Q = \gamma_1 < \gamma_2 < \cdots < \gamma_{N(Q)} = 1 \). Let \( I \) be a subinterval of \([0,1]\). Denote by 
\[ x_{ij} \equiv \gamma_i + \gamma_j \pmod{1} \]
and 
\[ G_Q := \{x_{ij}: 1 \leq i, j \leq N(Q)\} = \mathcal{F}_Q + \mathcal{F}_Q \pmod{1}, \]
the set of sum of Farey sequences of order \( Q \) counted with multiplicity, \( G = (G_Q)_Q \) and \( G_I(Q) := G_Q \cap I \).
Let \( \#G_I(Q) \) be the cardinality of \( G_I(Q) \). It is known from Lemma 1 that
\begin{equation}
N = \frac{\#G_I(Q)}{|I|} = \frac{9Q^4}{\pi^4} + O(Q^3(\log Q)^{3/2}).
\end{equation}
Our goal is to estimate the quantity 
\[ S_{Q,I,H,G}(\wedge) := \# \left\{ (x,y) \in G_I(Q) \times G_I(Q): x - y \in \left(\frac{0,\wedge}{N}\right) + \mathbb{Z} \right\} \]
for any positive real number \( \wedge \) as \( Q \to \infty \). In fact we prove a more general result.

Lemma 3. Given the functions \( G, H \in \mathcal{C}^1(\mathbb{R}) \) with \( \text{Supp} (G) \subseteq (0,1) \) and \( \text{Supp} (H) \subseteq (0,\wedge) \) for some \( \wedge > 0 \), define 
\[ h(y) = \sum_{n \in \mathbb{Z}} H(N(y + n)), \quad g(y) = \sum_{n \in \mathbb{Z}} G(y + n), \]
and let 
\[ S_{Q,I,H,G} = \sum_{x,y \in G_Q} h(x - y)g(x)g(y). \]
Then we have
\[ S_{Q,I,H,G} = \frac{9Q^4}{\pi^4} \left( \int_0^1 G(z)^2 \, dz \right) \int_0^\wedge H(x)g_2(x) \, dx + E_{Q,I,H,G}, \]
where for any \( x > 0 \),
\begin{equation}
g_2(x) = \frac{c}{\pi^2 x^2} \sum_{1 \leq k \leq \frac{\pi^2 x}{9}} \psi(k) \log^3 \frac{\pi^4 x}{9k}.
\end{equation}
PAIR CORRELATION OF SUMS OF RATIONALS WITH BOUNDED HEIGHT

Here

\[ c := \prod_{p \text{ prime}} \left( 1 - \frac{2}{p(p+1)} \right) \left( 1 - \frac{3}{p(p+2)} \right), \]

\(\psi\) is the multiplicative function defined in (1) and for any \(\eta > 0\),

\[ E_{Q, I, H, G} \ll_{I, H, G, \eta} Q^{4 - \frac{1}{18} + \eta}. \]

Note that assuming Lemma 3 with the error term \(Q^{4 - \frac{1}{18} + \eta}\), for \(0 < \eta < \frac{1}{18}\), we may obtain

\[ \lim_{Q \to \infty} \frac{S_{Q, I, H, G}}{\# G_1(Q)} = \lim_{Q \to \infty} \frac{S_{Q, I, H, G}}{\frac{9H}{\pi^4} Q^4} = \int_0^1 G(z)^2 \frac{\mathrm{d}z}{|I|} \int_0^\wedge H(x) g_2(x) \, \mathrm{d}x. \]

Let the smooth function \(G\) approach \(\chi_I\), the characteristic function of the interval \(I\), so that

\[ \frac{\int_0^1 G(z)^2 \, \mathrm{d}z}{|I|} \to 1. \]

Also let the smooth function \(H\) approach \(\chi_{(0, \wedge)}\), the characteristic function of the interval \((0, \wedge)\). By a standard approximation argument, we see that the pair correlation function of \(G\) along the subinterval \(I\) of \([0, 1]\) exists and is independent of the location and length of the subinterval. This completes the proof of Theorem 1.

**Proof of Lemma 3.** The proof of lemma 3 will require several steps. Throughout the proof, all constants implied by the big “\(O\)” or “\(\ll\)” notation may depend on the functions \(H\) and \(G\).

3.1. **Fourier series expansion and Poisson summation formula.** If the Fourier series expansion of the functions \(h\) and \(g\) are given by

\[ h(y) = \sum_{n \in \mathbb{Z}} c_n e(ny) \]

and

\[ g(y) = \sum_{n \in \mathbb{Z}} a_n e(ny) \]
for $y \in \mathbb{R}$, then it follows that

\[ S_{Q, I, H, G} = \sum_{\gamma_1, \gamma_2, \gamma_1', \gamma_2' \in \mathcal{F}_Q} \sum_m c_m e(m(\gamma_1 + \gamma_2) - m(\gamma_1' + \gamma_2')) \times \]
\[ \sum_n a_n e(n(\gamma_1 + \gamma_2)) \sum_r a_r e(r(\gamma_1' + \gamma_2')) \]
\[ = \sum_{m,n,r} c_{m,n} a_r \sum_{\gamma_1 \in \mathcal{F}_Q} e((m + n)\gamma_1) \sum_{\gamma_2 \in \mathcal{F}_Q} e((m + n)\gamma_2) \times \]
\[ \sum_{\gamma_1' \in \mathcal{F}_Q} e((r - m)\gamma_1') \sum_{\gamma_2' \in \mathcal{F}_Q} e((r - m)\gamma_2') \]
\[ = \sum_{m,n,r} c_{m,n} a_r \left( \sum_{\gamma \in \mathcal{F}_Q} e((m + n)\gamma) \right)^2 \left( \sum_{\gamma \in \mathcal{F}_Q} e((r - m)\gamma) \right)^2. \]

Therefore

\[ S_{Q, I, H, G} = \sum_{m,n,r} c_{m,n} a_r \left( \sum_{1 \leq d \leq Q, \atop d \mid m+n} dM \left( \frac{Q}{d} \right) \right)^2 \left( \sum_{1 \leq d \leq Q, \atop d \mid r-m} dM \left( \frac{Q}{d} \right) \right)^2. \]

Changing the summation indices using $m + n = m', r - m = n', m = r'$, we have $m = r', n = m' - r', r = n' + r'$. Consequently in terms of $m', n', r'$, we have

\[ S_{Q, I, H, G} = \sum_{m', n', r'} c_{r'} a_{m'-r'} a_{n'+r'} \left( \sum_{1 \leq d \leq Q, \atop d \mid m'} dM \left( \frac{Q}{d} \right) \right)^2 \left( \sum_{1 \leq d \leq Q, \atop d \mid n'} dM \left( \frac{Q}{d} \right) \right)^2 \]
\[ = \sum_{1 \leq d_1, d_2, d_3, d_4 \leq Q} d_1 \cdots d_4 M \left( \frac{Q}{d_1} \right) \cdots M \left( \frac{Q}{d_4} \right) \sum_{r,m,n \in \mathbb{Z}, \atop d_1 \mid m, d_2 \mid m, \atop d_3 \mid n, d_4 \mid n} c_{r} a_{m-r} a_{n+r}. \]
Using an argument similar to that from [8] with Poisson summation formula, the inner sum

\[
\sum_{r \in \mathbb{Z}} c_r \sum_{m \in \mathbb{Z}} a_{[d_1,d_2]m-r} \sum_{n \in \mathbb{Z}} a_{[d_3,d_4]n+r}
\]

is given as

\[
= \sum_{m,n} G\left(\frac{m}{[d_1,d_2][d_3,d_4]}\right) G\left(\frac{n}{[d_3,d_4]}\right) \sum_{r} c_r \left(\left(\frac{m}{[d_1,d_2]} - \frac{n}{[d_3,d_4]}\right)r\right)
\]

\[
= \sum_{m,n} G\left(\frac{m}{[d_1,d_2][d_3,d_4]}\right) G\left(\frac{n}{[d_3,d_4]}\right) \sum_{r} H\left(N \left(r + \frac{m}{[d_1,d_2]} - \frac{n}{[d_3,d_4]}\right)\right).
\]

3.2. Further Reductions. We need several reductions to convert the expression of \(S_{Q,I,H,G}\)
to a manageable form.

3.2.1. First Reduction. First of all, note that since \(\text{Supp}(G) \subset (0,1)\), \(\text{Supp}(H) \subset (0,\wedge)\),
\([d_1,d_2],[d_3,d_4] \leq Q^2\) and \(N \sim \frac{9}{\pi^2}Q^4\), if \(r \neq 0\), then as \(Q\) is sufficiently large, we have

\[
H\left(N \left(r + \frac{m}{[d_1,d_2]} - \frac{n}{[d_3,d_4]}\right)\right) = 0.
\]

Hence we may assume that \(r = 0\).

3.2.2. Second Reduction. For positive integers \(d_1,d_2,d_3,d_4\), let \(\bar{u} = ([d_1,d_2],[d_3,d_4])\) and

\[
e_1 = \frac{[d_1,d_2]}{\bar{u}}, \quad e_2 = \frac{[d_3,d_4]}{\bar{u}}.
\]

Since \((e_1,e_2) = 1\), there is a unique integer \(\bar{a}_2\) such that \(0 < \bar{a}_2 < e_1, \bar{a}_2 e_2 \equiv 1 \pmod{e_1}\).

Choose \(\bar{a}_1 = (1 - \bar{a}_2 e_2)/e_1\), so that \(\bar{a}_1 e_1 + \bar{a}_2 e_2 = 1\). Changing the summation indices with

\[
m' = e_2 m - e_1 n, \quad n' = \bar{a}_1 m + \bar{a}_2 n,
\]

we have \(m = \bar{a}_2 m' + e_1 n', \quad n = -\bar{a}_1 m' + e_2 n'\), and hence

\[
\sum_{m,n \in \mathbb{Z}} G\left(\frac{m}{[d_1,d_2]}\right) G\left(\frac{n}{[d_3,d_4]}\right) H\left(N \left(\frac{m}{[d_1,d_2]} - \frac{n}{[d_3,d_4]}\right)\right) =
\]

\[
\sum_{m',n' \in \mathbb{Z}} G\left(\frac{\bar{a}_2 m'}{[d_1,d_2]} + \frac{n'}{\bar{u}}\right) G\left(\frac{-\bar{a}_1 m'}{[d_3,d_4]} + \frac{n'}{\bar{u}}\right) H\left(\frac{Nm'}{[d_1,d_2,d_3,d_4]}\right).
\]

Using

\[
M\left(\frac{Q}{d}\right) = \sum_{r \leq Q/d} \mu(r),
\]

and changing the order of summation we rewrite \(S_{Q,I,H,G}\) as

\[
S_{Q,I,H,G} = \sum_{1 \leq r_1, r_2, r_3, r_4 \leq Q} \mu(r_1) \mu(r_2) \mu(r_3) \mu(r_4) \sum_{d_1 \leq Q/r_1} \frac{d_1 d_2 d_3 d_4}{[d_1,d_2][d_3,d_4]} \times
\]

\[
\vdots
\]

\[
\sum_{d_4 \leq Q/r_4} G\left(\frac{\bar{a}_2 m}{[d_1,d_2]} + \frac{n}{\bar{u}}\right) G\left(\frac{-\bar{a}_1 m}{[d_3,d_4]} + \frac{n}{\bar{u}}\right) H\left(\frac{Nm}{[d_1,d_2,d_3,d_4]}\right).
\]
3.2.3. Third Reduction. For positive integers $d_1, d_2, d_3, d_4$, denote
\[
\delta = \frac{d_1 d_2 d_3 d_4}{[d_1, d_2, d_3, d_4]}.
\]
If $Q$ is sufficiently large, then
\[
\frac{9Q^4}{\pi^4} (1 - \epsilon) < N < \frac{9Q^4}{\pi^4} (1 + \epsilon),
\]
for any $0 < \epsilon < 1$. Since $\text{Supp} (H) \subset (0, \wedge)$, to have a non-zero contribution from $H$, we need
\[
0 < mr_1 r_2 r_3 r_4 \frac{N}{Q^4} \cdot \delta < mr_1 r_2 r_3 r_4 \frac{N}{d_1 d_2 d_3 d_4} \cdot \delta
\]
which reduces to the condition
\[
mr_1 r_2 r_3 r_4 \delta < \frac{\pi^4 \wedge}{9(1 - \epsilon)}.
\]
Denoting
\[
C_\wedge = \frac{\pi^4 \wedge}{9},
\]
and choosing $\epsilon$ sufficiently small, we have
\[
1 \leq mr_1 r_2 r_3 r_4 \delta \leq C_\wedge.
\]

3.2.4. Fourth Reduction. Fix $m, r_1, r_2, r_3, r_4$ and $\delta = \frac{d_1 d_2 d_3 d_4}{[d_1, d_2, d_3, d_4]}$ bounded by $C_\wedge$. Since $\bar{u}|\delta$, and $\text{Supp} (G) \subset [0, 1]$, to have a non-zero contribution from $G$, we need
\[
0 < \frac{\bar{a}_2 m}{[d_1, d_2]} + \frac{n}{\bar{u}} < 1.
\]
There are only finitely many integers $n$ satisfying this inequality. Denote by $\mathcal{A}$ the finite set consisting of all possible values of such $n$. Changing the order of summation we obtain that
\[
S_{Q, H, G} = \sum_{\substack{mr_1 r_2 r_3 r_4 \delta \leq C_\wedge, \\ n \in \mathcal{A}, \\ d_i \leq Q/\bar{r}_i, 1 \leq i \leq 4, \\ \frac{d_1 d_2 d_3 d_4}{[d_1, d_2, d_3, d_4]} = \delta, \\ (|d_1, d_2|, |d_3, d_4|) = \bar{u}, \\ \frac{N m \delta}{d_1 d_2 d_3 d_4}} \mu(r_1) \mu(r_2) \mu(r_3) \mu(r_4) \delta \sum_{\frac{\bar{a}_2 m}{[d_1, d_2]} + \frac{n}{\bar{u}}} G \left( \frac{\bar{a}_2 m}{[d_1, d_2]} + \frac{n}{\bar{u}} \right) H \left( \frac{N m \delta}{d_1 d_2 d_3 d_4} \right).
\]
3.2.5. **Fifth Reduction.** Since \( \tilde{a}_1 e_1 + \tilde{a}_2 e_2 = 1 \), we have

\[
\frac{\tilde{a}_1}{[d_3, d_4]} + \frac{\tilde{a}_2}{[d_1, d_2]} = \tilde{u} + \frac{\tilde{u}}{[d_1, d_2] \cdot [d_3, d_4]},
\]

and it follows that

\[
\left| \frac{\tilde{a}_2 m}{[d_1, d_2]} + \frac{n}{u} \right| = \left| \frac{\tilde{a}_1 m}{[d_1, d_2]} - \frac{n}{u} \right| = \frac{m \delta}{d_1 d_2 d_3 d_4} \leq \frac{C}{d_1 d_2 d_3 d_4},
\]

and

\[
G \left( \frac{\tilde{a}_1 m}{[d_1, d_2]} + \frac{n}{u} \right) = G \left( \frac{\tilde{a}_2 m}{[d_1, d_2]} + \frac{n}{u} \right) + O \left( \frac{1}{d_1 d_2 d_3 d_4} \right).
\]

As a result of this reduction, we get

\[
S_{Q, \Lambda, H, G} = \sum_{m, r_1, r_2, r_3, r_4, \delta \leq C} \mu(r_1) \mu(r_2) \mu(r_3) \mu(r_4) \delta \cdot \sum_{n \in A} E_0,
\]

where the inner sum is given by

\[
\sum_{\substack{d_i \leq Q/r_i, 1 \leq i \leq 4, \\
\frac{d_1, d_2, d_3, d_4}{\delta}, \\
([d_1, d_2], [d_3, d_4]) = \tilde{u})} \frac{1}{u} G \left( \frac{\tilde{a}_2 m}{[d_1, d_2]} + \frac{n}{u} \right)^2 H \left( \frac{N \delta}{d_1 d_2 d_3 d_4} \right).
\]

Moreover the error term \( E_0 \) can be estimated as

\[
E_0 \ll \sum_{1 \leq d_1, d_2, d_3, d_4 \leq Q} \frac{1}{u \cdot d_1 d_2 d_3 d_4} \ll (\log Q)^4 \ll \eta Q^n,
\]

for any \( \eta > 0 \).

3.2.6. **Sixth Reduction.** Fix integers \( m, r_1, r_2, r_3, r_4, \delta, n \) and \( u \). Define

\[
P_\delta = \{ a \in \mathbb{N} : \text{ for any prime } p, p | a \Rightarrow p | \delta \}.
\]

For positive integers \( d_1, d_2, d_3, d_4 \), \( d_i \leq Q/r_i \), \( 1 \leq i \leq 4 \) with \( \frac{d_1, d_2, d_3, d_4}{[d_1, d_2, d_3, d_4]} \) and \( ([d_1, d_2], [d_3, d_4]) = \tilde{u} \), factoring \( d_1, d_2, d_3, d_4 \) as \( d_1 = a_1 q_1, d_2 = a_2 q_2, d_3 = a_3 q_3, d_4 = a_4 q_4 \) with \( a_i \in P_\delta \) and \( (q_i, \delta) = 1 \) for \( 1 \leq i \leq 4 \) together with \([d_1, d_2] = [a_1, a_2] q_1 q_2, \tilde{u} = ([d_1, d_2], [d_3, d_4]) = ([a_1, a_2], [a_3, a_4]), and \( \frac{d_1, d_2, d_3, d_4}{[d_1, d_2, d_3, d_4]} = \delta \) implies that

\[
\frac{a_1 a_2 a_3 a_4}{[a_1, a_2, a_3, a_4]} = \delta, \quad \frac{q_1 q_2 q_3 q_4}{[q_1, q_2, q_3, q_4]} = 1.
\]
Here \((q_i, q_j) = 1\) for \(i \neq j\). Using these observations we can rewrite \(\sum\) in (5) as

\[
\sum = \sum_{a_i \leq Q/r_i, 1 \leq i \leq 4, a_i, a_2, a_3, a_4 \in \mathbb{P}_\delta, (a_i, a_2, a_3, a_4) = \delta, (q_i, q_j) = 1, i \neq j, (q_i, \delta) = 1, (\delta, \delta) = 1, (a_i \delta, a_2 \delta) = 1, (a_3 \delta, a_4 \delta) = 1} \frac{1}{\bar{u}} \cdot \sum_{1}.
\]

where the inner sum in (8) is given by

\[
\sum_{1} = \sum_{q_i \leq Q/a_i, r_i, 1 \leq i \leq 4, (q_i, q_j) = 1, i \neq j, (q_i, \delta) = 1} G \left( \frac{1}{\bar{u}} \left( \frac{\tilde{a}_2 m}{a_i a_2 q_1 q_2} + n \right) \right)^2 H \left( \frac{N m \delta}{a_1 a_2 a_3 a_4 q_1 q_2 q_3 q_4} \right).
\]

3.2.7. Seventh Reduction. Next fix positive integers \(a_1, a_2, a_3, a_4 \in \mathbb{P}_\delta\). Let

\[
a = \left[ \frac{a_1, a_2}{\bar{u}} \right], \quad b = \left[ \frac{a_3, a_4}{\bar{u}} \right], \quad \text{so that } (a, b) = 1 \quad \text{and} \quad a|\delta, b|\delta, \bar{u}|\delta.
\]

Define the functions

\[
f(x) = G \left( \frac{1}{\bar{u}} \cdot (m x + n) \right)^2, \quad h(x, y, z, w) = H \left( \frac{N \lambda}{x y z w} \right),
\]

where

\[
\lambda = \frac{m \delta}{a_1 a_2 a_3 a_4} \neq 0.
\]

We have

\[
e_1 = \left[ \frac{d_1, d_2}{\bar{u}} \right] = \left[ \frac{a_1, a_2}{\bar{u}} q_1 q_2 \right] = a q_1 q_2 \leq Q^2,
\]

\[
e_2 = \left[ \frac{d_3, d_4}{\bar{u}} \right] = \left[ \frac{a_3, a_4}{\bar{u}} q_3 q_4 \right] = b q_3 q_4 \leq Q^2.
\]

and

\[
0 < \tilde{a}_2 < a q_1 q_2, \quad \tilde{a}_2 (b q_3 q_4) \equiv 1 \pmod{a q_1 q_2}.
\]

Denoting

\[
\delta_i = a_i r_i \geq 1, \quad 1 \leq i \leq 4,
\]

we can rewrite (9) in the form

\[
\sum_{1} = \sum_{q_i \leq Q/\delta_i, 1 \leq i \leq 4, (q_i, q_j) = 1, i \neq j, (q_i, \delta) = 1} f \left( \frac{\tilde{a}_2}{a q_1 q_2} \right) h(q_1, q_2, q_3, q_4).
\]

3.3. Further Estimations. We will need some further estimations in several stages.
3.3.1. **First Step.** We know that

\[(16) \quad ||f||_\infty = O(1), \quad ||Df||_\infty = O(1).\]

Choosing \(0 < \epsilon < 1/2\), one has

\[
\frac{9Q^4}{2\pi^4} < \frac{9Q^4}{\pi^4}(1 - \epsilon) < N < \frac{9Q^4}{\pi^4}(1 + \epsilon) < \frac{27Q^4}{2\pi^4},
\]

for \(Q\) sufficiently large. Since \(\text{Supp} (H) \subset (0, \wedge)\), and \(h(x, y, z, w) \neq 0\) for \(0 < x \leq Q/\delta_1, 0 < y \leq Q/\delta_2, 0 < z \leq Q/\delta_3\) and \(0 < w \leq Q/\delta_4\), we must have that \(0 < \frac{N\lambda_{xyzw}}{xyzw} < \wedge\). This implies that

\[
\frac{Q}{\delta_1} \geq x > \frac{N\lambda_{xyzw}}{\wedge xyzw} \geq \frac{9Q^4m\delta}{2\pi^4a_1a_2a_3a_4} = \frac{9m\delta r_1r_2r_3r_4 Q}{2\pi^4\wedge \delta_1}.
\]

Similar lower bounds can be obtained for \(y, z, w\) too. Denoting

\[(17) \quad c_\lambda = \frac{9m\delta r_1r_2r_3r_4}{2\pi^4\wedge},
\]

we have

\[(18) \quad h(x, y, z, w) \neq 0 \implies \begin{align*}
    c_\lambda \cdot Q/\delta_1 &\leq x \leq Q/\delta_1, \\
    c_\lambda \cdot Q/\delta_2 &\leq y \leq Q/\delta_2, \\
    c_\lambda \cdot Q/\delta_3 &\leq z \leq Q/\delta_3, \\
    c_\lambda \cdot Q/\delta_4 &\leq w \leq Q/\delta_4.
\end{align*}
\]

Next for the function \(h\), using

\[(19) \quad ||h||_\infty = O(1),
\]

and

\[
\left| \frac{\partial h}{\partial x}(x, y, z, w) \right| = \left| H' \left( \frac{N\lambda_{xyzw}}{xyzw} \right) \right| \cdot \frac{N\lambda_{xyzw}}{xyzw} \cdot \frac{1}{x},
\]

from (18), we obtain that

\[(20) \quad \left| \frac{\partial h}{\partial x}(x, y, z, w) \right| \leq ||DH||_\infty \cdot \wedge \cdot \frac{\delta_1}{c_\lambda Q} \ll \frac{\delta_1}{Q}.
\]

Similarly,

\[(21) \quad \left| \frac{\partial h}{\partial y}(x, y, z, w) \right| \ll \frac{\delta_2}{Q},
\]

\[(22) \quad \left| \frac{\partial h}{\partial z}(x, y, z, w) \right| \ll \frac{\delta_3}{Q},
\]

\[(23) \quad \left| \frac{\partial h}{\partial w}(x, y, z, w) \right| \ll \frac{\delta_4}{Q}.
\]
3.3.2. **Second Step.** We rewrite (15) in the form

\[(24) \quad \sum_1 = \sum_{q_1 \leq Q/\delta_1} \sum_{q_3 \leq Q/\delta_3} f\left(\frac{\tilde{a}_2}{aq_1q_2}\right) h(q_1, q_2, q_3, q_4).\]

Fixing \(q_1, q_2\), we may denote the inner sum in (24) as

\[(25) \quad \sum_2 = \sum_{q_3 \leq Q/\delta_3} f\left(\frac{\tilde{a}_2}{aq_1q_2}\right) h(q_1, q_2, q_3, q_4).
\]

Let \(K_3, K_4\) be large positive integers to be chosen later and \(T_3, T_4\) be real numbers such that

\[T_3K_3 = \frac{Q}{\delta_3}, \quad T_4K_4 = \frac{Q}{\delta_4}.
\]

Therefore (25) becomes

\[(26) \quad \sum_2 = \sum_{1 \leq k_3 \leq K_3, \ (k_3-1)T_3 < q_3 \leq k_3T_3} \sum_{1 \leq k_4 \leq K_4, \ (k_4-1)T_4 < q_4 \leq k_4T_4} f\left(\frac{\tilde{a}_2}{aq_1q_2}\right) h(q_1, q_2, q_3, q_4).
\]

Since \((k_3-1)T_3 < q_3 \leq k_3T_3 = \frac{Q}{\delta_3} K_3\), \((k_4-1)T_4 < q_4 \leq k_4T_4 = \frac{Q}{\delta_4} K_4\), using (22) and (23) we have

\[|h(q_1, q_2, q_3, q_4) - h\left(q_1, q_2, \frac{Q}{\delta_3} K_3, \frac{Q}{\delta_4} K_4\right)| \ll \frac{(\delta_3 + \delta_4)(T_3 + T_4)}{Q}.
\]

Inserting this into (26), we deduce that

\[(27) \quad \sum_2 = \sum_{1 \leq k_3 \leq K_3, \ (k_3-1)T_3 < q_3 \leq k_3T_3} \sum_{1 \leq k_4 \leq K_4, \ (k_4-1)T_4 < q_4 \leq k_4T_4} f\left(\frac{\tilde{a}_2}{aq_1q_2}\right) + E'_2,
\]

where the error term \(E'_2\) in (27) can be estimated as

\[(28) \quad E'_2 \ll \frac{Q}{\delta_3} \frac{Q}{\delta_4} \left(\frac{(\delta_3 + \delta_4)(T_3 + T_4)}{Q}\right) \ll (T_3 + T_4)Q.
\]
3.3.3. Third Step. For fixed \( q_1, q_2, k_3, k_4 \), let \( K' \) be a large positive integer to be chosen later and let \( T' \) be a real number such that

\[ T'K' = aq_1q_2 \leq Q^2. \]

We can now rewrite the inner sum of the main term of \( \sum_3 \) from (27) as

\[
\sum_3 = \sum_{1 \leq k' \leq K'} \sum_{(k_3-1)T_3 < q_1 \leq k_3T_3, (k_4-1)T_4 < q_2 \leq k_4T_4, (q_3,q_4,q_3q_4\delta)=1, (k' - 1)T' < \tilde{a}_2 \leq k'T', \tilde{a}_2 - b_3q_4 \equiv 1 \pmod{aq_1q_2}} f\left(\frac{\tilde{a}_2}{aq_1q_2}\right). \tag{29}
\]

For \((k' - 1)T' < \tilde{a}_2 \leq k'T'\), we have

\[
\frac{(k' - 1)T'}{aq_1q_2} < \frac{\tilde{a}_2}{aq_1q_2} \leq \frac{k'T'}{aq_1q_2} = \frac{k'}{K'},
\]

so that

\[
\left| \frac{\tilde{a}_2}{aq_1q_2} - \frac{k'}{K'} \right| \leq \frac{T'}{aq_1q_2} = \frac{1}{K'},
\]

and

\[
\left| f\left(\frac{\tilde{a}_2}{aq_1q_2}\right) - f\left(\frac{k'}{K'}\right)\right| \leq \|Df\|_{\infty} \cdot \left| \frac{\tilde{a}_2}{aq_1q_2} - \frac{k'}{K'} \right| \ll \frac{1}{K'}.
\]

Therefore (29) becomes

\[
\sum_3 = \sum_{1 \leq k' \leq K'} f\left(\frac{k'}{K'}\right) \sum_{(k_3-1)T_3 < q_1 \leq k_3T_3, (k_4-1)T_4 < q_2 \leq k_4T_4, (q_3,q_4,q_3q_4\delta)=1, (k' - 1)T' < \tilde{a}_2 \leq k'T', \tilde{a}_2 - b_3q_4 \equiv 1 \pmod{aq_1q_2}} 1 + E'_3, \tag{30}
\]

where the error term \( E'_3 \) in (30) can be estimated as

\[
E'_3 \ll \frac{T_3T_4}{K'} = \frac{Q^2}{\delta_3\delta_4K'K_3K_4}. \tag{31}
\]

3.4. A Counting Lemma. For fixed \( q_1, q_2, k_3, k_4 \), our next goal is to estimate the inner sum of the main term of \( \sum_3 \) from (30), which can be written in the simpler form

\[
\sum_4 = \sum_{m \in I, n \in J, (m,n)=(mn,\delta)=1, \frac{mn}{q} \in \alpha, \beta}} 1,
\]

with \( I = ((k_3 - 1)T_3, k_3T_3] \subset (0, Q/\delta_3], J = ((k_4 - 1)T_4, k_4T_4] \subset (0, Q/\delta_4], \)

\[ q = aq_1q_2 \leq Q^2, \]
\[ \alpha = \left( \frac{k' - 1}{q} \right)^T, \beta = \frac{k'}{q}, m = q_3, n = q_4 \] where \( b \) is a fixed integer satisfying \( (b, q) = 1 \). Here for an integer \( x \) such that \( (x, q) = 1 \), we denote by \( \bar{x} \) the multiplicative inverse of \( x \) modulo \( q \), i.e. \( 0 < \bar{x} < q \) and \( \bar{x}x \equiv 1 \pmod{q} \).

### 3.4.1. First Step.

Defining the set
\[
V := \left\{ x_{m,n} = \frac{bmn}{q} : m \in I, n \in J, (m, n) = (mn, q\delta) = 1 \right\},
\]
we have
\[
\sum_4 = \# \left( V \cap (\alpha, \beta) \right).
\]
We will obtain the formula
\[
\sum_4 = \# \left( V \cap (\alpha, \beta) \right) = \frac{6T_3T_4}{\pi^2K^\prime} \prod_{p\mid \delta q_1q_2} \left( 1 - \frac{2}{p + 1} \right) + E_4
\]
where the error term \( E_4 \) is to be estimated later. To this end first note that
\[
\# V = \sum_{m \in I, n \in J, (m, n) = (mn, q\delta) = 1} 1 = \sum_{m \in I, (m, q\delta) = 1} \sum_{n \in J, (n, mq\delta) = 1} 1 = \sum_{m \in I, (m, q\delta) = 1} \sum_{d \mid m} \mu(d)
\]
\[
= \sum_{m \in I, (m, q\delta) = 1} \sum_{d \mid m} \mu(d) \sum_{d \mid n} 1 = \sum_{m \in I, (m, q\delta) = 1} \sum_{d \mid m} \mu(d) \left( \frac{|J|}{d} + O(1) \right)
\]
\[
= |J| \sum_{m \in I, (m, q\delta) = 1} \frac{\mu(d)}{d} + O_\eta(|I|Q^\eta) = |J| \sum_{m \in I, (m, q\delta) = 1} \frac{\varphi(mq\delta)}{mq\delta} + O_\eta(|I|Q^\eta)
\]
\[
= \frac{|J| \varphi(q\delta)}{q\delta} \sum_{m \in I, (m, q\delta) = 1} \frac{\varphi(m)}{m} + O_\eta(|I|Q^\eta).
\]

We observe that
\[
\sum_{m \in I, (m, q\delta) = 1} \frac{\varphi(m)}{m} = \sum_{m \in I, (m, q\delta) = 1} \sum_{d \mid m} \frac{\mu(d)}{d} = \sum_{d \leq Q, (d, q\delta) = 1} \frac{\mu(d)}{d} \sum_{m \in I, (m, q\delta) = 1} 1
\]
\[
= \sum_{d \leq Q, (d, q\delta) = 1} \frac{\mu(d)}{d} \sum_{m \in I, (m, q\delta) = 1} 1.
\]
Recall the elementary result that if the function \( f \in C^1(\mathbb{R}) \) has compact support, \( I \) is a finite interval, \( A \) is a fixed positive integer and \( l \) an integer, then

\[
\left| \sum_{l \in I} f(l) - \frac{\varphi(A)}{A} \int_I f(x) \, dx \right| \leq \sigma_0(A) \big(||Df||_\infty |I| + 2||f||_\infty\big),
\]

where the number of divisors function satisfies

\[\sigma_0(A) = \sum_{d \mid A} 1 \ll \epsilon \ A^\epsilon \]

for every fixed \( \epsilon > 0 \). Using this result we have

\[
\sum_{m \in I, \ (m,q\delta) = 1} \frac{\varphi(m)}{m} = \sum_{d \leq Q, \ (d,q\delta) = 1} \frac{\mu(d)}{d} \left( \frac{\varphi(q\delta)}{q\delta} \int_0^1 1 \, dt + O_\eta \left((q\delta)^\eta\right) \right)
\]

\[
= \sum_{d \leq Q, \ (d,q\delta) = 1} \frac{\mu(d)}{d} \left( \frac{\varphi(q\delta) |I|}{d} + O_\eta (Q^\eta) \right)
\]

\[
= \frac{\varphi(q\delta)}{q\delta} |I| \sum_{d \leq Q, \ (d,q\delta) = 1} \frac{\mu(d)}{d^2} + O_\eta (Q^\eta).
\]

Completing the convergent sum above gives

\[
\sum_{d \leq Q, \ (d,q\delta) = 1} \frac{\mu(d)}{d^2} = \sum_{d \geq 1, \ (d,q\delta) = 1} \frac{\mu(d)}{d^2} - \sum_{d > Q, \ (d,q\delta) = 1} \frac{\mu(d)}{d^2}
\]

\[
= \prod_p \left( 1 - \frac{1}{p^2} \right) \frac{1}{\prod_{\nu|q\delta} \left( 1 - \frac{1}{p^2} \right)} + O \left( \frac{1}{Q} \right).
\]

Since

\[
\prod_{p \text{ prime}} \left( 1 - \frac{1}{p^2} \right) = \frac{6}{\pi^2},
\]

and \( |I| < Q \), we obtain

\[
\sum_{m \in I, \ (m,q\delta) = 1} \frac{\varphi(m)}{m} = \frac{\varphi(q\delta)}{q\delta} |I| \frac{6}{\pi^2} \frac{1}{\prod_{\nu|q\delta} \left( 1 - \frac{1}{p^2} \right)} + O_\eta (Q^\eta).
\]
Inserting this into the above expression for \( \#V \) finally gives

\[
\#V = |J| \frac{\varphi(q \delta)}{q \delta} \left( \frac{\varphi(q \delta)}{q \delta} |I| \frac{6 \pi^2}{\prod_{p|q \delta} \left( 1 - \frac{1}{p^2} \right)} + O_\eta(Q^n) \right) + O_\eta(|I|Q^n)
\]

\[
= |I| \cdot |J| \frac{6 \prod_{p|q \delta} \left( 1 - \frac{1}{p^2} \right)^2}{\pi^2} + O_\eta(|J|Q^n) + O_\eta(|I|Q^n)
\]

\[
= \frac{6 |I| \cdot |J|}{\pi^2} \prod_{p|q \delta} \left( 1 - \frac{2}{p + 1} \right) + O_\eta((|I| + |J|) \cdot Q^n).
\]

### 3.4.2. Second Step.

By the Erdős-Turán inequality ([31]),

\[
\left| \left( V \cap \left( \alpha, \beta \right) \right) - (\beta - \alpha) \#V \right| \ll \frac{\#V}{L} + \sum_{1 \leq K \leq L} \frac{1}{K} \sum_{x_{m,n} \in V} e(Kx_{m,n}) \ll \frac{|I| \cdot |J|}{L} + \sum_{1 \leq K \leq L} S_K.
\]

Here \( L \) is a large real number to be chosen later and

\[
S_K = \left| \sum_{x_{m,n} \in V} e(Kx_{m,n}) \right|.
\]

Define, for \( 1 \leq K \leq L \),

\[
S(I, J, \delta, q, l) = \sum_{m \in I, \, (m, q \delta) = 1} \sum_{n \in J, \, (n, q \delta) = 1} e \left( \frac{l \bar{m} \bar{n}}{q} \right),
\]

and note that taking \( x_{m,n} = \frac{\bar{m} \bar{n}}{q} \), \( S_K \) can be rewritten as

\[
(34) \quad S_K = \left| \sum_{m \in I, \, (m, q \delta) = 1} \sum_{n \in J, \, (n, q \delta) = 1} e \left( \frac{K \bar{m} \bar{n}}{q} \right) \right| = \left| \sum_{m \in I, \, (m, q \delta) = 1} \sum_{n \in J, \, (n, q \delta) = 1} e \left( \frac{K \bar{m} \bar{n}}{q} \right) \mu(d) \right| \sum_{d|n, \, d|m} \mu(d)
\]

\[
(35) \quad = \left| \sum_{d \leq Q, \, (d, q \delta) = 1} \mu(d) \sum_{m \in \mathbb{Z}, \, (m, q \delta) = 1} \sum_{n \in \mathbb{Z}, \, (n, q \delta) = 1} e \left( \frac{K \bar{d} \bar{m} \bar{n}}{q} \right) \right|
\]
\[ \mu(d) \sum_{d \leq Q, (d,q\delta) = 1} \left| \sum_{d \leq Q, (d,q\delta) = 1} \mu(d) S \left( \frac{I}{d}, \frac{J}{d}, \delta, q, Kbd^2 \right) \right| \cdot \mu(d) \sum_{d \leq Q, (d,q\delta) = 1} \left| \sum_{d \leq Q, (d,q\delta) = 1} \mu(d) S \left( \frac{I}{d}, \frac{J}{d}, \delta, q, Kbd^2 \right) \right| \]

We use the trivial estimate
\[ \left| S \left( \frac{I}{d}, \frac{J}{d}, \delta, q, Kbd^2 \right) \right| \leq \left( \frac{|I|}{d} + 1 \right) \left( \frac{|J|}{d} + 1 \right) = \frac{|I| \cdot |J|}{d^2} + \frac{|I|}{d} + \frac{|J|}{d} + 1, \]
and let \( R \) be a real number to be chosen later with \( 0 < R < |I| \), to deduce that
\[ \left(36\right) \left| \sum_{R < d \leq Q, (d,q\delta) = 1} \mu(d) S \left( \frac{I}{d}, \frac{J}{d}, \delta, q, Kbd^2 \right) \right| \leq \sum_{R < d \leq Q} \left( \frac{|I| \cdot |J|}{d^2} + \frac{|I|}{d} + \frac{|J|}{d} + 1 \right) \]
\[ \leq |I| \cdot |J| \sum_{d > R} \frac{1}{d^2} + \left( |I| + |J| \right) \sum_{d > R} \frac{1}{d} + Q \]
\[ \ll \frac{|I| \cdot |J|}{R} + (|I| + |J|) \log Q + Q. \]

The main difficulty comes from small values of \( d \), namely if \( 1 \leq d \leq R \), then we write
\[ S = S \left( \frac{I}{d}, \frac{J}{d}, \delta, q, Kbd^2 \right) = \sum_{u \in \frac{I}{d}, (u,q\delta) = 1} \sum_{v \in \frac{J}{d}, (v,q\delta) = 1} e \left( Kbd^2 \bar{u} \bar{v} \right). \]

Applying Hölder’s inequality and noting that \((u,q\delta) = 1 \) implies \((u,q) = 1 \), we have
\[ \left| S \right|^4 \leq \left( \sum_{u \in \frac{I}{d}, (u,q\delta) = 1} 1 \right)^3 \left( \sum_{v \in \frac{J}{d}, (v,q\delta) = 1} 1 \right) \left( \sum_{u \in \frac{I}{d}, (u,q) = 1} \sum_{v \in \frac{J}{d}, (v,q\delta) = 1} e \left( Kbd^2 \bar{u} \bar{v} \right) \right)^4 \]
\[ \ll \frac{|I|}{q} \sum_{v \in \frac{J}{d}, (v,q\delta) = 1} \sum_{1 \leq u \leq d} e \left( Kbd^2 \bar{u} (\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \right). \]
As $u$ runs through a reduced residue system modulo $q$, then so is $\overline{bd}^2 u$, so that (37) becomes

\[ |S|^4 \ll \left( \frac{|I|}{d} \right)^4 \frac{1}{q} \sum_{v_1, v_2, v_3, v_4 \in \frac{I}{d}, (v_i, q) = 1} \sum_{s|u, s|q} e \left( Ku(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \right) \]

\[ = \left( \frac{|I|}{d} \right)^4 \frac{1}{q} \sum_{v_1, v_2, v_3, v_4 \in \frac{I}{d}, (v_i, q) = 1, 1 \leq u \leq q} \sum_{s|u, s|q} e \left( Ku(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \right) \sum_{s|u} \mu(s) \]

\[ = \left( \frac{|I|}{d} \right)^4 \frac{1}{q} \sum_{s|q} \mu(s) \sum_{v_1, v_2, v_3, v_4 \in \frac{I}{d}, (v_i, q) = 1, 1 \leq t \leq \frac{q}{s}} e \left( Kt(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \right) \]

Using the fact that

\[ \sum_{1 \leq t \leq \frac{q}{s}} e \left( Kt(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \right) = \frac{q}{s} \]

when

\[ K(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \equiv 0 \pmod{\frac{q}{s}} \]

and zero otherwise, we have from (38) that

\[ |S|^4 \ll \left( \frac{|I|}{d} \right)^4 \sum_{s|q} \frac{1}{s} \times \]

\[ \# \left\{ (v_1, v_2, v_3, v_4) \mid v_i \in \frac{I}{d}, (v_i, q) = 1 \right\} \]

\[ K(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \equiv 0 \pmod{\frac{q}{s}} \}

A similar argument for the case $q > \frac{|I|}{d}$ gives

\[ |S|^4 \ll \left( \frac{|I|}{d} \right)^3 q \sum_{s|q} \frac{1}{s} \times \]

\[ \# \left\{ (v_1, v_2, v_3, v_4) \mid v_i \in \frac{I}{d}, (v_i, q) = 1 \right\} \]

\[ K(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \equiv 0 \pmod{\frac{q}{s}} \} \]

Since $\frac{|I|}{d} < Q$ and $q \leq Q^2$, (39) and (40) can be combined under the single estimate

\[ |S|^4 \ll \left( \frac{|I|}{d} \right)^3 Q^2 \sum_{s|q} \frac{1}{s} \times \]

\[ \# \left\{ (v_1, v_2, v_3, v_4) \mid v_i \in \frac{I}{d}, (v_i, q) = 1 \right\} \]

\[ K(\bar{v}_1 + \bar{v}_2 - \bar{v}_3 - \bar{v}_4) \equiv 0 \pmod{\frac{q}{s}} \} \]

We need to control the number of all admissible tuples $(v_1, v_2, v_3, v_4)$ appearing in (41). Although it is possible to obtain reasonable upper bounds for individual $q$, the quality of
these bounds would not be good enough to arrive at an error term which is \( o(Q^4) \). Therefore we prefer to average over all \( q \leq Q^2 \). Clearly the condition

\[
K(\overline{v}_1 + \overline{v}_2 - \overline{v}_3 - \overline{v}_4) \equiv 0 \pmod{\frac{q}{s}}
\]

for \( s \mid q \) implies

\[
K(\nu_1 v_2(v_3 + v_4) - v_3 v_4(v_1 + v_2)) \equiv 0 \pmod{\frac{q}{s}}.
\]

Consequently we have

\[
\sum_{q \leq Q^2} \sum_{s \mid q} \frac{1}{s} \times \left\{ \left( v_1, v_2, v_3, v_4 \right) \bigg| \begin{array}{c} v_i \in \frac{1}{d}, \quad (v_i, q\delta) = 1 \\ K(\overline{v}_1 + \overline{v}_2 - \overline{v}_3 - \overline{v}_4) \equiv 0 \pmod{\frac{q}{s}} \end{array} \right\}
\]

\[
\leq \sum_{q \leq Q^2} \sum_{s \mid q} \frac{1}{s} \times \left\{ \left( v_1, v_2, v_3, v_4 \right) \bigg| \begin{array}{c} v_i \in \frac{1}{d}, \quad (v_i, q\delta) = 1 \\ K(v_1v_2(v_3 + v_4) - v_3v_4(v_1 + v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array} \right\}
\]

\[
= \sum_{s \leq Q^2} \sum_{q \leq Q^2} \frac{1}{s} \left\{ \left( v_1, v_2, v_3, v_4 \right) \bigg| \begin{array}{c} v_i \in \frac{1}{d}, \quad (v_i, q\delta) = 1 \\ K(v_1v_2(v_3 + v_4) - v_3v_4(v_1 + v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array} \right\}.
\]

Fixing \( s \leq Q^2 \) temporarily, we observe that

\[
\sum_{q \leq Q^2} \sum_{s \mid q} \frac{1}{s} \left\{ \left( v_1, v_2, v_3, v_4 \right) \bigg| \begin{array}{c} v_i \in \frac{1}{d}, \quad (v_i, q\delta) = 1 \\ K(v_1v_2(v_3 + v_4) - v_3v_4(v_1 + v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array} \right\}
\]

\[
\leq \sum_{q \leq Q^2} \frac{q}{s} \left\{ \left( q, v_1, v_2, v_3, v_4 \right) \bigg| \begin{array}{c} q \leq Q^2, \quad v_i \in \frac{1}{d}, \quad s \mid q \\ K(v_1v_2(v_3 + v_4) - v_3v_4(v_1 + v_2)) \equiv 0 \pmod{\frac{q}{s}} \end{array} \right\}.
\]

In order to find a useful upper bound for the number of admissible tuples \((q, v_1, v_2, v_3, v_4)\), we have to distinguish two cases. First of all, if \( v_1v_2(v_3 + v_4) \neq v_3v_4(v_1 + v_2) \), then using the fact that

\[
\frac{q}{s} \mid K(v_1v_2(v_3 + v_4) - v_3v_4(v_1 + v_2))
\]

and

\[
0 \neq |K(v_1v_2(v_3 + v_4) - v_3v_4(v_1 + v_2))| \leq 2Q^5,
\]

it follows that the number of such integers \( \frac{q}{s} \) is bounded by the number of divisors of \( K(v_1v_2(v_3 + v_4) - v_3v_4(v_1 + v_2)) \), which is \( \ll \eta Q^n \) for \( \eta > 0 \). Since \( s \) is fixed, for each tuple \((v_1, v_2, v_3, v_4)\), the number of admissible \( q \) is again \( \ll \eta Q^n \). In conclusion, the number of all admissible tuples \((q, v_1, v_2, v_3, v_4)\) is

\[
\ll \eta \left( \frac{|\mathcal{J}|}{d} \right)^4 Q^n.
\]
In the other case, if \( v_1 v_2 (v_3 + v_4) = v_3 v_4 (v_1 + v_2) \), then fix \( v_3, v_4 \in \frac{1}{d} \cap \mathbb{Z} \) and put
\[
\frac{1}{v_3} + \frac{1}{v_4} = \frac{a}{b},
\]
where \( a, b \) are integers with \( (a, b) = 1, |a|, |b| \leq Q^2 \). We consider solutions of the equation
\[
\frac{1}{v_1} + \frac{1}{v_2} = \frac{a}{b}
\]
for \( v_1, v_2 \in \frac{1}{d} \cap \mathbb{Z} \). Equivalently, we may write \( b(v_1 + v_2) = av_1 v_2 \). Taking \( v_1 = \tilde{d}, v_2 = \tilde{d} n, v_2 = \tilde{d} m \) with \( (m, n) = 1 \), gives
\[
b(m + n) = a \tilde{d} m n.
\]
Using \( (m, m + n) = (n, m + n) = 1 \), we have \( m | b, n | b \), and the number of such pairs \( (m, n) \) is \( \ll_n b^n \ll Q^n \). Since \( \tilde{d} | b(m + n) \), the number of \( \tilde{d} \) for fixed \( m, n \) is \( \ll Q^n \). Therefore for fixed \( v_3, v_4 \), the number of such pairs \( (v_1, v_2) \) is also \( \ll Q^n \). Observing that there are at most \( Q^2 \) choices for \( q \) and at most \( \ll \left( \frac{|J|}{d} \right)^2 \) choices for the pairs \( (v_3, v_4) \), the number of admissible tuples \( (q, v_1, v_2, v_3, v_4) \) in this case is
\[
\ll_n \left( \frac{|J|}{d} \right)^2 Q^{2 + \eta}.
\]
Using \( \frac{|J|}{d} < Q \) and combining the two cases, we see that
\[
\# \left\{ (q, v_1, v_2, v_3, v_4) \mid q \leq Q^2, v_i \in \frac{1}{d} \cap \mathbb{Z}, s \mid q, K(v_1 v_2 (v_3 + v_4) - v_3 v_4 (v_1 + v_2)) \equiv 0 \pmod{\frac{q}{s}} \right\} \ll_n \left( \frac{|J|}{d} \right)^2 Q^{2 + \eta}
\]
for \( \eta > 0 \). Combining (42), (43) and (44), we deduce
\[
\sum_{q \leq Q^2} \sum_{s \mid q} \frac{1}{s} \# \left\{ (v_1, v_2, v_3, v_4) \mid v_i \in \frac{1}{d} \cap \mathbb{Z}, (v_i, q) = 1, K(\tilde{v}_1 + \tilde{v}_2 - \tilde{v}_3 - \tilde{v}_4) \equiv 0 \pmod{\frac{q}{s}} \right\} \ll_n \left( \frac{|J|}{d} \right)^2 Q^{2 + \eta} \sum_{s \leq Q^2} \frac{1}{s} \ll_n \left( \frac{|J|}{d} \right)^2 Q^{2 + \eta} \log Q \ll_n \left( \frac{|J|}{d} \right)^2 Q^{2 + \eta}
\]
for \( \eta > 0 \). Let \( 0 < \sigma < 1 \) be a parameter which we will fix later. As a result of (45), the number of \( q \leq Q^2 \) such that
\[
\sum_{s \mid q} \frac{1}{s} \# \left\{ (v_1, v_2, v_3, v_4) \mid v_i \in \frac{1}{d} \cap \mathbb{Z}, (v_i, q) = 1, K(\tilde{v}_1 + \tilde{v}_2 - \tilde{v}_3 - \tilde{v}_4) \equiv 0 \pmod{\frac{q}{s}} \right\} > \left( \frac{|J|}{d} \right)^{3 - \sigma}
\]
is
\[
\ll_n \frac{Q^{2 + \eta}}{\left( \frac{|J|}{d} \right)^{1 - \sigma}}.
\]
Let \( B_{\sigma,K,J,d}(Q) \) be the set of all \( q \leq Q^2 \) such that (46) holds. Clearly we have

\[
|B_{\sigma,K,J,d}(Q)| \ll Q^{2+\eta} \left( \frac{|J|}{d} \right)^{1-\sigma}.
\]

Also let \( G_{\sigma,K,J,d}(Q) \) be the complementary set of \( B_{\sigma,K,J,d}(Q) \) in \((0,Q^2]\). If \( q \leq Q^2 \) and \( q \in G_{\sigma,K,J,d}(Q) \), then it follows from (41) that

\[
|S|^4 \leq \left( \frac{|I|}{d} \right)^3 Q^2 \left( \frac{|J|}{d} \right)^{3-\sigma},
\]

and consequently that

\[
|S| \leq Q^{\frac{1}{2}} \left( \frac{|I|}{d} \right)^{\frac{3}{4}} \left( \frac{|J|}{d} \right)^{\frac{3-\sigma}{4}} = Q^{\frac{1}{2}} \frac{|I|^{\frac{3}{4}} \cdot |J|^{\frac{3-\sigma}{4}}}{d^{\frac{6-\sigma}{4}}}.
\]

Therefore if \( q \in \cap_{d \leq R} G_{\sigma,K,J,d}(Q) \), then

\[
\left| \sum_{d \leq R, (d,qd) = 1} \mu(d)S \left( \frac{I}{d}, \frac{J}{d}, \delta, q, Kbd^2 \right) \right| \leq \sum_{d \leq R} Q^{\frac{1}{2}} \frac{|I|^{\frac{3}{4}} \cdot |J|^{\frac{3-\sigma}{4}}}{d^{\frac{6-\sigma}{4}}},
\]

since

\[
\frac{6-\sigma}{4} > \frac{5}{4} > 1.
\]

In conclusion, for

\[
q \in \bigcap_{1 \leq K \leq L} \bigcap_{d \leq R} G_{\sigma,K,J,d}(Q),
\]

one has from (34), (35) and (49) that

\[
S_K \leq \left| \sum_{R \leq d \leq Q, (d,qd) = 1} \mu(d)S \left( \frac{I}{d}, \frac{J}{d}, \delta, q, Kbd^2 \right) \right| + \left| \sum_{d \leq R, (d,qd) = 1} \mu(d)S \left( \frac{I}{d}, \frac{J}{d}, \delta, q, Kbd^2 \right) \right|
\]

\[
\ll \left( \frac{|I| \cdot |J|}{R} + (|I| + |J|) \log Q + Q \right) + \left( Q^{\frac{1}{2}} |I|^{\frac{3}{4}} \cdot |J|^{\frac{3-\sigma}{4}} \right).
\]

Finally we obtain for such \( q \) that

\[
\# \left( V \bigcap (\alpha,\beta) \right) - (\beta - \alpha) \# V \ll \frac{|I| \cdot |J|}{L} + \sum_{1 \leq K \leq L} \frac{S_K}{K}
\]

\[
\ll \frac{|I| \cdot |J|}{L} + \left( \frac{|I| \cdot |J|}{R} + (|I| + |J|) \log Q + Q \right) + Q^{\frac{1}{2}} \frac{|I|^{\frac{3}{4}} \cdot |J|^{\frac{3-\sigma}{4}}}{R} \log L.
\]
3.4.3. **Third Step.** Recall that

\[ |I| = T_3, \ |J| = T_4, \ q = aq_1q_2, \ \text{and} \ \beta - \alpha = \frac{T'}{aq_1q_2} = \frac{1}{K'}. \]

Choosing

\[ R = L < T_3 < Q, \]

we see that

\[ \log R = \log L \leq \log Q \ll \eta Q. \]

In this way (51) becomes

\[ \left| \# \left( V \cap (\alpha, \beta) \right) - (\beta - \alpha) \# V \right| \ll \eta \left( \frac{T_3T_4}{L} + Q + Q^2T_3^3 \cdot T_4^{3+\sigma} \right) Q^n. \]

Recall that

\[ \# V = \frac{6T_3T_4}{\pi^2} \prod_{p \mid \delta q} \left( 1 - \frac{2}{p + 1} \right) + O_\eta \left( (T_3 + T_4) \cdot Q^n \right) \]

and

\[ (\beta - \alpha) \cdot \# V = \frac{6T_3T_4}{\pi^2K'} \prod_{p \mid \delta q} \left( 1 - \frac{2}{p + 1} \right) + O_\eta \left( \frac{T_3 + T_4}{K'} \cdot Q^n \right). \]

Since \( K_3T_3 = Q/\delta_3, K_4T_4 = Q/\delta_4 \) and \( a \mid \delta, \) from (32) we obtain, as promised in the beginning of Section 3.4, that

\[ \sum_4 = \# \left( V \cap (\alpha, \beta) \right) = \frac{6T_3T_4}{\pi^2K'} \prod_{p \mid \delta q_1q_2} \left( 1 - \frac{2}{p + 1} \right) + E_4 \]

\[ = \frac{6}{\pi^2\delta_3\delta_4} \frac{Q^2}{K_3K_4K'} \prod_{p \mid \delta q_1q_2} \left( 1 - \frac{2}{p + 1} \right) + E_4, \]

where the error term \( E_4 \) is estimated as

\[ E_4 \ll \eta \left( \frac{T_3 + T_4}{K'} \cdot Q^n + \left( \frac{T_3T_4}{L} + Q + Q^2T_3^3 \cdot T_4^{3+\sigma} \right) Q^n \right) \]

\[ \ll \eta \left( \frac{T_3T_4}{L} + Q + Q^2T_3^3 \cdot T_4^{3+\sigma} \right) Q^n. \]

3.5. **Estimation of Error Terms.**
3.5.1. **First Step.** Denoting

\[ U_\sigma := \bigcap \left\{ G_{\sigma,K,\frac{J}{d}}(Q) \middle| \begin{align*} 1 \leq d &\leq L, \quad 1 \leq K \leq L, \\
1 \leq k_4 &\leq K_4, \quad J = ((k_4 - 1)T_1,k_4T_4) \right\}, \]

for \( aq_1 q_2 \in U_\sigma \), and gathering \( \sum_3, \sum_4 \) from (30), (52) we have

\[
\sum_3 = \sum_{1 \leq k' \leq K'} f\left( \frac{k'}{K'} \right) \left( \frac{6Q^2}{\pi^2 \delta_3 \delta_4 K_3 K_4 K'} \prod_{p_i \mid q_1 q_2} \left( 1 - \frac{2}{p+1} \right) + E_4 \right) + E'_3
\]

\[
= \frac{6Q^2}{\pi^2 \delta_3 \delta_4 K_3 K_4} \prod_{p_i \mid q_1 q_2} \left( 1 - \frac{2}{p+1} \right) \frac{1}{K'} \sum_{1 \leq k' \leq K'} f\left( \frac{k'}{K'} \right) + E''_3,
\]

where

\[
E''_3 \ll K' E_4 + E'_3,
\]

and using the estimates for \( E_4 \) and \( E'_3 \) from (31), (53) we get

\[
(55) \quad E''_3 \ll \eta \left( \frac{T_3 T_4}{L} + Q + Q^{\frac{1}{2}} T_3^{\frac{3}{4}} \cdot T_4^{\frac{3}{4} - \sigma} \right) K' Q' + \frac{Q^2}{\delta_3 \delta_4 K_3 K_4 K'}.
\]

Recall that if the function \( f \in C^1(\mathbb{R}) \) and \( K \) is any positive integer, then

\[
(56) \quad \left| \frac{1}{K} \sum_{k=1}^{K} f\left( \frac{k}{K} \right) - \int_0^1 f(x)dx \right| \leq \frac{||Df||_\infty}{K}.
\]

Using this elementary result and (16), one has

\[
\frac{1}{K'} \sum_{1 \leq k' \leq K'} f\left( \frac{k'}{K'} \right) = \int_0^1 f(x)dx + O\left( \frac{1}{K'} \right),
\]

so that

\[
\sum_3 = \frac{6Q^2}{\pi^2 \delta_3 \delta_4 K_3 K_4} \prod_{p_i \mid q_1 q_2} \left( 1 - \frac{2}{p+1} \right) \int_0^1 f(x)dx + E_3,
\]

where we can estimate \( E_3 \) as

\[
(57) \quad E_3 \ll E''_3 + \frac{Q^2}{\delta_3 \delta_4 K_3 K_4 K'}.
\]
3.5.2. **Second Step.** Going back to $\sum_2$ from (27) and using the error term, for $aq_1q_2 \in U_\sigma$ we have

$$
\sum_2 = \sum_{1 \leq k_3 \leq K_3, 1 \leq k_4 \leq K_4} h\left(q_1, q_2, \frac{Q k_3}{\delta_3 K_3}, \frac{Q k_4}{\delta_4 K_4}\right) \times
$$

$$
\left(\frac{6Q^2}{\pi^2 \delta_3 \delta_4 K_3 K_4} \prod_{p|\delta q_1 q_2} \left(1 - \frac{2}{p + 1}\right) \int_0^1 f(x)dx + E_3\right) + E'_2
$$

$$
= \frac{6Q^2}{\pi^2 \delta_3 \delta_4} \int_0^1 f(x)dx \prod_{p|\delta q_1 q_2} \left(1 - \frac{2}{p + 1}\right) \times
$$

$$
\frac{1}{K_3 K_4} \sum_{1 \leq k_3 \leq K_3, 1 \leq k_4 \leq K_4} h\left(q_1, q_2, \frac{Q k_3}{\delta_3 T_3}, \frac{Q k_4}{\delta_4 T_4}\right) + E''_2,
$$

where

$$
E''_2 \ll K_3 K_4 E_3 + E'_2 \ll K_3 K_4 + (T_3 + T_4)Q.
$$

Applying (56) two times to the sum

$$
\frac{1}{K_3 K_4} \sum_{1 \leq k_3 \leq K_3, 1 \leq k_4 \leq K_4} h\left(q_1, q_2, \frac{Q k_3}{\delta_3 T_3}, \frac{Q k_4}{\delta_4 T_4}\right),
$$

we get

$$
\frac{1}{K_3 K_4} \sum_{1 \leq k_3 \leq K_3, 1 \leq k_4 \leq K_4} h\left(q_1, q_2, \frac{Q k_3}{\delta_3 T_3}, \frac{Q k_4}{\delta_4 T_4}\right) = \iint_{[0,1]^2} h\left(q_1, q_2, \frac{Q}{\delta_3 z}, \frac{Q}{\delta_4 w}\right) dz dw + O\left(\frac{1}{K_3} + \frac{1}{K_4}\right).
$$

Therefore

$$
\sum_2 = \frac{6Q^2}{\pi^2 \delta_3 \delta_4} \left(\int_0^1 f(x)dx\right) \prod_{p|\delta q_1 q_2} \left(1 - \frac{2}{p + 1}\right) \times
$$

$$
\iint_{[0,1]^2} h\left(q_1, q_2, \frac{Q}{\delta_3 z}, \frac{Q}{\delta_4 w}\right) dz dw + E_2,
$$

where

$$
E_2 \ll Q^2 \left(\frac{1}{K_3} + \frac{1}{K_4}\right) + E''_2.
$$
3.5.3. **Third Step.** Now returning to \( \sum_1 \) from (24), we may write it as

\[
\sum_1 = \sum_{q_1 \leq Q/\delta_1, \atop q_2 \leq Q/\delta_2, \atop (q_1, q_2) = 1, \atop (q_2, \delta) = 1, \atop a_{q_1q_2} \not\in U_\sigma} \left( \sum_{a_{q_1q_2} \in U_\sigma} + \sum_{a_{q_1q_2} \not\in U_\sigma} \right) = \sum_1' + \sum_1''.
\]

As we know the complement of \( U_\sigma \) is given by

\[
U_\sigma := \bigcup \left\{ B_{\sigma,K,J}(Q) \mid 1 \leq d \leq L, \quad 1 \leq K \leq L, \quad 1 \leq k_4 \leq K_4, \quad J = ((k_4 - 1)T_4, k_4T_4) \right\}
\subset (0, Q^2],
\]

and using (47),

\[
\#U_\sigma \ll Q^2 = K_4 L \frac{Q^2}{T_4^{1-\sigma}} \sum_{d \leq L} d^{1-\sigma} \ll K_4 L^{3-\sigma} \cdot \frac{Q^2}{T_4^{1-\sigma}}.
\]

Since the number of divisors of every integer in \( U_\sigma \) is \( \ll Q^\eta \), the number of triples \( (a, q_1, q_2) \) with \( a q_1 q_2 \in U_\sigma \) fixed is \( \ll Q^{3\eta} \) for \( \eta > 0 \) and therefore

\[
\# \left\{ (a, q_1, q_2) \in \mathbb{N} : a q_1 q_2 \in U_\sigma \right\} \ll Q^\eta \cdot \#U_\sigma,
\]

for every fixed \( \eta > 0 \). It follows that

\[
E_1'' = \sum_1'' \ll Q^\eta \cdot \#U_\sigma \cdot \frac{Q^2}{\delta_1 \delta_2} \ll Q^4 \cdot \frac{Q^4}{T_4^{1-\sigma} \delta_1 \delta_2}.
\]

Combining this with the result of the Second Step, for fixed \( a \), we have

\[
\sum_1' = \sum_{q_1 \leq Q/\delta_1, \atop q_2 \leq Q/\delta_2, \atop (q_1, q_2) = 1, \atop (q_2, \delta) = 1, \atop a_{q_1q_2} \in U_\sigma} \left[ \frac{6Q^2}{\pi^2 \delta_3 \delta_4} \left( \int_0^1 f(x)dx \right) \prod_{p | a_{q_1q_2}} \left( 1 - \frac{2}{p + 1} \right) \times \right.
\]

\[
\left. \iint_{[0,1]^2} h(z, w) \frac{Q}{\delta_3} \frac{Q}{\delta_4} \right] dz dw + E_2',
\]

\[
\int_{[0,1]^2} \frac{h(z, w) \frac{Q}{\delta_3} \frac{Q}{\delta_4}}{dz dw + E_2},
\]

\[
\text{PAIR CORRELATION OF SUMS OF RATIONALS WITH BOUNDED HEIGHT 29}
\]

\[
\sum_1 = \sum_{q_1 \leq Q/\delta_1, \atop q_2 \leq Q/\delta_2, \atop (q_1, q_2) = 1, \atop (q_2, \delta) = 1, \atop a_{q_1q_2} \not\in U_\sigma} \left( \sum_{a_{q_1q_2} \in U_\sigma} + \sum_{a_{q_1q_2} \not\in U_\sigma} \right)
\]

\[
= \sum_1' + \sum_1''.
\]
Therefore one has
\[
\sum_1 = \sum_{q_1 \leq Q/\delta_1, \quad q_2 \leq Q/\delta_2, \quad (q_1, q_2, \delta) = 1, \quad (q_2, \delta) = 1,}
\frac{6Q^2}{\pi^2 \delta_3 \delta_4} \left( \int_0^1 f(x) dx \right) \prod_{p \mid \delta_1, \delta_2} \left( 1 - \frac{2}{p + 1} \right) \times
\int_{[0,1]^2} \prod_{q_1 \leq Q/\delta_1, \quad q_2 \leq Q/\delta_2, \quad (q_1, q_2, \delta) = 1, \quad (q_2, \delta) = 1,} \prod_{p \mid \delta_q_1 q_2} \left( 1 - \frac{2}{p + 1} \right) h \left( q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) dz \, dw,
\]
where
\[
E_1 \ll E_1'' + Q^2 E_2.
\]
In conclusion we have
\[
\sum_1 = E_1 + \frac{6Q^2}{\pi^2 \delta_3 \delta_4} \left( \int_0^1 f(x) dx \right) \times
\int_{[0,1]^2} \prod_{q_1 \leq Q/\delta_1, \quad q_2 \leq Q/\delta_2, \quad (q_1, q_2, \delta) = 1, \quad (q_2, \delta) = 1,} \prod_{p \mid \delta_q_1 q_2} \left( 1 - \frac{2}{p + 1} \right) h \left( q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) dz \, dw,
\]
where \( E_1 \) is estimated above in (61).

3.5.4. **Fourth Step.** Let us now complete the estimation of the error term \( E_1 \). Note that using \( T_4 K_4 = \frac{Q}{\delta_4} \) in (59), we have
\[
E_1'' \ll \eta \frac{K_4 L^{3-\sigma} Q^{4+\eta}}{T_4^{1-\sigma} \delta_1 \delta_2} \ll \eta K_4^{2-\sigma} L^{3-\sigma} Q^{3+\sigma+\eta}.
\]
Also from (58),
\[
E_2 \ll \frac{Q^2}{K_3} + \frac{Q^2}{K_4} + E_2'',
\]
where by (57)
\[
E_2'' \ll K_3 K_4 E_3 + (T_3 + T_4) Q.
\]
Finally combining (54) and (56) one has
\[
E_3 \ll \eta \left( \frac{T_3 T_4}{L} + Q + Q^2 T_3^3 T_4^3 \right) K' Q^n + \frac{Q^2}{\delta_3 \delta_4 K_3 K_4 K''}.
\]
Gathering all of the above estimates together,

\[ E_1 \ll E''_1 + Q^2 E_2 \]

\[ \ll \eta K_2^{2-\sigma} L^{3-\sigma} Q^{3+\sigma+\eta} + \frac{Q^4}{K_3} + \frac{Q^4}{K_4} + \frac{Q^4}{K'} + Q^n \cdot \left( Q^4 \frac{K'}{L} + K_3 K_4 K' Q^3 + K' K_3 \frac{1+\sigma}{4} K_4 \frac{1+\sigma}{4} Q^{4-\frac{\sigma}{4}} \right) . \]

Choosing

\[ L = Q^{\sigma_1}, \quad K_3 = K_4 = K' \approx Q^{\sigma_2}, \]

one obtains

\[ E_1 \ll \eta Q^{(2-\sigma)\sigma_2 + (3-\sigma)\sigma_1 + 3+\sigma+\eta} + Q^{+\sigma_2 + Q^n} \cdot \left( Q^{4+\sigma_2 - \sigma_1} + Q^{3+3\sigma_2} + Q^{\frac{6+\sigma}{4} \sigma_2 + 4 - \frac{\sigma}{4}} \right) . \]

Taking \( \sigma_1 = 2\sigma_2 \), this reduces to

\[ E_1 \ll \eta Q^n \left( Q^{3+\sigma(1-3\sigma_2) + 8\sigma_2} + Q^{1-\sigma_2} + Q^{3+3\sigma_2} + Q^{\frac{6+\sigma}{4} \sigma_2 + 4 - \frac{\sigma}{4}} \right) . \]

In order to balance all the terms above, it suffices to solve the system

\[ \begin{cases} 3 + \sigma(1 - 3\sigma_2) + 8\sigma_2 & = 4 - \sigma_2, \\ \frac{6+\sigma}{4} \sigma_2 + 4 - \frac{\sigma}{4} & = 4 - \sigma_2, \end{cases} \]

with \( 0 < \sigma < 1 \) to obtain that

\[ \sigma_2 = \frac{10 - \sqrt{61}}{39} \approx \frac{1}{17.8} > \frac{1}{18} . \]

For convenience we may take \( \sigma_2 = \frac{1}{18}, \sigma_1 = \frac{1}{9}, \sigma = \frac{46}{77} \) and arrive at the concluding estimate for \( E_1 \) as

\[ (63) \quad E_1 \ll \eta Q^{4 - \frac{\sigma}{4} - \frac{\eta}{4}} . \]

3.6. Final Reductions. Let

\[ f_1(n) := \prod_{p \mid n} \left( 1 - \frac{2}{p+1} \right) , \]

and note that \( f_1(n) \) is a multiplicative function with \( |f_1(n)| \leq 1 \). Our next objective is to eliminate the dependence on \( q_1, q_2 \) in the main term of \( \sum_1 \) from (61). To this end, we
consider the sum over $q_1, q_2$ appearing in $\sum_1$ and put
\[
W = \sum_{q_1 \leq Q/\delta_1, \ q_2 \leq Q/\delta_2, \ (q_1, q_2, \delta) = 1} f_1(\delta q_1 q_2) h\left( q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right)
\]
\[
= \sum_{1 \leq q_1 \leq Q/\delta_1, \ (q_1, \delta) = 1} f_1(\delta q_1) \sum_{q_2 \leq Q/\delta_2, \ (q_2, q_1, \delta) = 1} f_1(q_2) h\left( q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right).
\]

If we define the Dirichlet convolution of $\mu$ and $f_1$ as
\[
g_1(m) = (\mu * f_1)(m) = \sum_{d|m} \mu(d) f_1\left( \frac{m}{d} \right),
\]
then $f_1 = 1 * g_1$ so that for any prime $p$,
\[
g_1(p^n) = \begin{cases} 
- \frac{2}{p+1} & : n = 1; \\
0 & : n \geq 2.
\end{cases}
\]

Therefore one has, $|g_1(p)| < \frac{2}{p}$, and
\[
|g_1(d)| \leq \frac{2^{\omega(d)}}{d} \text{ for every } d \geq 1,
\]
where $\omega(d)$ is the number of distinct prime divisors of $d$. In this way the inner sum of $W$ becomes
\[
W_{q_1} = \sum_{q_2 \leq Q/\delta_2, \ (q_2, q_1, \delta) = 1} f_1(q_2) h\left( q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right)
\]
\[
= \sum_{q_2 \leq Q/\delta_2, \ (q_2, q_1, \delta) = 1} h\left( q_1, q_2, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) \sum_{D/q_2} g_1(D)
\]
\[
= \sum_{D \leq Q/\delta_2, \ (D, q_1, \delta) = 1} g_1(D) \sum_{m \leq \frac{Q}{\delta_2}, \ (m, q_1, \delta) = 1} h\left( q_1, mD, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right).
\]

For fixed $q_1, z, w$ and $D$, define $F(m) := h(q_1, mD, z, w)$, and note that $||F||_\infty \ll 1$, and from (21),
\[
|F'(m)| = D \cdot \left| \frac{\partial h}{\partial y} \right| \ll \frac{D \delta_2}{Q}.
\]
Applying the result (33),
\[
\sum_{m \leq \frac{Q}{\delta}, \ (m,q_1,\delta)=1} h \left(q_1, mD, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) = \frac{\varphi(\delta q_1)}{\delta q_1} \int_0^{\frac{Q}{\delta_2}} h \left(q_1, Dy, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) dy + E_F
\]
\[
= \frac{\varphi(\delta q_1)}{\delta q_1} \frac{Q}{D\delta_2} \int_0^1 h \left(q_1, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) dy + E_F,
\]
where the error term $E_F$ is estimated as
\[
E_F \ll \eta \left( \frac{\delta q_1}{Q} \right)^n \left( \frac{D\delta_2}{Q} \cdot \frac{Q}{D\delta_2} + 1 \right) \ll \eta Q^n.
\]
Consequently,
\[
W_{q_1} = \sum_{D \leq Q/\delta_2, \ (D,q_1,\delta)=1} g_1(D) \left( \frac{\varphi(\delta q_1)}{\delta q_1} \frac{Q}{\delta_2 D} \int_0^1 h \left(q_1, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) dy + E_F \right)
\]
\[
= \frac{Q}{\delta_2} \frac{\varphi(\delta q_1)}{\delta q_1} \left( \int_0^1 h \left(q_1, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) dy \right) \left( \sum_{D \leq Q/\delta_2, \ (D,q_1,\delta)=1} g_1(D) \right) + E'_{q_1},
\]
where the error term $E'_{q_1}$ is estimated as
\[
E'_{q_1} \ll \sum_{D \leq Q/\delta_2, \ (D,q_1,\delta)=1} |g_1(D)| \cdot E_F \ll \eta \left( \log Q \right)^2 \cdot Q^n \ll \eta Q^n,
\]
since $g_1(D) \neq 0$ only for square-free $D$, $2^{\omega(D)} = \sigma_0(D)$ for such $D$ and
\[
\sum_{D \leq Q/\delta_2} |g_1(D)| \leq \sum_{D \leq Q/\delta_2} \frac{\sigma_0(D)}{D} \ll (\log Q)^2
\]
by partial summation. Completing the convergent sum on $D$, we have
\[
\sum_{D \leq Q/\delta_2, \ (D,q_1,\delta)=1} \frac{g_1(D)}{D} = \sum_{D \geq 1, \ (D,q_1,\delta)=1} \frac{g_1(D)}{D} + O \left( \sum_{D > Q/\delta_2} \frac{\mu^2(D)\sigma_0(D)}{D^2} \right)
\]
\[
= \prod_p \left( 1 + \frac{g_1(p)}{p} \right) + O \left( \frac{\delta_2}{Q^{1-\eta}} \right)
\]
\[
= \prod_p \left( 1 - \frac{2}{p(p+1)} \right) + O \left( \frac{\delta_2}{Q^{1-\eta}} \right).
\]
Using this we may rewrite $W_{q_1}$ as

$$W_{q_1} = \frac{Q \varphi(\delta q_1)}{\delta_2} \prod_{p|\delta q_1} \left(1 - \frac{2}{p(p+1)}\right) \left(\int_0^1 h \left(q_1, \frac{Q}{\delta_2}, y, \frac{Q}{\delta_3}, z, \frac{Q}{\delta_4}, w\right) dy\right) + E_{q_1},$$

where the error term $E_{q_1}$ is still estimated as

$$E_{q_1} \ll \eta Q^n.$$

Recall that

$$W = \sum_{1 \leq q_1 \leq Q/\delta_1, \ (q_1, \delta) = 1} f_1(\delta q_1)W_{q_1}.$$

Let

$$f_2(n) := \prod_{p|n} \left(1 - \frac{3}{p+2}\right), \ n \in \mathbb{N}$$

and note that $f_2$ is a multiplicative function with $|f_2(n)| \leq 1$. It follows that

$$f_1(n) \varphi(n) \frac{1}{n} \prod_{p|n} \left(1 - \frac{2}{p(p+1)}\right) = \prod_{p|n} \left(\frac{1 - \frac{2}{p+1}}{1 - \frac{2}{p(p+1)}}\right) = \prod_{p|n} \left(1 - \frac{3}{p+2}\right) = f_2(n),$$

and rewriting $W$ gives

$$W = \frac{c_1 Q}{\delta_2} \int_0^1 \sum_{1 \leq q_1 \leq Q/\delta_1, \ (q_1, \delta) = 1} f_2(\delta q_1) h \left(q_1, \frac{Q}{\delta_2}, y, \frac{Q}{\delta_3}, z, \frac{Q}{\delta_4}, w\right) dy + E_W',$$

where

$$c_1 = \prod_{p \text{ prime}} \left(1 - \frac{2}{p(p+1)}\right) > 0,$$

and the error term can be estimated as

$$E_W' \ll \frac{Q}{\delta_1} \cdot E_{q_1} \ll \eta Q^{1+\eta}.$$

Similarly, if we define $g_2 := \mu * f_2$, then $f_2 = 1 * g_2$ and for any prime $p$,

$$g_2(p^n) = \begin{cases} -\frac{3}{p+2} : & n = 1; \\ 0 : & n \geq 2, \end{cases}$$

so that for every $d \geq 1$,

$$|g_2(d)| \leq \frac{3\omega(d)}{d} \leq \frac{2^{2\omega(d)}}{d}.$$
We may now rewrite the inner sum inside the integral for $W$ as

$$W_1 = \sum_{q_1 \leq Q/\delta_1, (q_1, \delta) = 1} f_2(\delta) f_2(q_1) h \left( \frac{q_1}{\delta_2}, \frac{Q}{\delta_3} y, \frac{Q}{\delta_4} z, \frac{Q}{\delta_4} w \right)$$

$$= f_2(\delta) \sum_{q_1 \leq Q/\delta_1, (q_1, \delta) = 1} h \left( \frac{q_1}{\delta_2}, \frac{Q}{\delta_3} y, \frac{Q}{\delta_4} z, \frac{Q}{\delta_4} w \right) \sum_{D|q_1} g_2(D)$$

$$= f_2(\delta) \sum_{D \leq Q/\delta_1, (D, \delta) = 1} g_2(D) \sum_{m \leq \frac{Q}{\delta_1}, (m, \delta) = 1} h \left( \frac{mD}{\delta_2}, \frac{Q}{\delta_3} y, \frac{Q}{\delta_4} z, \frac{Q}{\delta_4} w \right).$$

Using (33) and (20), we obtain

$$\sum_{m \leq \frac{Q}{\delta_1}, (m, \delta) = 1} h \left( \frac{mD}{\delta_2}, \frac{Q}{\delta_3} y, \frac{Q}{\delta_4} z, \frac{Q}{\delta_4} w \right) = \frac{\varphi(\delta)}{\delta} \frac{Q}{D \delta_1} \times$$

$$\int_0^1 h \left( \frac{Q}{\delta_1} x, \frac{Q}{\delta_2} y, \frac{Q}{\delta_3} z, \frac{Q}{\delta_4} w \right) \, dy + O(1),$$

and completing the convergent sum on $D$,

$$\sum_{D \leq Q/\delta_1, (D, \delta) = 1} \frac{g_2(D)}{D} = \sum_{D \geq 1, (D, \delta) = 1} \frac{g_2(D)}{D} + O \left( \sum_{D > Q/\delta_1} \frac{\mu(D) \sigma_0^2(D)}{D^2} \right)$$

$$= \frac{\Pi_p \left( 1 + \frac{g_2(p)}{p} \right)}{\Pi_p \delta \left( 1 + \frac{g_2(p)}{p} \right)} + O \left( \frac{\delta_1}{Q^{1-\eta}} \right)$$

$$= \frac{\Pi_p \left( 1 - \frac{3}{p(p+2)} \right)}{\Pi_p \delta \left( 1 - \frac{3}{p(p+2)} \right)} + O \left( \frac{\delta_1}{Q^{1-\eta}} \right).$$

Moreover using

$$|g_2(D)| \leq \frac{\sigma_0^2(D)}{D}$$

and the well-known estimate

$$\sum_{m \leq t} \sigma_0^2(m) \ll t (\log t)^3,$$

we obtain by partial summation that

$$\sum_{D \leq Q/\delta_1, (D, \delta) = 1} |g_2(D)| \ll (\log Q)^4 \ll_Q Q^\eta.$$
In conclusion we have
\[ W_1 = f_2(\delta) \frac{Q \varphi(\delta)}{\delta} \prod_p \left( 1 - \frac{3}{\delta p(p+2)} \right) \int_0^1 h \left( \frac{Q}{\delta_1 x}, \frac{Q}{\delta_2 y}, \frac{Q}{\delta_3 z}, \frac{Q}{\delta_4 w} \right) dx + E^*, \]
where the error term \( E^* \) is estimated as
\[ E^* \ll \eta Q^\eta. \]
This completes the estimation of all error terms. Denoting
\[ c_2 = \prod_{p \text{ prime}} \left( 1 - \frac{3}{p(p+2)} \right) > 0, \]
and the multiplicative function
\[ c(n) = f_2(n) \frac{\varphi(n)}{n} \prod_{p|n} \left( 1 - \frac{3}{p(p+2)} \right) = \prod_{p|n} \left( 1 - \frac{4}{p+3} \right), \]
we rewrite \( \sum_1 \) from (61) as
\[ \sum_1 = \frac{6c_1c_2c(\delta)Q^4}{\pi^2 \delta_1 \delta_2 \delta_3 \delta_4} \left( \int_0^1 f(x) dx \right) \times \int_{[0,1]^4} h \left( \frac{Q}{\delta_1 x}, \frac{Q}{\delta_2 y}, \frac{Q}{\delta_3 z}, \frac{Q}{\delta_4 w} \right) dx dy dz dw + E. \]
Since \( E_1 \ll \eta Q^{1-\frac{1}{18}+\eta}, Q^2 E'_W \ll \eta Q^{3+\eta} \) and \( Q^3 E^* \ll \eta Q^{3+\eta} \), the final error term \( E \) can be estimated as
\[ E \ll E_1 + Q^2 E'_W + Q^3 E^* \ll \eta Q^{4-\frac{1}{18}+\eta} + Q^{3+\eta} \ll \eta Q^{4-\frac{1}{18}+\eta}, \]
for \( 0 < \eta < \frac{1}{18} \).
Using (18) we have
\[ h \left( \frac{Q}{\delta_1 x}, \frac{Q}{\delta_2 y}, \frac{Q}{\delta_3 z}, \frac{Q}{\delta_4 w} \right) \neq 0 \implies c_\wedge \leq x, y, z, w \leq 1, \]
and combining (11),(12) and (14),
\[ h \left( \frac{Q}{\delta_1 x}, \frac{Q}{\delta_2 y}, \frac{Q}{\delta_3 z}, \frac{Q}{\delta_4 w} \right) = H \left( \frac{Nm\delta r_1 r_2 r_3 r_4}{Q^4 xyzw} \right). \]
By (2), we have
\[ \frac{N}{Q^4} = \frac{9}{\pi^4} + O_\eta \left( Q^{-1+\eta} \right), \]
so that
\[ \left| H \left( \frac{Nm\delta r_1 r_2 r_3 r_4}{Q^4 xyzw} \right) - H \left( \frac{9m\delta r_1 r_2 r_3 r_4}{\pi^4 xyzw} \right) \right| \ll \eta Q^{-1+\eta}. \]
Letting \( c = c_1c_2 \) and rewriting \( \sum_1 \), we get
\[
\sum_1 = \frac{6cc(\delta)Q^4}{\pi^2 \delta_1 \delta_2 \delta_3 \delta_4} \left( \int_0^1 f(x)dx \right) \times \int_{[0,1]^4} H \left( \frac{9m\delta r_1 r_2 r_3 r_4}{\pi^4 xyzw} \right) dx \, dy \, dz \, dw + E,
\]
where
\[
E \ll Q^{1-\frac{1}{n}+\eta}.
\]
Recall that, by (11),
\[
\int_0^1 f(x)dx = \int_0^1 G \left( \frac{mx + n}{\tilde{u}} \right)^2 \, dx = \frac{\tilde{u}}{m} \int_{\frac{n}{m}}^1 G(z)^2 \, dz
\]
\[
= \frac{\tilde{u}}{m} \left( \int_{\frac{n}{m}}^{\frac{n+1}{m}} G(z)^2 \, dz + \cdots + \int_{\frac{n+m-1}{m}}^{\frac{n+1}{m}} G(z)^2 \, dz \right),
\]
and
\[
\sum_{n \in \mathbb{Z}} \int_0^1 f(x)dx = \frac{\tilde{u}}{m} \cdot \int_0^1 G(z)^2 \, dx = \tilde{u} \int_0^1 G(z)^2 \, dx.
\]
Recall that \( \mathcal{A} \) is the finite set of all integers \( n \) satisfying
\[
0 < \frac{\tilde{a}_2 m}{[d_1, d_2]} + \frac{n}{\tilde{u}} < 1.
\]
Summing over all \( n \in \mathcal{A} \), we obtain from (8) that
\[
\sum_{a_i \leq Q/\gamma, 1 \leq i \leq 4, \atop a_1, a_2, a_3, a_4 \in P_5, \atop \frac{a_1 a_2 a_3 a_4}{a_1 a_2 a_3 a_4} = \delta, \atop ([a_1, a_2], [a_3, a_4]) = \tilde{u}} \frac{1}{u} \left\{ \frac{6cc(\delta)Q^4}{\pi^2 \delta_1 \delta_2 \delta_3 \delta_4} \tilde{u} \left( \int_0^1 G(z)^2 \, dz \right) \times \int_{[0,1]^4} H \left( \frac{9m\delta r_1 r_2 r_3 r_4}{\pi^4 xyzw} \right) dx \, dy \, dz \, dw + E \right\}
\]
where \( \sum_1 \) is defined as in (9). If \( \delta = 1 \), then \( P_5 = \{1\} \) and \( a_1 = a_2 = a_3 = a_4 = 1 \). Otherwise \( \delta \geq 2 \), and we consider the prime factorization of \( \delta \) as
\[
\delta = p_1^{e_1}p_2^{e_2} \cdots p_k^{e_k}, \quad e_1, \ldots, e_k \geq 1.
\]
Writing the prime factorizations of \( a_i \) as
\[
a_i = p_1^{e_i1}p_2^{e_i2} \cdots p_k^{e_ik}, \quad 1 \leq i \leq 4,
\]
the condition $\frac{a_1 a_2 a_3 a_4}{a_1, a_2, a_3, a_4} = \delta$ implies that
\begin{equation}
(69) \quad e_1 j + e_2 j + e_3 j + e_4 j - \max\{e_1 j, e_2 j, e_3 j, e_4 j\} = e_j,
\end{equation}
for $1 \leq j \leq k$. Since $a_i \leq Q/r_i \leq Q$, it follows that
\begin{equation}
(70) \quad e_i j \log 2 \leq e_i j \log p_1 + \cdots e_i k \log p_k \leq \log Q - \log r_i,
\end{equation}
and
\begin{equation}
0 \leq e_i j \leq \frac{\log Q}{\log 2},
\end{equation}
for any $1 \leq i \leq 4$ and $1 \leq j \leq k$. Since $\delta$ is absolutely bounded and $\omega(\delta) = k$, $k$ is absolutely bounded and
\begin{equation}
\sum_{a_i \leq Q/r_i, 1 \leq i \leq 4, \quad a_1, a_2, a_3, a_4 \in P_\delta, \quad a_1 a_2 a_3 a_4 = \delta,}
1 \leq \left(\frac{\log Q}{\log 2}\right)^{4k} \ll (\log Q)^{4k} \ll Q^{\eta}.
\end{equation}
Since $\delta_i = a_i r_i, 1 \leq i \leq 4$, we also have from (65)
\begin{equation}
(70) \quad \sum_{a_i \leq Q/r_i, 1 \leq i \leq 4, \quad a_1, a_2, a_3, a_4 \in P_\delta, \quad a_1 a_2 a_3 a_4 = \delta,}
\frac{6cc(\delta)Q^4}{\pi^2 \delta_1 \delta_2 \delta_3 \delta_4}
\left(\int_0^1 G(z)^2 dz\right) \times \int\int_{[0,1]^4} H \left(\frac{9m\delta r_1 r_2 r_3 r_4}{\pi^4 xyzw}\right) dx dy dz dw + E
\end{equation}
\begin{equation}
= \frac{6cc(\delta)}{\pi^2}
\left(\int_0^1 G(z)^2 dz\right) \int\int_{[0,1]^4} H \left(\frac{9m\delta r_1 r_2 r_3 r_4}{\pi^4 xyzw}\right) dx dy dz dw \times
\end{equation}
\begin{equation}
\frac{Q^4}{r_1 r_2 r_3 r_4}
\sum_{a_i \leq Q/r_i, 1 \leq i \leq 4, \quad a_1, a_2, a_3, a_4 \in P_\delta, \quad a_1 a_2 a_3 a_4 = \delta,}
\frac{1}{a_1 a_2 a_3 a_4} + E,
\end{equation}
where, by (64)
\begin{equation}
E \ll Q^{4 - \frac{1}{2} + \eta}.
\end{equation}
Our next goal is to estimate the sum
\begin{equation}
A_{\delta, r_1, r_2, r_3, r_4}(Q) = \sum_{a_i \leq Q/r_i, 1 \leq i \leq 4, \quad a_1, a_2, a_3, a_4 \in P_\delta, \quad a_1 a_2 a_3 a_4 = \delta,}
\frac{1}{a_1 a_2 a_3 a_4}.
\end{equation}
Again we may assume $\delta \geq 2$ and (67), (68), (69). Since $A_{\delta,r_1,r_2,r_3,r_4}(Q)$ is increasing as $Q \to \infty$, we deduce that

$$A_{\delta,r_1,r_2,r_3,r_4}(Q) \leq \sum_{e_{ij}} \frac{1}{p_1^{e_{11}+e_{21}+e_{31}+e_{41}} p_2^{e_{12}+e_{22}+e_{32}+e_{42}} \cdots p_k^{e_{1k}+e_{2k}+e_{3k}+e_{4k}}}$$

$$= \left( \sum_{e=0}^{\infty} \frac{1}{p_1^e} \right)^4 \cdots \left( \sum_{e=0}^{\infty} \frac{1}{p_k^e} \right)^4$$

$$= \left( \prod_{j=1}^{k} \frac{1}{1 - p_j^{-1}} \right)^4 < \infty,$$

and consequently

$$\lim_{Q \to \infty} A_{\delta,r_1,r_2,r_3,r_4}(Q) = A(\delta)$$

for some constant $A(\delta)$ depending only on $\delta$. Denoting the condition (69) as $(e_{ij}) \in PP$, we obtain

$$A(p^m) = \sum_{e_{ij} \geq 0, (e_{ij}) \in PP} \frac{1}{p_1^{e_{11}+e_{21}+e_{31}+e_{41}} p_2^{e_{12}+e_{22}+e_{32}+e_{42}} \cdots p_k^{e_{1k}+e_{2k}+e_{3k}+e_{4k}}}.$$
and

\[
\Omega'_1 = \sum_{e_{11} \log p_1 + \cdots + e_{1k} \log p_k > \log Q - \log r_1} \frac{1}{p_1^{e_{11}} \cdots p_k^{e_{1k}}}.
\]

We can estimate \(\Omega'_1\) as

\[
\Omega'_1 < \sum_{e_{11} \log p_1 > \log Q - \log r_1} \frac{1}{p_1^{e_{11}} \cdots p_k^{e_{1k}}} + \cdots + \sum_{e_{1k} \log p_k > \log Q - \log r_1} \frac{1}{p_1^{e_{11}} \cdots p_k^{e_{1k}}}
\]

\[
+ \sum_{e_{11} \log p_1 \leq \log Q - \log r_1, \ldots, e_{1k} \log p_k \leq \log Q - \log r_1} \frac{1}{p_1^{e_{11}} \cdots p_k^{e_{1k}}}
\]

\[= \Omega_{1,1} + \cdots + \Omega_{1,k} + \Omega''_1.\]

Note that for \(\Omega_{1,1}, e_{12}, \ldots, e_{1k}\) run through all nonnegative integers and since

\[
\frac{1}{p_1^{e_{11}}} < \frac{r_1}{Q},
\]

it follows that

\[
\Omega_{1,1} = \left( \prod_{j=2}^{k} \frac{1}{1 - p_j^{-1}} \right) \sum_{e_{11} \log p_1 > \log Q - \log r_1} \frac{1}{p_1^{e_{11}}} \leq \left( \prod_{j=2}^{k} \frac{1}{1 - p_j^{-1}} \right) \frac{r_1}{Q} \leq \frac{1}{Q}.
\]

Similarly,

\[\Omega_{1,j} \ll \frac{1}{Q}, \text{ for } 1 \leq j \leq k.\]

On the other hand, \(\Omega''_1\) can be estimated as

\[
\Omega''_1 \leq \sum_{e_{11} \log p_1 \leq \log Q - \log r_1, \ldots, e_{1k} \log p_k \leq \log Q - \log r_1} \frac{r_1}{Q}
\]

\[
\leq \frac{r_1}{Q} \left( \frac{\log Q - \log r_1}{\log p_1} + 1 \right) \cdots \left( \frac{\log Q - \log r_1}{\log p_k} + 1 \right)
\]

\[\ll \frac{(\log Q)^k}{Q} \ll \eta Q^{-1 + \eta}.
\]

Combining all these estimates finally gives

\[\Omega_1 \ll \eta Q^{-1 + \eta}.
\]
PAIR CORRELATION OF SUMS OF RATIONALS WITH BOUNDED HEIGHT 41

Similar estimates hold true for \( \Omega_2, \Omega_3 \) and \( \Omega_4 \), and we have

\[
A_{\delta, r_1, \ldots, r_4}(Q) = A(\delta) + O_{\eta} (Q^{-1+\eta}).
\]

Returning to \( \sum \) from (5) and also to \( S_{Q,I,H,G} \), we have in conclusion that

\[
S_{Q,I,H,G} = \frac{6cQ^4}{\pi^2} \left( \int_0^1 G(z)^2 dz \right) \sum_{m_1 r_2 r_3 r_4 \delta \leq C_\lambda} \frac{\mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4)}{r_1 r_2 r_3 r_4} c(\delta) A(\delta) \delta \int \int H \left( \frac{9m_\delta r_1 r_2 r_3 r_4}{\pi^4 xyzw} \right) dx dy dz dw + E,
\]

where, by (64)

\[
E \ll \eta Q^{4-\frac{1}{18}+\eta} + Q^{3+\eta} \ll \eta Q^{4-\frac{1}{18}+\eta},
\]

for any \( 0 < \eta < \frac{1}{18} \).

3.7. Completion of the proof. To complete the proof of Lemma 3 and to arrive at the simple formula promised for the pair correlation function in Theorem 1, we need to do further calculations. Recall that \( C_\lambda = \frac{\pi^4 \Lambda}{9} \) and write \( S_{Q,I,H,G} \) as

\[
(72) \quad \frac{6cQ^4}{\pi^2} \left( \int_0^1 G(z)^2 dz \right) S_\lambda + E,
\]

where

\[
(73) \quad S_\lambda = \sum_{1 \leq k \leq \frac{\pi^4 \Lambda}{9}} \frac{H_k}{k} \sum_{m_1 r_2 r_3 r_4 \delta = k} \frac{k\mu(r_1)\mu(r_2)\mu(r_3)\mu(r_4)}{r_1 r_2 r_3 r_4} c(\delta) A(\delta) \delta,
\]

and

\[
H_k = \int \int \int \left( \frac{9k}{\pi^4 xyzw} \right) dx dy dz dw.
\]

First, for fixed \( k \) with \( 1 \leq k \leq C_\lambda \), let

\[
\Omega = [0, 1]^4 \cap \left\{ (x, y, z, w) \in \mathbb{R}^4 : xyzw > \frac{9k}{\pi^4 \Lambda} \right\}.
\]

Changing variables by \( x' = \frac{9k}{\pi^4 xyzw}, y' = y, z' = z, w' = w \), \( \Omega \) is mapped to the region

\[
\Omega' = \left\{ (x', y', z', w') \in \mathbb{R}^4 : 0 \leq y', z', w' \leq 1, \frac{9k}{\pi^4 y' z' w'} \leq x' < \Lambda \right\}.
\]

Using the Jacobian of the transformation

\[
\left| \frac{\partial(x, y, z, w)}{\partial(x', y', z', w')} \right| = \frac{9k}{\pi^4 x'^2 y' z' w'},
\]

we have

\[
H_k = \int \Omega H \left( \frac{9k}{\pi^4 xyzw} \right) dx dy dz dw = \int \Omega' H(x') \frac{9k}{y' z' w'} \frac{1}{x'^2} dx' dy' dz' dw'.
\]
Changing the dummy variables $x', y', z', w'$ back to $x, y, z, w$, this further gives

$$H_k = \frac{9k}{\pi^4} \int_{\frac{9k}{\pi^4}}^{\pi} H(x) x^2 U \left( \frac{9k}{\pi^4} x \right) \, dx,$$

where the function $U : (0, 1] \rightarrow [0, \infty)$ is given by

$$U(t) = \int_t^1 \int_{\frac{y}{t}}^1 \int_{\frac{w}{y}}^1 dw \, dz \, dy = \int_t^1 \int_{\frac{1}{t}}^1 \left( \log z + \log \frac{y}{t} \right) \frac{dz}{z} \, dy = \int_t^1 \frac{1}{2} \log^2 \left( \frac{y}{t} \right) \, dy = \frac{1}{6} \log^3 \left( \frac{1}{t} \right).$$

Thus

$$H_k = \frac{3k}{2\pi^4} \int_{\frac{9k}{\pi^4}}^{\pi} H(x) x^2 \log^3 \frac{\pi^4 x}{9k} \, dx.$$

Next, we define a multiplicative function $B$ by letting $B(n) = c(n) A(n) n^2$ for any $n \in \mathbb{N}$ and a multiplicative function $\psi$ by the convolution

$$\psi = \mu * \mu * \mu * \mu * I_d * B,$$

where $I_d$ is the identity function. It is easy to see that

$$\sum_{m_1 r_1 r_2 r_3 r_4 \delta = k} \frac{k \mu(r_1) \mu(r_2) \mu(r_3) \mu(r_4)}{r_1 r_2 r_3 r_4} c(\delta) A(\delta) \delta$$

$$= \sum_{m_1 r_1 r_2 r_3 r_4 \delta = k} m \mu(r_1) \mu(r_2) \mu(r_3) \mu(r_4) c(\delta) A(\delta) \delta^2$$

$$= \mu * \mu * \mu * \mu * I_d * B = \psi(k).$$

Returning to $S_\wedge$ from (72), we get

$$S_\wedge = \sum_{1 \leq k \leq \frac{\pi}{9\sigma}} \frac{3 \psi(k)}{2\pi^4} \left( \int_{\frac{9\sigma}{\pi^4}}^{\pi} \frac{H(x)}{x^2} \log^3 \frac{\pi^4 x}{9k} \, dx \right)$$

$$= \frac{3}{2\pi^4} \left( \int_0^{\pi} \frac{H(x)}{x^2} \sum_{1 \leq k \leq \frac{\pi}{9\sigma}} \psi(k) \max \left\{ 0, \log^3 \frac{\pi^4 x}{9k} \right\} \, dx \right)$$

$$= \frac{3}{2\pi^4} \left( \int_0^{\pi} \frac{H(x)}{x^2} \sum_{1 \leq k \leq \frac{\pi}{9\sigma}} \psi(k) \log^3 \frac{\pi^4 x}{9k} \, dx \right).$$
From (72) we finally obtain

\[ S_{Q,1,H,G} = \frac{9Q^4}{\pi^4} \left( \int_0^1 G(z)^2 \, dz \right) \left( \int_0^\infty \frac{H(x)}{x^2} \sum_{1 \leq k \leq e_1^+} \frac{c\psi(k)}{\pi^2} \log^3 \frac{\pi^4 x}{9k} \, dx \right) + E, \]

where

\[ E \ll \eta \, Q^{4-\frac{1}{11}+\eta}. \]

Our last step is to compute the Zeta-function for \( \psi \) explicitly. For any \( k, m \geq 1 \), define a function \( H \) by

\[ H(k, m) = \sum_{0 \leq n_1, n_2, n_3, n_4 \leq k \atop n_1 + n_2 + n_3 + n_4 = m} 1. \]

By the definition of \( A \) from (71), we see that for any prime \( p \) and integer \( m \geq 1 \),

\[ A(p^m) = \sum_{e_1, e_2, e_3, e_4 \geq 0 \atop e_1 + e_2 + e_3 + e_4 = m} \frac{1}{p^{e_1 + e_2 + e_3 + e_4}} = \sum_{k=1}^{\infty} \frac{1}{p^{k+m}} \sum_{e_1 + e_2 + e_3 + e_4 = k+m \atop \max(e_1, e_2, e_3, e_4) = k} 1 = \sum_{k=1}^{\infty} \frac{1}{p^{k+m}} \left( \sum_{0 \leq e_1, e_2, e_3, e_4 \leq k \atop e_1 + e_2 + e_3 + e_4 = k+m} 1 - \sum_{0 \leq e_1, e_2, e_3, e_4 \leq k-1 \atop e_1 + e_2 + e_3 + e_4 = k+m} 1 \right) = \sum_{k=1}^{\infty} \frac{H(k, k + m) - H(k - 1, k + m)}{p^{k+m}}. \]

Then for \( s \in \mathbb{C} \) with \( \text{Re}(s) \) sufficiently large and for any prime number \( p \),

\[ H_p(s) = 1 + \sum_{m=1}^{\infty} \frac{B(p^m)}{p^{ms}} = 1 + \sum_{m=1}^{\infty} \frac{p^{2m} \left( 1 - \frac{4}{p(p+3)} \right)}{p^{ms}} A(p^m) = 1 + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{p^{2m} \left( 1 - \frac{4}{p(p+3)} \right)}{p^{ms} p^{k+m}} \left( H(k, k + m) - H(k - 1, k + m) \right) = 1 + \left( 1 - \frac{4}{p(p+3)} \right) \sum_{k=1}^{\infty} p^{k(s-2)} \sum_{m=k+1}^{\infty} \left( \frac{H(k, m)}{p^{m(s-1)}} - \frac{H(k - 1, m)}{p^{m(s-1)}} \right). \]

It is easy to see that

\[ \sum_{m=0}^{\infty} H(k, m) q^m = \sum_{0 \leq n_1, n_2, n_3, n_4 \leq k} q^{n_1 + n_2 + n_3 + n_4} = \left( \frac{1 - q^{k+1}}{1 - q} \right)^4, \]
for $0 \leq m \leq k$, 
\[
H(k, m) = \sum_{n_1, n_2, n_3, n_4 \geq 0, n_1 + n_2 + n_3 + n_4 = m} 1 = \binom{m + 3}{3},
\]
and for $k \geq 1$, 
\[
H(k, k) - H(k - 1, k) = \sum_{0 \leq n_1, n_2, n_3, n_4 \leq k, n_1 + n_2 + n_3 + n_4 = k} 1 = 4.
\]

Therefore, denoting $q = p^{1-s}$, the inner sum of $H_p(s)$ on $m$ is 
\[
\left( \sum_{m=0}^{\infty} \frac{H(k, m)}{p^{m(s-1)}} - \sum_{m=0}^{k} \frac{H(k, m)}{p^{m(s-1)}} \right) - \left( \sum_{m=0}^{\infty} \frac{H(k-1, m)}{p^{m(s-1)}} - \sum_{m=0}^{k} \frac{H(k-1, m)}{p^{m(s-1)}} \right)
\]
\[
= - \left( \frac{1 - q^{k+1}}{1 - q} \right)^4 - \sum_{m=0}^{k} \binom{m + 3}{3} q^m - \left( \frac{1 - q^k}{1 - q} \right)^4 - \sum_{m=0}^{k} \binom{m + 3}{3} q^m - H(k - 1, k) q^k
\]
\[
= - \left( \frac{1 - q^{k+1}}{1 - q} \right)^4 - \left( \frac{1 - q^k}{1 - q} \right)^4 - 4q^k.
\]

Replacing $q$ by $p^{1-s}$ we obtain that 
\[
H_p(s) = 1 + \left( 1 - \frac{4}{p(p+3)} \right) \sum_{k=1}^{\infty} p^{k(s-2)} \times
\]
\[
\left( \frac{1 - p^{(1-s)(k+1)}}{1 - p^{1-s}} \right)^4 - \left( \frac{1 - p^{(1-s)k}}{1 - p^{1-s}} \right)^4 - 4p^{(1-s)k}
\]
\[
= 1 + \frac{(p-1)(p+4)}{p(p+3)} \left\{ - \frac{4}{p-1} + \frac{\zeta(s-1)}{\zeta^4(s)} \sum_{k=1}^{\infty} \frac{(1 - p^{(1-s)(k+1)})^4 - (1 - p^{(1-s)k})^4}{(1 - p^{1-s})p^k(2-s)} \right\}.
\]

Since 
\[
\psi = \mu * \mu * \mu * \mu * I_d * B,
\]
it is clear that 
\[
\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta^4(s)} \prod_{p \text{ prime}} H_p(s).
\]

This completes the proof of Lemma 3, and therefore also the proof of Theorem 1.
References


**Emre Alkan:** Department of Mathematics, Koc University, Rumelifeneri Yolu, 34450, Sarıyer, İstanbul, TURKEY.
*E-mail address:* ealkan@ku.edu.tr

**Maosheng Xiong:** Department of Mathematics, Eberly College of Science, Pennsylvania State University, McAllister Building, University Park, PA 16802 USA
*E-mail address:* xiong@math.psu.edu
ALEXANDRU ZAHARESCU: INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1–764, 70700 BUCHAREST, ROMANIA, AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, ALTGELD HALL, 1409 W. GREEN STREET, URBANA, IL 61801 USA

E-mail address: zaharesc@math.uiuc.edu