# QUOTIENTS OF VALUES OF THE DEDEKIND ETA FUNCTION 

EMRE ALKAN, MAOSHENG XIONG, AND ALEXANDRU ZAHARESCU


#### Abstract

Inspired by Riemann's work on certain quotients of the Dedekind Eta function, in this paper we investigate the value distribution of quotients of values of the Dedekind Eta function in the complex plane, using the form $\frac{\eta\left(A_{j} z\right)}{\eta\left(A_{j-1} z\right)}$, where $A_{j-1}$ and $A_{j}$ are matrices whose rows are the coordinates of consecutive visible lattice points in a dilation $X \Omega$ of a fixed region $\Omega$ in $\mathbb{R}^{2}$, and $z$ is a fixed complex number in the upper half plane. In particular, we show that the limiting distribution of these quotients depends heavily on the index of Farey fractions which was first introduced and studied by Hall and Shiu. The distribution of Farey fractions with respect to the value of the index dictates the universal limiting behavior of these quotients. Motivated by chains of these quotients, we show how to obtain a generalization, due to Zagier, of an important formula of Hall and Shiu on the sum of the index of Farey fractions.


## 1. Introduction

The Dedekind Eta function $\eta(z)$ is defined on the upper half plane by the infinite product,

$$
\eta(z)=e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty}\left(1-e^{2 \pi i n z}\right)
$$

Historically, this function was first studied by Jacobi in his seminal work, "Fundamenta Nova". In his study, Jacobi assumed that $\operatorname{Im} z>0$. The behaviour of $\eta(z)$ in the limiting case $\operatorname{Im} z=0$ was studied by Riemann in some of his unpublished works, which were edited by Dedekind and Weber in 1874 after Riemann's untimely death.

[^0]The first unpublished note written by Riemann contained 68 formulas. According to Dedekind (see [20]), the problem that Riemann was interested in is the study of the behaviour of $\eta(z)$ under Möbius transformations of the upper half plane. More precisely, Dedekind was able to prove that

$$
\eta\left(\frac{a z+b}{c z+d}\right)=e^{\frac{\pi i(a+d)}{12 c}-\pi i s(d, c)} \sqrt{-i(c z+d)} \eta(z)
$$

Here $a, b, c, d$ are integers satisfying $a d-b c=1$, and $s(.,$.$) is what is now called$ the Dedekind sum, given by

$$
s(h, k)=\sum_{s}\left(\left(\frac{s}{k}\right)\right)\left(\left(\frac{h s}{k}\right)\right)
$$

where the summation on $s$ runs through a complete residue system modulo $k$, and $((x))$ is the sawtooth function defined by

$$
((x))= \begin{cases}x-[x]-\frac{1}{2}, & x \text { is not an integer } \\ 0, & \text { otherwise }\end{cases}
$$

The reciprocity laws for Dedekind sums and their generalizations have been studied by various authors such as Rademacher and Grosswald [20], Carlitz [10], [11], [12], [13], [14], and Berndt [4], [5], [6]. In modern terminology, Dedekind's transformation formula shows that $\eta(z)$ is a holomorphic cusp form of weight $\frac{1}{2}$ for the full modular group $S L_{2}(\mathbb{Z})$. In fact, Riemann did considerably more work on certain quotients of values of the Dedekind Eta function, such as $\frac{\eta(2 z)}{\eta\left(\frac{1+z}{2}\right)}, \frac{\eta\left(\frac{z}{2}\right)}{\eta\left(\frac{1+z}{2}\right)}, \frac{\eta^{2}\left(\frac{1+z}{2}\right)}{\eta(z)}$, and their logarithms (see [22], pp 498-510). Another aspect of Riemann's work concerned eight formulae involving Dedekind sums, the proof of which appeared for the first time in [21]. Recently, more general quotients of the Dedekind Eta function, which lie in certain congruence subgroups of the full modular group, were investigated by Martin and Ono (see [18] and [19]).

In the present paper we study the distribution of quotients of values of $\eta(z)$ at certain chains inside the $S L_{2}(\mathbb{Z})$ orbit of a given element $z$ in the upper half plane
$\mathbb{H}$. In order to determine the distribution of these quotients of values, we first show that (see formula (3) below) quotients of values are closely related with the index of a Farey fraction, a concept which was first introduced and studied by Hall and Shiu [16] (see also [9]). More precisely, for any three consecutive Farey fractions $\frac{d}{c}<\frac{b}{a}<\frac{s}{r}$ of order $Q$, the index of $\frac{b}{a}$, denoted by $v_{Q}\left(\frac{b}{a}\right)$, is defined as

$$
v_{Q}\left(\frac{b}{a}\right)=\frac{c+r}{a}=\frac{d+s}{b}
$$

The index of a Farey fraction turns out to be a positive integer. Geometrically, the index of $\frac{b}{a}$ is twice the area of the triangle with vertices $(0,0),(c, d)$ and $(r, s)$. Hall and Shiu showed that the index satisfies the inequalities

$$
\left[\frac{2 Q+1}{a}\right]-1 \leq v_{Q}\left(\frac{b}{a}\right) \leq\left[\frac{2 Q}{a}\right]
$$

and that the only possible values of $v_{Q}\left(\frac{b}{a}\right)$ are $\left[\frac{2 Q}{a}\right]$ and $\left[\frac{2 Q}{a}\right]-1$. Their investigation of the frequency of the upper and lower values for the index led them to the remarkable formula

$$
\sum_{\gamma \in F_{Q}} v_{Q}(\gamma)=3 N(Q)-1
$$

where $F_{Q}$ is the set of all Farey fractions of order $Q$ and

$$
\left|F_{Q}\right|=N(Q)=\sum_{j=1}^{Q} \phi(j)
$$

They also proved the asymptotic formula

$$
\sum_{\gamma^{\prime} \in F_{Q}} v_{Q}\left(\gamma^{\prime}\right)^{2}=\frac{24 Q^{2}}{\pi^{2}}\left(\log 2 Q-\frac{\zeta^{\prime}(2)}{\zeta(2)}-\frac{17}{8}+2 \gamma\right)+O\left(Q \log ^{2} Q\right)
$$

where $\zeta$ is the Riemann Zeta function and $\gamma$ is Euler's constant. Recent work on the index for certain subsets of $F_{Q}$ defined by mild arithmetical constraints was done in [1] and [2]. In [1] it is shown that higher moments of the index are biased towards some arithmetic progressions by studying the asymptotics of the index along Farey fractions whose denominators form a progression. In [2] further asymptotic results
on the index are obtained for Farey fractions whose denominators are square-free and form a progression. Going back to producing the required chains inside the $S L_{2}(\mathbb{Z})$ orbit, we consider dilations $X \Omega$ of a given region $\Omega \subseteq \mathbb{R}^{2}$, with $X \rightarrow \infty$. For each $X$ we look at the finite sequence of visible lattice points inside $X \Omega$. Note that if $\left(a_{j}, b_{j}\right)$ and $\left(a_{j+1}, b_{j+1}\right)$ are consecutive (with respect to the increasing slope of the lines connecting them to the origin) visible points in $X \Omega$, then the matrix

$$
A_{j}:=\left(\begin{array}{ll}
a_{j} & b_{j} \\
a_{j+1} & b_{j+1}
\end{array}\right)
$$

is an element of $S L_{2}(\mathbb{Z})$, and in this way we can produce a chain of elements inside the $S L_{2}(\mathbb{Z})$ orbit of $z$. On the other hand we might have such points $\left(a_{j}, b_{j}\right)$ in $X \Omega$ such that $\left(a_{j+1}, b_{j+1}\right)$ is not in $X \Omega$, but as we will show below, by imposing certain restrictions on the region $\Omega$, that such exceptional points are rather few in number and therefore can be neglected in our discussion. Next, for any two consecutive matrices $A_{j-1}$ and $A_{j}$, we consider the quotient $\frac{\eta\left(A_{j} z\right)}{\eta\left(A_{j-1} z\right)}$. Our goal is to understand how these quotients are distributed inside the complex plane, for a fixed region $\Omega \subseteq \mathbb{R}^{2}$ and a fixed complex number $z$ in the upper half plane, as $X$ tends to infinity. To describe the behavior of these quotients in a more precise manner, we first set some notations. In what follows we denote by $\mathscr{M}$ the set of open plane domains $\Omega$ of the type

$$
\begin{equation*}
\Omega=\left\{(r, \theta): r<\rho(\theta), \theta_{1}<\theta<\theta_{2}\right\} \tag{1}
\end{equation*}
$$

expressed in polar coordinates, for some bounded positive continuous function $\rho$, and $-\frac{\pi}{2} \leq \theta_{1}<\theta_{2}<\frac{\pi}{2}$. Given $X>0$, let $\mathcal{A}_{\Omega}(X)$ be the set of integer points $(a, b) \in X \Omega$ with relatively prime coordinates, written as

$$
\mathcal{A}_{\Omega}(X)=\left\{P_{i}=\left(a_{i}, b_{i}\right): 1 \leq i \leq N\right\}
$$

Figure 1. The set $\mathcal{W}_{z, \Omega}(X)$ for $z=i, X=100$, and $\Omega$ is the rectangle $(0,1) \times(-1,1), \#\left(\mathcal{W}_{z, \Omega}(X)\right)=12173$.
where the rays $\overrightarrow{O P_{1}}, \overrightarrow{O P_{2}}, \ldots, \overrightarrow{O P_{N}}$ are arranged with counterclockwise order around the origin, and denote

$$
\mathcal{B}_{\Omega}(X):=\left\{A_{j}=\left(\begin{array}{ll}
a_{j} & b_{j} \\
a_{j+1} & b_{j+1}
\end{array}\right): 1 \leq j \leq N-1\right\} .
$$

Figure 2. The set $\mathcal{W}_{z, \Omega}(X)$ for $z=2 i+3, X=100$, and $\Omega$ is the half disc $\left\{(x, y): x^{2}+y^{2}<1, x>0\right\}, \#\left(\mathcal{W}_{z, \Omega}(X)\right)=9541$.

Fix $z \in \mathbb{H}$ and consider the set

$$
\mathcal{W}_{z, \Omega}(X):=\left\{\frac{\eta\left(A_{j} z\right)}{\eta\left(A_{j-1} z\right)}: 2 \leq j \leq N-1\right\}
$$

Taking $X=100$, the location of elements of the set $W_{z, \Omega}(X)$ is shown in Figures 1 and 2 for two different regions $\Omega$ and two different values of $z$. These figures and other similar numerical data appear to suggest that $W_{z, \Omega}(X)$ has a universal limiting distribution as $X \rightarrow \infty$, which is independent of the region $\Omega \in \mathscr{M}$ and of the point $z \in \mathbb{H}$. This limiting distribution appears to be supported on the union of a sequence of segments $J_{1}, J_{2}, J_{3}, \ldots$ in the complex plane. Also, the proportion of elements of $W_{z, \Omega}(X)$ which fall around a given segment $J_{k}$ seems to have a limit $\rho_{k}>0$ as $X \rightarrow \infty$, again independent of $\Omega$ and $z$. We will prove that all these observations hold true. Moreover, we show that inside each segment $J_{k}$ one has a limiting measure, which is absolutely continuous with respect to the Lebesgue measure along $J_{k}$, and we identify the corresponding density function $h_{k}$. Our main result is as follows :

Theorem 1. Let $M$ be the set of all open planar regions $\Omega$ of the type

$$
\left\{(r, \theta): r<\rho(\theta), \theta_{1}<\theta<\theta_{2}\right\}
$$

where $\rho$ is a continuous, positive and bounded function with $-\frac{\pi}{2} \leq \theta_{1}<\theta_{2}<\frac{\pi}{2}$. Let $\mathcal{A}_{\Omega}(X)$ be the set of all integer points $\left(a_{j}, b_{j}\right) \in X \Omega$ with relatively prime coordinates arranged with counterclockwise order around the origin for $1 \leq j \leq N$. Let $\mathcal{B}_{\Omega}(X)$ be the set of all matrices of form

$$
A_{j}=\left(\begin{array}{ll}
a_{j} & b_{j} \\
a_{j+1} & b_{j+1}
\end{array}\right)
$$

for $1 \leq j \leq N-1$ and denote $T=\left\{(x, y) \in \mathbb{R}^{2}: 0 \leq x, y \leq 1, x+y>1\right\}$,

$$
\mathcal{W}_{z, \Omega}(X)=\left\{\frac{\eta\left(A_{j} z\right)}{\eta\left(A_{j-1} z\right)}: 2 \leq j \leq N-1\right\}
$$

Then for any $z \in \mathbb{H}$, any region $\Omega \in M$ and any bounded continuous function $f: \mathbb{C} \rightarrow \mathbb{C}$, we have

$$
\lim _{X \rightarrow \infty} \frac{1}{\#\left(\mathcal{A}_{\Omega}(X)\right)} \sum_{w \in \mathcal{W}_{z, \Omega}(X)} f(w)=2 \iint_{\mathscr{T}} f\left(e^{\frac{\pi i}{12}\left(3-\left[\frac{y+1}{x}\right]\right)} \sqrt{\frac{y}{x}}\right) \mathrm{d} x \mathrm{~d} y
$$

Using Theorem 1 we obtain the promised universal limiting behavior of the quotients of values of the Dedekind Eta function.

Theorem 2. Assuming the notations of Theorem 1, for any $z \in \mathbb{H}$, any domain $\Omega \in \mathscr{M}$, and any bounded continuous function $f: \mathbb{C} \rightarrow \mathbb{C}$, we have

$$
\lim _{X \rightarrow \infty} \frac{1}{\#\left(\mathcal{A}_{\Omega}(X)\right)} \sum_{w \in \mathcal{W}_{z, \Omega}(X)} f(w)=\sum_{k=1}^{\infty} \int_{\sqrt{\frac{k-1}{2}}}^{\sqrt{\frac{k+1}{2}}} f\left(e^{\frac{\pi i}{12}(3-k)} t\right) h_{k}(t) \mathrm{d} t
$$

where

$$
h_{1}(t)= \begin{cases}2 t\left(1-\frac{1}{\left(1+t^{2}\right)^{2}}\right) & : 0 \leq t \leq \sqrt{\frac{1}{2}} \\ 2 t\left(1-\frac{1}{\left(2-t^{2}\right)^{2}}\right) & : \sqrt{\frac{1}{2}}<t \leq 1\end{cases}
$$

and

$$
h_{k}(t)=\left\{\begin{array}{cll}
2 t\left(\frac{1}{\left(k-t^{2}\right)^{2}}-\frac{1}{\left(1+t^{2}\right)^{2}}\right) & : \sqrt{\frac{k-1}{2}} \leq t \leq \sqrt{\frac{k}{2}} \\
2 t\left(\frac{1}{t^{4}}-\frac{1}{\left(k+1-t^{2}\right)^{2}}\right) & : \sqrt{\frac{k}{2}}<t \leq \sqrt{\frac{k+1}{2}}
\end{array}\right.
$$

for any $k \geq 2$.

The above theorems show that $W_{z, \Omega}(X)$ has indeed a universal limiting distribution as $X \rightarrow \infty$, which is independent of $\Omega$ and $z$, and which is supported on the union of a sequence of segments in the complex plane. These segments are contained in 24 rays from the origin corresponding to the directions formed by the 24 -th roots of unity, and for each $k \geq 1$ the proportion of points from $W_{z, \Omega}(X)$ around the $k$-th segment has a limit $\rho_{k}$ as $X \rightarrow \infty$, which is given by

$$
\rho_{k}=\int_{\sqrt{\frac{k-1}{2}}}^{\sqrt{\frac{k+1}{2}}} h_{k}(t) \mathrm{d} t
$$

Using the explicit formulas for $h_{k}$ from Theorem 2, it is easy to see that

$$
\rho_{1}=\frac{1}{3} \quad \text { and } \quad \rho_{k}=\frac{8}{k(k+1)(k+2)} \quad \text { for } k \geq 2
$$

It turns out that these limiting proportions $\rho_{k}$ are exactly two times the asymptotic proportion of the Farey fractions of order $Q$ with index $k$ as $Q$ tends to infinity. Motivated by the connection between quotients of values of the Dedekind Eta function and the index of Farey fractions, one can obtain a generalization, due to Zagier, of the important formula of Hall and Shiu [16] on the sum of the index of Farey fractions of order $Q$.

Theorem 3. Let $\frac{1}{Q}=\gamma_{1}<\gamma_{2}<\ldots .<\gamma_{N(Q)}=1$ be the sequence of Farey fractions of order $Q \geq 2$. If $v_{Q}\left(\gamma_{j}\right)$ is the index of $\gamma_{j}$ and $s(.,$.$) is the Dedekind sum, then$ for any $1 \leq k \leq N(Q)-1$, we have

$$
\sum_{j \leq k} v_{Q}\left(\gamma_{j}\right)=3 k+3-Q-\frac{a+d}{c}+12 s(d, c)
$$

where $\gamma_{k}=\frac{b}{a}$ and $\gamma_{k+1}=\frac{d}{c}$.
In section 2 we offer a conceptual explanation, based on the quotients of values of the Dedekind Eta function, of the existence of the formula in Theorem 3 relating partial sums of the index with the Dedekind sum. Taking $k=N(Q)-1$ in Theorem 3 , and noting that $s(1,1)=0$, where $\gamma_{N(Q)-1}=\frac{Q-1}{Q}, \gamma_{N(Q)}=\frac{1}{1}$, we see that

$$
\sum_{j \leq N(Q)-1} v_{Q}\left(\gamma_{j}\right)=3 N(Q)-2 Q-1
$$

Moreover $v_{Q}\left(\frac{1}{1}\right)$ is two times the area of the triangle with vertices $(0,0),(Q, Q-1)$ and $(Q, Q+1)$ which is $2 Q$. Hence by adding $v_{Q}\left(\frac{1}{1}\right)=2 Q$, we can recover the formula

$$
\sum_{\gamma \in F_{Q}} v_{Q}(\gamma)=3 N(Q)-1
$$

of Hall and Shiu. For completeness we give a proof of Theorem 3 in section 4.

## 2. Preliminary Results

For consecutive visible points $P_{j}=\left(a_{j}, b_{j}\right), P_{j+1}=\left(a_{j+1}, b_{j+1}\right) \in X \Omega$ with $a_{j}, a_{j+1}>0$, if the triangle $\triangle_{O P_{j} P_{j+1}}$ is entirely in $\Omega$, then the area of this triangle is $\frac{1}{2}$, and consequently $a_{j} b_{j+1}-a_{j+1} b_{j}=1$. Therefore

$$
A_{j}=\left(\begin{array}{ll}
a_{j} & b_{j} \\
a_{j+1} & b_{j+1}
\end{array}\right) \in S L_{2}(\mathbb{Z})
$$

By the modular transformation formula of $\eta$-function, for any $z \in \mathbb{H}$,

$$
\eta\left(A_{j} z\right)=e^{\frac{\pi i\left(a_{j}+b_{j+1}\right)}{12 a_{j+1}}-\pi i s\left(b_{j+1}, a_{j+1}\right)} \sqrt{-i\left(a_{j+1} z+b_{j+1}\right)} \eta(z)
$$

where the function $s(\cdot, \cdot)$ is the Dedekind sum, and for $z \neq 0, \sqrt{z}=e^{\frac{1}{2} \log z}$. Here we choose the main branch for $\operatorname{logarithm}$, so that $\log z=\log |z|+i \arg z$ for $-\pi<\arg z<\pi$. It follows that

$$
\frac{\eta\left(A_{j} z\right)}{\eta\left(A_{j-1} z\right)}=e^{\frac{\pi i}{12}\left(\frac{a_{j}+b_{j+1}}{a_{j+1}}-\frac{a_{j-1}+b_{j}}{a_{j}}\right)-\pi i\left(s\left(b_{j+1}, a_{j+1}\right)-s\left(b_{j}, a_{j}\right)\right)} \sqrt{\frac{a_{j+1} z+b_{j+1}}{a_{j} z+b_{j}}} .
$$

Since $b_{j+1} \equiv \bar{a}_{j}\left(\bmod a_{j+1}\right)$, where $\bar{q}$ is the multiplicative inverse of $q$ modulo $a_{j+1}$ such that $1 \leq \bar{q} \leq a_{j+1}$, using the periodicity of Dedekind sums gives,

$$
s\left(b_{j+1}, a_{j+1}\right)=s\left(\bar{a}_{j}, a_{j+1}\right)=\sum_{s}\left(\left(\frac{s}{a_{j+1}}\right)\right)\left(\left(\frac{\bar{a}_{j} s}{a_{j+1}}\right)\right) .
$$

As $s$ runs over a complete residue system modulo $a_{j+1}, a_{j} s$ also runs over a complete residue system, hence replacing $s$ by $a_{j} s$ we have

$$
\begin{aligned}
s\left(b_{j+1}, a_{j+1}\right) & =s\left(\bar{a}_{j}, a_{j+1}\right)=\sum_{a_{j} s}\left(\left(\frac{a_{j} s}{a_{j+1}}\right)\right)\left(\left(\frac{\bar{a}_{j} a_{j} s}{a_{j+1}}\right)\right) \\
& =\sum_{s\left(\bmod a_{j+1}\right)}\left(\left(\frac{s}{a_{j+1}}\right)\right)\left(\left(\frac{a_{j} s}{a_{j+1}}\right)\right)=s\left(a_{j}, a_{j+1}\right) .
\end{aligned}
$$

Moreover, noting that $a_{j+1} b_{j} \equiv-1\left(\bmod a_{j}\right)$ and $b_{j} \equiv-\bar{a}_{j+1}\left(\bmod a_{j}\right)$, we obtain by a similar calculation that

$$
s\left(b_{j}, a_{j}\right)=s\left(-\bar{a}_{j+1}, a_{j}\right)=-s\left(\bar{a}_{j+1}, a_{j}\right)=-s\left(a_{j+1}, a_{j}\right)
$$

Consequently, using the reciprocity law for Dedekind sums, we deduce that
$s\left(b_{j+1}, a_{j+1}\right)-s\left(b_{j}, a_{j}\right)=s\left(a_{j}, a_{j+1}\right)+s\left(a_{j+1}, a_{j}\right)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a_{j}}{a_{j+1}}+\frac{a_{j+1}}{a_{j}}+\frac{1}{a_{j} a_{j+1}}\right)$.
Using this it is easy to derive the formula

$$
\begin{equation*}
\frac{\eta\left(A_{j} z\right)}{\eta\left(A_{j-1} z\right)}=e^{\frac{\pi i}{12}\left(3-\frac{a_{j-1}+a_{j+1}}{a_{j}}\right)} \sqrt{\frac{a_{j+1} z+b_{j+1}}{a_{j} z+b_{j}}} \tag{2}
\end{equation*}
$$

Note that if $\frac{b_{j-1}}{a_{j-1}}<\frac{b_{j}}{a_{j}}<\frac{b_{j+1}}{a_{j+1}}$ are consecutive Farey fractions of order $Q$, then formula (2) can be rewritten as

$$
\begin{equation*}
\frac{\eta\left(A_{j} z\right)}{\eta\left(A_{j-1} z\right)}=e^{\frac{\pi i}{12}\left(3-v_{Q}\left(\frac{b_{j}}{a_{j}}\right)\right) \sqrt{\frac{a_{j+1} z+b_{j+1}}{a_{j} z+b_{j}}}} \tag{3}
\end{equation*}
$$

where $v_{Q}\left(\frac{b_{j}}{a_{j}}\right)=\frac{a_{j-1}+a_{j+1}}{a_{j}}$ is the index of $\frac{b_{j}}{a_{j}}$. As an application of this formula, let $\Omega$ be the triangular region with vertices $(0,0),(1,0)$ and $(1,1)$. Consider the region $Q \Omega$ where $Q$ is a positive integer. Let $A_{\Omega}(Q)$ be the set of all integer points $\left(a_{j}, b_{j}\right)$, $1 \leq j \leq N(Q)$ with relatively prime coordinates where $N(Q)$ is the number of Farey fractions of order $Q$. Let

$$
\frac{1}{Q}=\frac{b_{1}}{a_{1}}=\gamma_{1}<\frac{b_{2}}{a_{2}}=\gamma_{2}<\ldots<\frac{b_{k}}{a_{k}}=\gamma_{k}<\ldots<\gamma_{N(Q)}=1
$$

be the Farey fractions of order $Q$. For $1 \leq j \leq k \leq N(Q)-1$, consider the matrices $B_{j}=\left(\begin{array}{ll}a_{j-1} & b_{j-1} \\ a_{j} & b_{j}\end{array}\right)$ in $S L_{2}(\mathbb{Z})$ (with the convention that $\left(a_{0}, b_{0}\right)=(1,0)$ ). Then using (3), we have

$$
\frac{\eta\left(B_{j} z\right)}{\eta\left(B_{j+1} z\right)}=e^{\frac{\pi i}{12}\left(v_{Q}\left(\gamma_{j}\right)-3\right)} \sqrt{\frac{a_{j} z+b_{j}}{a_{j+1} z+b_{j+1}}}
$$

for $1 \leq j \leq k$. Multiplying all of these equations and noting that the left side is a telescoping product, we obtain

$$
\frac{\eta\left(B_{1} z\right)}{\eta\left(B_{k+1} z\right)}=e^{\frac{\pi i}{12}\left(\sum_{j \leq k} v_{Q}\left(\gamma_{j}\right)-3 k\right)} \sqrt{\frac{a_{1} z+b_{1}}{a_{k+1} z+b_{k+1}}} .
$$

Since the left side of this formula is obtained as a telescoping product, we are interested to see if the partial sum of the index of Farey fractions, $\sum_{j \leq k} v_{Q}\left(\gamma_{j}\right)$, also behaves like a telescoping sum. Indeed applying the modular transformation formula of the Dedekind Eta function to $\eta\left(B_{1} z\right)$ and $\eta\left(B_{k+1} z\right)$ and using them in the above formula, we can easily guess a formula for $\sum_{j \leq k} v_{Q}\left(\gamma_{j}\right)$. This is the main motivation behind Theorem 3.

We need the following variation of a result from [7]. For any subinterval $\mathbf{I}=[\alpha, \beta]$ of $[0,1]$, denote $\mathbf{I}_{a}=[(1-\beta) a,(1-\alpha) a]$ and consider

$$
\mathbb{Z}_{p r}^{2}:=\left\{(x, y) \in \mathbb{Z}^{2}: \operatorname{gcd}(x, y)=1\right\} .
$$

If $f$ is a continuously differentiable, $C^{1}$, function defined on a bounded region $\Omega$ in $\mathbb{R}^{2}$, then we put

$$
\|f\|_{\infty, \Omega}=\sup _{(x, y) \in \Omega}|f(x, y)|
$$

and

$$
\|D f\|_{\infty, \Omega}=\left\|f_{x}\right\|_{\infty, \Omega}+\left\|f_{y}\right\|_{\infty, \Omega}
$$

We also define

$$
\|f\|_{\infty}=\sup _{(x, y) \in \mathbb{R}^{2}}|f(x, y)|
$$

and

$$
\|D f\|_{\infty}=\left\|f_{x}\right\|_{\infty}+\left\|f_{y}\right\|_{\infty}
$$

Lemma 1. Let $\Omega \subset[1, R] \times[1, R]$ be a convex region and let $f$ be a $C^{1}$ function on $\Omega$. For any subinterval $\mathbf{I} \subset[0,1]$ one has

$$
\sum_{\substack{(a, b) \in \Omega \cap \mathbb{Z}_{p r}^{2}, \bar{b} \in \mathbf{I}_{a}}} f(a, b)=\frac{6|\mathbf{I}|}{\pi^{2}} \iint_{\Omega} f(x, y) \mathrm{d} x \mathrm{~d} y+\mathcal{F}_{R, \Omega, f, \mathbf{I}}
$$

where

$$
\begin{gathered}
\mathcal{F}_{R, \Omega, f, \mathbf{I}}<_{\delta} \quad m_{f}\|f\|_{\infty, \Omega} R^{3 / 2+\delta}+\|f\|_{\infty, \Omega} R \log R \\
+\|D f\|_{\infty, \Omega} \operatorname{Area}(\Omega) \log R
\end{gathered}
$$

for any $\delta>0$, where $\bar{b}$ denotes the multiplicative inverse of $b(\bmod a)$, so that $1 \leq \bar{b} \leq a, b \bar{b} \equiv 1(\bmod a)$, and $m_{f}$ is an upper bound for the number of intervals of monotonicity of each of the functions $y \mapsto f(x, y)$.

This is essentially Lemma 8 in [7], where Weil type estimates ([23], [15]) for certain weighted incomplete Kloosterman sums play a crucial role in its proof.

Next we recall some results on Farey fractions. For an exposition of their basic properties, the reader is referred to [17]. Let $\mathscr{F}_{Q}=\left\{\gamma_{1}, \ldots, \gamma_{N(Q)}\right\}$ be the Farey sequence of order $Q$ with $1 / Q=\gamma_{1}<\gamma_{2}<\cdots<\gamma_{N(Q)}=1$. It is well-known that

$$
N(Q)=\sum_{j=1}^{Q} \phi(j)=\frac{3 Q^{2}}{\pi^{2}}+O(Q \log Q)
$$

Write $\gamma_{i}=a_{i} / q_{i}$ in reduced form, with $a_{i}, q_{i} \in \mathbb{Z}, 1 \leq a_{i} \leq q_{i} \leq Q$ and $\operatorname{gcd}\left(a_{i}, q_{i}\right)=1$. For any two consecutive Farey fractions $a_{i} / q_{i}<a_{i+1} / q_{i+1}$, one has $a_{i+1} q_{i}-a_{i} q_{i+1}=1$ and $q_{i}+q_{i+1}>Q$. Conversely, if $q$ and $q^{\prime}$ are two coprime integers in $\{1, \ldots, Q\}$ with $q+q^{\prime}>Q$, then there are unique integers $a \in\{1, \ldots, q\}$ and $a^{\prime} \in\left\{1, \ldots, q^{\prime}\right\}$ for which $a^{\prime} q-a q^{\prime}=1$ so that $a / q<a^{\prime} / q^{\prime}$ are consecutive Farey fractions of order $Q$. Therefore, the pairs of coprime integers $\left(q, q^{\prime}\right)$ with $q+q^{\prime}>Q$ are in one-to-one correspondence with the pairs of consecutive Farey fractions of order $Q$.

Lemma 2. Let $\mathbf{I}=[\alpha, \beta]$ be a subinterval of $[0,1]$, and assume that the function $f: \mathbb{R}^{2} \rightarrow \mathbb{C}$ is continuously differentiable with $\|f\|_{\infty}<\infty,\|D f\|_{\infty}<\infty$. Extend the definition of $f$ to $\mathbb{C}$ by assigning $f(x+i y)=f(x, y)$ for $(x, y) \in \mathbb{R}^{2}$. Then for any $\delta>0$, as $Q \rightarrow \infty$, we have

$$
\begin{aligned}
S_{Q, \mathbf{I}, f} & =\sum_{\frac{a}{q} \in \mathscr{F}_{Q} \cap \mathbf{I}} f\left(e^{\frac{\pi i}{12}\left(3-\left[\frac{q^{\prime}+Q}{q}\right]\right)} \sqrt{\frac{q^{\prime}}{q}}\right) \\
& =\frac{6|\mathbf{I}|}{\pi^{2}} Q^{2} \iint_{\mathscr{T}} f\left(e^{\frac{\pi i}{12}\left(3-\left[\frac{y+1}{x}\right]\right)} \sqrt{\frac{y}{x}}\right) \mathrm{d} x \mathrm{~d} y+O_{\delta, f}\left(Q^{2-\frac{1}{4}+\delta}\right) .
\end{aligned}
$$

Proof. For consecutive Farey fractions $\frac{a}{q}<\frac{a^{\prime}}{q^{\prime}}$ one has $a^{\prime} q-a q^{\prime}=1$, and $a \equiv-\overline{q^{\prime}}$ $(\bmod q)$, where the integer $\bar{x}(1 \leq \bar{x} \leq q)$ is the multiplicative inverse of $x(\bmod q)$ for any integer $x$ with $\operatorname{gcd}(x, q)=1$. Since $1<a<q$, we have $a=q-\bar{q}^{\prime}$ and

$$
\frac{a}{q}=1-\frac{\bar{q}^{\prime}}{q} \in \mathscr{F}_{Q} \bigcap \mathbf{I} \Longleftrightarrow \overline{q^{\prime}} \in \mathbf{I}_{q}=[(1-\beta) q,(1-\alpha) q] .
$$

Denote

$$
g(x, y)=f\left(e^{\frac{\pi i}{12}\left(3-\left[\frac{y+1}{x}\right]\right)} \sqrt{\frac{y}{x}}\right),(x, y) \in \mathbb{R}^{2} .
$$

Fixing $0<\epsilon<1$, one has

$$
S_{Q, \mathbf{I}, f}=\sum_{\substack{\frac{a}{q} \in \mathscr{F}_{Q} \cap \mathbf{I}}} g\left(\frac{q}{Q}, \frac{q^{\prime}}{Q}\right)=\sum_{\substack{\frac{a}{q} \in \mathscr{F}_{Q} \cap \mathbf{I} \\ q, q^{\prime}>\in Q}} g\left(\frac{q}{Q}, \frac{q^{\prime}}{Q}\right)+\sum_{\substack{\frac{a}{q} \in \mathscr{F}_{Q} \cap \mathbf{I} \\ q \text { or } q^{\prime} \leq \epsilon Q}} g\left(\frac{q}{Q}, \frac{q^{\prime}}{Q}\right) .
$$

The second sum above is $\lll_{f} \epsilon Q^{2}$. Denoting

$$
\Omega_{\epsilon}=\left\{(x, y) \in \mathbb{R}^{2}: \epsilon<x, y \leq 1, x+y>1\right\}
$$

the first sum above can be written as

$$
\begin{equation*}
\sum_{\substack{\left(q, q^{\prime}\right) \in Q \Omega_{\epsilon} \cap \mathbb{Z}_{p r}^{2} \\ \overline{q^{\prime}} \in \mathbf{I}_{q}}} g\left(\frac{q}{Q}, \frac{q^{\prime}}{Q}\right) . \tag{4}
\end{equation*}
$$

Since for any $(x, y) \in \Omega_{\epsilon},\left[\frac{y+1}{x}\right] \leq \frac{y+1}{x}<\frac{2}{\epsilon}$, we may let

$$
\Omega_{\epsilon, k}=\left\{(x, y) \in \Omega_{\epsilon}:\left[\frac{y+1}{x}\right]=k\right\}
$$

for each integer $k \geq 1$, so that

$$
\Omega_{\epsilon}=\bigcup_{1 \leq k \leq 2 / \epsilon} \Omega_{\epsilon, k} .
$$

Notice that for each $k \geq 1, g$ is differentiable in $\Omega_{\epsilon, k}$, and

$$
\|g\|_{\infty, \Omega_{\epsilon, k}} \ll{ }_{f} 1,\|D g\|_{\infty, \Omega_{e, k}} \ll{ }_{f} \frac{1}{\epsilon^{3 / 2}}
$$

Applying Lemma 2 to (4) we obtain for any $\delta>0$ that,

$$
\begin{aligned}
S_{Q, \mathbf{I}, f}= & \sum_{1 \leq k \leq 2 / \epsilon} \sum_{\substack{\left(q, q^{\prime}\right) \in Q \Omega_{\epsilon, k} \cap \mathbb{Z}_{p r}^{2} \\
\overline{q^{\prime} \in \mathbf{I}_{q}}}} g\left(\frac{q}{Q}, \frac{q^{\prime}}{Q}\right)+O_{f}\left(\epsilon Q^{2}\right) \\
= & \sum_{1 \leq k \leq 2 / \epsilon}\left(\frac{6|\mathbf{I}|}{\pi^{2}} \iint_{Q \Omega_{\epsilon, k}} g\left(\frac{x}{Q}, \frac{y}{Q}\right) \mathrm{d} x \mathrm{~d} y+\right. \\
& \left.O_{\delta, f}\left(Q^{\frac{3}{2}+\delta}+\frac{Q \log Q \operatorname{Area}\left(\Omega_{\epsilon, k}\right)}{\epsilon^{3 / 2}}\right)\right)+O_{f}\left(\epsilon Q^{2}\right) \\
= & \frac{6|\mathbf{I}|}{\pi^{2}} Q^{2} \iint_{\Omega_{\epsilon}} g(x, y) \mathrm{d} x \mathrm{~d} y+O_{\delta, f}\left(\frac{Q^{\frac{3}{2}+\delta}}{\epsilon}+\frac{Q \log Q}{\epsilon^{3 / 2}}+\epsilon Q^{2}\right) .
\end{aligned}
$$

Since

$$
\iint_{\mathscr{T}} g(x, y) \mathrm{d} x \mathrm{~d} y=\iint_{\Omega_{\epsilon}} g(x, y) \mathrm{d} x \mathrm{~d} y+O_{f}\left(\epsilon^{2}\right)
$$

where

$$
T=\left\{(x, y) \in \mathbb{R}^{2}: 0 \leq x, y \leq 1, x+y>1\right\}
$$

is a triangular region, we can choose $\epsilon=Q^{-1 / 4}$ to obtain

$$
S_{Q, \mathbf{I}, f}=\frac{6|\mathbf{I}|}{\pi^{2}} Q^{2} \iint_{\mathscr{T}} g(x, y) \mathrm{d} x \mathrm{~d} y+O_{\delta, f}\left(Q^{2-\frac{1}{4}+\delta}\right) .
$$

This completes the proof of Lemma 2.

## 3. Proof of Theorem 1 and Theorem 2

Assuming Theorem 1, let us first see how we can deduce Theorem 2. Recall that

$$
T=\left\{(x, y) \in \mathbb{R}^{2}: 0 \leq x, y \leq 1, x+y>1\right\}
$$

and for each $k \geq 1$ define the sets

$$
T_{k}=\left\{(x, y) \in T:\left[\frac{y+1}{x}\right]=k\right\} .
$$

Since $T_{k}, k \geq 1$ form a partition of $T$, we have

$$
\iint_{\mathscr{T}} f\left(e^{\frac{\pi i}{12}\left(3-\left[\frac{y+1}{x}\right]\right)} \sqrt{\frac{y}{x}}\right) \mathrm{d} x \mathrm{~d} y=\sum_{k=1}^{\infty} \iint_{\mathscr{T}_{k}} f\left(e^{\frac{\pi i}{12}(3-k)} \sqrt{\frac{y}{x}}\right) \mathrm{d} x \mathrm{~d} y .
$$

Using the change of variable $t=\sqrt{\frac{y}{x}}$, Theorem 2 follows at once from Theorem 1. Therefore it suffices to prove only Theorem 1.

Let $z, \Omega$ and $f$ be as in the statement of the Theorem 1 . Since $\Omega \in \mathscr{M}$, it is known that (for details see [8])

$$
\#\left(\mathcal{A}_{\Omega}(X)\right) \asymp \frac{6 A(\Omega)}{\pi^{2}} X^{2}
$$

where $A(\Omega)$ is the area of $\Omega$.
Denote by $\bar{\Omega}$ the closure of $\Omega$. Let

$$
\Omega_{i}=\bar{\Omega} \bigcap\left\{(r, \theta): \frac{(i-1) \pi}{4} \leq \theta<\frac{i \pi}{4}\right\}, \quad i=-1,0,1,2
$$

and for each $i$, define $\mathcal{A}_{\Omega_{i}}(X), \mathcal{B}_{\Omega_{i}}(X)$ and $\mathcal{W}_{z, \Omega_{i}}(X)$ similarly by replacing $\Omega$ by $\Omega_{i}$. Defining

$$
S_{\Omega_{i}, f}(X)=\sum_{w \in \mathcal{W}_{z, \Omega_{i}}(X)} f(w)
$$

we see that

$$
\begin{equation*}
S_{\Omega, f}(X)=\sum_{i=-1}^{2} S_{\Omega_{i}, f}(X)+O_{f}(1) \tag{5}
\end{equation*}
$$

Let us consider $S_{\Omega_{1}, f}(X)$ first. Assume for the moment that

$$
\Omega_{1}=\left\{(r, \theta): r \leq \rho(\theta), \theta_{1} \leq \theta \leq \theta_{2}\right\}
$$

where $\rho$ is a bounded, positive and continuously differentiable function of $\theta$, and in addition, assume that if we redefine $f$ on $\mathbb{R}^{2}$ by assigning $f(x, y)=f(x+i y)$ for any $x, y \in \mathbb{R}$, then $f$ turns out to be continuously differentiable over $\mathbb{R}^{2}$ with $\|D f\|_{\infty}<\infty$. Fix a large integer $L>0$ and put

$$
\alpha=\frac{\theta_{2}-\theta_{1}}{L}, \quad \alpha_{i}=\theta_{1}+i \cdot \alpha, \quad 0 \leq i \leq L .
$$

Assume that for any $0 \leq i \leq L$, the ray $\theta=\alpha_{i}, \quad 0 \leq i \leq L$ intersects the boundary of $\Omega_{1}$ at the point $A_{i}$. At each point $A_{i+1}, 0 \leq i \leq L-1$, we draw a vertical line that intersects the ray $\theta=\alpha_{i}$ at the point $A_{i}^{\prime}$. One sees that

$$
\begin{aligned}
A_{i+1} & =\left(\rho\left(\alpha_{i+1}\right) \cos \left(\alpha_{i+1}\right), \rho\left(\alpha_{i+1}\right) \sin \left(\alpha_{i+1}\right)\right) \\
A_{i}^{\prime} & =\left(\rho\left(\alpha_{i+1}\right) \cos \left(\alpha_{i+1}\right), \rho\left(\alpha_{i+1}\right) \cos \left(\alpha_{i+1}\right) \tan \left(\alpha_{i}\right)\right),
\end{aligned}
$$

for $0 \leq i \leq L-1$. Let $\Omega_{1, i}$ be the $i$-th subregion of $\Omega_{1}$ lying inside the rays $\overrightarrow{O A_{i}}, \overrightarrow{O A_{i+1}}$ and $\triangle_{i}$ be the triangle $O A_{i+1} A_{i}^{\prime}$. We have

$$
A\left(\Omega_{1, i}\right)-A\left(\triangle_{i}\right) \ll_{\rho} \frac{1}{L^{2}},
$$

and it follows that

$$
\begin{aligned}
\#\left(\mathcal{A}_{\Omega_{1, i}}(X)\right) & \asymp \frac{6 A\left(\Omega_{1, i}\right) X^{2}}{\pi^{2}} \\
& =\frac{6 A\left(\triangle_{i}\right) X^{2}}{\pi^{2}}+O_{\rho}\left(\frac{X^{2}}{L^{2}}\right) .
\end{aligned}
$$

Therefore

$$
\begin{equation*}
S_{\Omega_{1, i}, f}(X) \asymp S_{\triangle_{i, f}}(X)+O_{\rho, f}\left(\frac{X^{2}}{L^{2}}\right) \tag{6}
\end{equation*}
$$

Fix any $i$ and let $\mathbf{I}_{i}=\left[\tan \left(\alpha_{i}\right), \tan \left(\alpha_{i+1}\right)\right] \subset[0,1]$. Denote $x_{i+1}=\rho\left(\alpha_{i+1}\right) \cos \left(\alpha_{i+1}\right)$ and define $X_{i}:=x_{i+1} X \ll_{\rho} X$. Suppose that the visible points inside $X \triangle_{i}$ are $P_{j}=\left(a_{j}, b_{j}\right), 1 \leq j \leq N=N\left(\triangle_{i}, X\right)$, where the rays $\overrightarrow{O P_{N}}, \overrightarrow{O P_{N-1}}, \ldots, \overrightarrow{O P_{1}}$ are
arranged with counterclockwise order around the origin. We see that $\frac{b_{1}}{a_{1}}<\frac{b_{2}}{a_{2}}<$ $\cdots<\frac{b_{N}}{a_{N}}$ are consecutive Farey fractions of order $X_{i}$ inside the interval $\mathbf{I}_{i}$. For any fraction $\frac{a}{q} \in \mathscr{F}_{X_{i}}$, denote by $\frac{a^{\prime}}{q^{\prime}}$ its next neighbor and by $\frac{a^{\prime \prime}}{q^{\prime \prime}}$ its previous neighbor. Recall that the index $v_{Q}\left(\frac{a}{q}\right)$ of the Farey fraction $\frac{a}{q}$ satisfies the equalities

$$
v_{Q}\left(\frac{a}{q}\right)=\frac{a^{\prime \prime}+a^{\prime}}{a}=\left[\frac{a^{\prime}+X_{i}}{a}\right] .
$$

If we let

$$
w_{\frac{a}{q}}=e^{\frac{\pi i}{12}\left(3-\frac{q^{\prime}+q^{\prime \prime}}{q}\right)} \sqrt{\frac{q^{\prime} z+a^{\prime}}{q z+a}}=e^{\frac{\pi i}{12}\left(3-\left[\frac{q^{\prime}+X_{i}}{q}\right]\right)} \sqrt{\frac{q^{\prime} z+a^{\prime}}{q z+a}},
$$

then

$$
S_{\triangle_{i}, f}(X)=\sum_{\frac{a}{q} \in \mathscr{F}_{X_{i}} \cap \mathbf{I}_{i}} f\left(w_{\frac{a}{q}}\right)+O_{f}(1)
$$

Next we take

$$
u_{\frac{a}{q}}=e^{\frac{\pi i}{12}\left(3-\left[\frac{q^{\prime}+X_{i}}{q}\right]\right)} \sqrt{\frac{q^{\prime}}{q}} .
$$

Since $q q^{\prime} \geq q+q^{\prime}>X_{i}$, and

$$
\frac{q^{\prime} z+a^{\prime}}{q z+a}=\frac{q^{\prime}}{q}\left(1+\frac{1}{q^{\prime}(q z+a)}\right)=\frac{q^{\prime}}{q}\left(1+O_{z}\left(\frac{1}{X_{i}}\right)\right)
$$

it follows that

$$
\sqrt{\frac{q^{\prime} z+a^{\prime}}{q z+a}}=\sqrt{\frac{q^{\prime}}{q}}\left(1+O_{z}\left(\frac{1}{X_{i}}\right)\right) .
$$

Consequently we have

$$
\left|f\left(w_{\frac{a}{q}}\right)-f\left(u_{\frac{a}{q}}\right)\right| \lll f_{f}\left|\sqrt{\frac{q^{\prime} z+a^{\prime}}{q z+a}}-\sqrt{\frac{q^{\prime}}{q}}\right| \ll z z^{\frac{q^{\prime}}{q}} \frac{1}{X_{i}} \leq \frac{1}{\sqrt{X_{i}}},
$$

and

$$
\begin{aligned}
S_{\triangle_{i}, f}(X) & =\sum_{\frac{a}{q} \in \mathscr{F}_{X_{i}} \cap \mathbf{I}_{i}} f\left(u_{\frac{a}{q}}\right)+O_{f, z}\left(\frac{\#\left(\mathscr{F}_{X_{i}} \cap \mathbf{I}_{i}\right)}{\sqrt{X_{i}}}\right)+O_{f}(1) \\
& =\sum_{\frac{a}{q} \in \mathscr{F}_{X_{i}} \cap \mathbf{I}_{i}} f\left(u_{\frac{a}{q}}\right)+O_{f, \rho, z}\left(X^{\frac{3}{2}}\right)
\end{aligned}
$$

If we denote

$$
u(x, y)=e^{\frac{\pi i}{12}\left(3-\left[\frac{y+1}{x}\right]\right)} \sqrt{\frac{y}{x}}
$$

then taking any fixed $\delta>0$ and letting $X \rightarrow \infty$, we obtain by Lemma 3 that

$$
S_{\triangle_{i}, f}(X)=\frac{6\left|\mathbf{I}_{i}\right|}{\pi^{2}} X_{i}^{2} \iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y+O_{\delta, \rho, f, z}\left(X^{2-\frac{1}{4}+\delta}\right) .
$$

Therefore we have

$$
\begin{aligned}
S_{\Omega_{1}, f}(X) & =\sum_{i=0}^{L-1} S_{\Omega_{1, i}, f}(X)+O_{f}(L) \\
& \asymp \sum_{i=0}^{L-1}\left(\frac{6\left|\mathbf{I}_{i}\right|}{\pi^{2}} X_{i}^{2} \iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y+O_{\delta, \rho, f, z}\left(X^{2-\frac{1}{4}+\delta}+\frac{X^{2}}{L^{2}}\right)\right)+O_{f}(L) \\
& =\frac{6 X^{2}}{\pi^{2}}\left(\iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y\right) \sum_{i=0}^{L-1}\left|\mathbf{I}_{i}\right| x_{i+1}^{2}+O_{\delta, \rho, f, z}\left(L X^{2-\frac{1}{4}+\delta}+\frac{X^{2}}{L}\right)
\end{aligned}
$$

Moreover using

$$
\begin{aligned}
\sum_{i=0}^{L-1}\left|\mathbf{I}_{i}\right| x_{i+1}^{2} & =\sum_{i=0}^{L-1}\left(\tan \left(\alpha_{i+1}\right)-\tan \left(\alpha_{i}\right)\right) \rho\left(\alpha_{i+1}\right)^{2} \cos \left(\alpha_{i+1}\right)^{2} \\
& =\sum_{i=0}^{L-1} \frac{1}{\cos \left(\xi_{i}\right)^{2}} \rho\left(\alpha_{i+1}\right)^{2} \cos \left(\alpha_{i+1}\right)^{2} \quad\left(\exists \xi_{i} \in\left[\alpha_{i}, \alpha_{i+1}\right]\right) \\
& =\int_{\theta_{1}}^{\theta_{2}} \rho^{2}(\theta) \mathrm{d} \theta+O_{\rho}\left(\frac{1}{L}\right)=2 A\left(\Omega_{1}\right)+O_{f}\left(\frac{1}{L}\right)
\end{aligned}
$$

gives us that

$$
S_{\Omega_{1}, f}(X) \asymp \frac{6 X^{2}}{\pi^{2}}\left(\iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y\right) 2 A\left(\Omega_{1}\right)+O_{\delta, \rho, f, z}\left(L X^{2-\frac{1}{4}+\delta}+\frac{X^{2}}{L}\right) .
$$

Choosing $L=Q^{\frac{1}{8}}$, we have

$$
S_{\Omega_{1}, f}(X) \asymp \frac{6 X^{2}}{\pi^{2}}\left(\iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y\right) 2 A\left(\Omega_{1}\right)+O_{\delta, \rho, f, z}\left(X^{2-\frac{1}{8}+\delta}\right)
$$

Finally choosing $0<\delta<\frac{1}{8}$ and letting $X \rightarrow \infty$, we obtain that

$$
\begin{equation*}
\lim _{X \rightarrow \infty} \frac{S_{\Omega_{1}, f}(X)}{\#\left(\mathcal{A}_{\Omega}(X)\right)}=\frac{2 A\left(\Omega_{1}\right)}{A(\Omega)} \iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y . \tag{7}
\end{equation*}
$$

Since continuous functions can be approximated by $C^{\infty}$ functions uniformly on a compact interval, using a standard approximation procedure we see that (7) also holds under the weaker assumption that $f$ and $\rho$ are continuous.

We treat $\Omega_{2}$ similarly, with a slight difference. Fix a large integer $L>0$, let $\alpha=\frac{\pi}{4 L}$ and $\alpha_{i}=\frac{\pi}{4}+i \alpha$. Assume that the ray $\theta=\alpha_{i}$ intersects the boundary of $\Omega_{1}$ at the point $A_{i}$. At each point $A_{i}$, we draw a horizontal line which intersects the ray $\theta=\alpha_{i+1}$ at the point $A_{i}^{\prime}$. We use the triangle $\triangle_{O A_{i} A_{i}^{\prime}}$ to estimate the subregion of $\Omega_{1}$ lying inside the rays $\theta=\alpha_{i}$ and $\theta=\alpha_{i+1}$. Following a similar argument as above and applying Lemma 3, we obtain that

$$
\lim _{X \rightarrow \infty} \frac{S_{\Omega_{2}, f}(X)}{\#\left(\mathcal{A}_{\Omega}(X)\right)}=\frac{2 A\left(\Omega_{2}\right)}{A(\Omega)} \iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y .
$$

Observing that $s(-a, b)=-s(a, b)$ for $\Omega_{-1}$ and $\Omega_{0}$, the computation of asymptotic formulas in these regions can be reduced to that of the regions $\Omega_{1}$ and $\Omega_{2}$. As a result we also have

$$
\lim _{X \rightarrow \infty} \frac{S_{\Omega_{i}, f}(X)}{\#\left(\mathcal{A}_{\Omega}(X)\right)}=\frac{2 A\left(\Omega_{i}\right)}{A(\Omega)} \iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y, \quad i=-1,0 .
$$

Lastly, from (5) we deduce that

$$
\begin{aligned}
\lim _{X \rightarrow \infty} \frac{S_{\Omega, f}(X)}{\#\left(\mathcal{A}_{\Omega}(X)\right)} & =\frac{2\left(A\left(\Omega_{-1}\right)+A\left(\Omega_{0}\right)+A\left(\Omega_{1}\right)+A\left(\Omega_{2}\right)\right)}{A(\Omega)} \iint_{\mathscr{T}} f \circ u(x, y) \mathrm{d} x \mathrm{~d} y \\
& =2 \iint_{\mathscr{T}} f\left(e^{\frac{\pi i}{12}\left(3-\left[\frac{y+1}{x}\right]\right)} \sqrt{\frac{y}{x}}\right) \mathrm{d} x \mathrm{~d} y .
\end{aligned}
$$

This completes the proof of Theorem 1.
Remark. One may consider the following more general problem. Fix a positive integer $d$ and let

$$
\mathcal{W}_{z, \Omega}^{(d)}(X):=\left\{\frac{\eta\left(A_{j} z\right)}{\eta\left(A_{j-d} z\right)}: d+1 \leq j \leq N-1\right\}
$$

For $d=1$, this set reduces to $\mathcal{W}_{z, \Omega}(X)$ whose distribution was established above. A similar distribution result can be proved for any $d$. The proof goes on the same lines as that of Theorem 2 with some additional technical complications which can
be handled by employing the machinery developed in [3], [7] and [8] to study the local spacing distribution of Farey fractions and visible points.

## 4. Proof of Theorem 3

The proof of Theorem 3 will be by induction on $k$. Note that if $k=1$, then $v_{Q}\left(\gamma_{1}\right)=v_{Q}\left(\frac{1}{Q}\right)=1$ and $\gamma_{2}=\frac{1}{Q-1}$ for $Q \geq 2$ so that taking $a=Q, d=1$ and $c=Q-1$ we see that the right side of the desired equality is

$$
6-Q-\frac{Q+1}{Q-1}+12 s(1, Q-1)
$$

By the reciprocity law for Dedekind sums we have

$$
s(1, Q-1)+s(Q-1,1)=-\frac{1}{4}+\frac{1}{12}\left(Q-1+\frac{2}{Q-1}\right)
$$

and it is easy to see that $s(Q-1,1)=0$. Using these values in the above expression we obtain that the right side of the desired equality is also 1 . Hence Theorem 3 holds for $k=1$ and induction starts. For the induction step, assume that $\gamma_{k}=\frac{b}{a}<$ $\gamma_{k+1}=\frac{d}{c}<\gamma_{k+2}=\frac{f}{e}$ are consecutive Farey fractions of order $Q$ and the desired formula holds for $k$, namely that

$$
\sum_{j \leq k} v_{Q}\left(\gamma_{j}\right)=3 k+3-Q-\frac{a+d}{c}+12 s(d, c)
$$

Our goal is to show that

$$
\sum_{j \leq k+1} v_{Q}\left(\gamma_{j}\right)=3(k+1)+3-Q-\frac{c+f}{e}+12 s(f, e)
$$

To complete the proof by induction, it suffices to show that the difference of both sides of the above equations are the same. Clearly the difference of left sides is $v_{Q}\left(\gamma_{k+1}\right)=\frac{a+e}{c}$. On the other hand, the difference of right sides is

$$
3-\frac{c+f}{e}+\frac{a+d}{c}+12(s(f, e)-s(d, c)) .
$$

Repeating the derivation in Section 2 and using the reciprocity law for Dedekind sums, we see that

$$
s(f, e)-s(d, c)=-\frac{1}{4}+\frac{1}{12}\left(\frac{e}{c}+\frac{c}{e}+\frac{1}{c e}\right) .
$$

Using this in the above expression and noting that $d e-f c=-1$, it follows that the difference of right sides is also $\frac{a+e}{c}$. This completes the induction and the proof of Theorem 3.

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Department of Mathematics, Koc University, Rumelifeneri Yolu, 34450 Sariyer, Istanbul, TURKEY

E-mail address: ealkan@ku.edu.tr

Department of Mathematics, University of Illinois at Urbana-Champaign, 273
Altgeld Hall, MC-382, 1049 W. Green Street, Urbana, Illinois 61801-2975, USA
E-mail address: xiong@math.uiuc.edu
E-mail address: zaharesc@math.uiuc.edu


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