On Character Sums with Distances on the Upper Half Plane over a Finite Field

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Abstract

For the finite field $\mathbb{F}_q$ of q elements (q odd) and a quadratic non-residue $\alpha \in \mathbb{F}_q$, we define the distance function

$$\delta\left(u + v\sqrt{\alpha}, x + y\sqrt{\alpha}\right) = \frac{(u-x)^2 - \alpha(v-y)^2}{vy}$$

on the upper half plane $\mathcal{H}_q = \{x + y\sqrt{\alpha} \mid x \in \mathbb{F}_q, y \in \mathbb{F}_q^*\} \subseteq \mathbb{F}_{q^2}$. For two sets $\mathcal{E}, \mathcal{F} \subset \mathcal{H}_q$ with $\#\mathcal{E} = E$, $\#\mathcal{F} = F$ and a non-trivial additive character ψ on $\mathbb{F}_q$, we give the following estimate

$$\left| \sum_{\mathbf{w} \in \mathcal{E}, \mathbf{z} \in \mathcal{F}} \psi(\delta(\mathbf{w}, \mathbf{z})) \right| \le \min\left\{ \sqrt{3} q^{5/4}, q + \sqrt{2qE} \right\} \sqrt{EF},$$

which is non-trivial if $EF/q^2 \to \infty$ as $q \to \infty$.

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1 Introduction

1.1 Background

Since the groundbreaking result of Jean Bourgain, Nets Katz and Terence Tao [4] on the sum-product problem in finite fields, there have been a burst of activity in the area of discrete analogues of classical combinatorial problems, which have recently culminated in a solution of the *Kakeya problem* over finite fields by Z. Dvir [6]. In particular, A. Iosevich and M. Rudnev [7] have introduced a finite field analogue of the Erdős distance problem. Namely, it is shown in [7] that there are absolute constants $c_1, c_2 > 0$ such that for any odd q and any set $\mathcal{E} \subseteq \mathbb{F}_q^n$ of cardinality $\#\mathcal{E} \ge c_1 q^{n/2}$ the number $D(\mathcal{E})$ of distances

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (x_i - y_i)^2$$

between all pairs of vectors

$$\mathbf{x} = (x_1, \ldots, x_n),\ \mathbf{y} = (y_1, \ldots, y_n) \in \mathcal{E},$$

satisfies

$$D(\mathcal{E}) \ge c_2 \min\left\{q, q^{-(n-1)/2} \#\mathcal{E}\right\}.$$

In fact one can also obtain a more general result for the number of pairwise distances between elements of two sets $\mathcal{E}, \mathcal{F} \subseteq \mathbb{F}_q^n$ (see also [16]).

The argument of A. Iosevich and M. Rudnev [7] is based on Fourier analysis on finite fields, more specifically, on analysis on "Q-spheres" in finite fields. L. A. Vinh [19] has revisited the problem in a more general setting. Instead of the Euclidean distance function, L. A. Vinh [19] considers any non-degenerate quadratic forms and obtains similar results. The new idea in [19] is to combine sharp eigenvalue estimate resulting from Ramanujan graphs related with non-degenerate quadratic forms and a result in graph theory. While this method, as pointed out in [19], is essentially equivalent to that of A. Iosevich and M. Rudnev [7], this approach is more elementary.

Furthermore, L. A. Vinh [19] has considered an analogue of the Erdős distance problem for points of the upper half plane $\mathcal{H}_q$ on the finite field $\mathbb{F}_q$, where the upper half plane $\mathcal{H}_q$ is

$$\mathcal{H}_q = \{x + y\sqrt{\alpha} \mid x \in \mathbb{F}_q, y \in \mathbb{F}_q^*\} \subseteq \mathbb{F}_{q^2},$$

for a fixed quadratic non-residue $\alpha \in \mathbb{F}_q$, and the distance between two points is

$$\delta\left(u + v\sqrt{\alpha}, x + y\sqrt{\alpha}\right) = \frac{(u-x)^2 - \alpha(v-y)^2}{vy}.$$

For two sets $\mathcal{E}, \mathcal{F} \subseteq \mathcal{H}_q$ he defines

$$\Delta_{\mathcal{H}_q}(\mathcal{E}, \mathcal{F}) = \#\{\delta(\mathbf{w}, \mathbf{z}) \mid \mathbf{w} \in \mathcal{E}, \mathbf{z} \in \mathcal{F}\},$$

and by using similar ideas he obtains

$$\Delta_{\mathcal{H}_q}(\mathcal{E}, \mathcal{F}) \geq \min\left\{\frac{\sqrt{\#\mathcal{E}\#\mathcal{F}}}{3q^{1/2}}, q-1\right\}.$$

More details can be found in the original paper of L. A. Vinh [19]. For various structural properties of $\mathcal{H}_q$, interested readers can see the series of papers [2, 3, 9, 15, 17, 18] and references therein. In particular, they naturally lead to new examples of *Ramanujan graphs* (see [5, 10, 11, 12, 13]).

1.2 Main Result

For an additive character ψ on $\mathbb{F}_q$ (q odd) and two sets $\mathcal{E}, \mathcal{F} \subseteq \mathcal{H}_q$, we define the character sum

$$S_\psi(\mathcal{E}, \mathcal{F}) = \sum_{\mathbf{w} \in \mathcal{E}, \mathbf{z} \in \mathcal{F}} \psi(\delta(\mathbf{w}, \mathbf{z}))$$

(see [14] for basic properties of additive characters).

Our main result is the following.

Theorem 1. *For arbitrary sets $\mathcal{E}, \mathcal{F} \subseteq \mathcal{H}_q$ with $\#\mathcal{E} = E$, $\#\mathcal{F} = F$ and any non-trivial additive character ψ on $\mathbb{F}_q$, we have*

$$S_\psi(\mathcal{E}, \mathcal{F}) \le \min\left\{q + \sqrt{2qE}, \sqrt{3}q^{5/4}\right\}\sqrt{EF}.$$

Clearly, the bound of Theorem 1 is non-trivial if $EF/q^2 \to \infty$ as $q \to \infty$.

Our method is a combination of Fourier analysis on finite fields of A. Iosevich and M. Rudnev [7] with the graph theory method of L. A. Vinh [19]. We note that each of the above methods can be used independently, but their combination leads to a stronger result.

We remark that if we consider any non-degenerate quadratic form Q on the n-dimensional vector space $\mathbb{F}_q^n$, then for any two sets $\mathcal{E}, \mathcal{F} \subset \mathbb{F}_q^n$ with $\#\mathcal{E} = E$, $\#\mathcal{F} = F$ and any non-trivial additive character ψ on $\mathbb{F}_q$, we can obtain

$$\left|\sum_{\mathbf{w}\in\mathcal{E},\mathbf{z}\in\mathcal{F}} \psi(Q(\mathbf{w}, \mathbf{z}))\right| \le \sqrt{q^n EF},$$

which for $n = 2$ looks a little better than Theorem 1 (although of course it applies to a different distance functions so the results are incomparable).

2 Preparations

2.1 Some Auxiliary Character Sums

First we need to evaluate the following character sums.

For a additive character ψ of $\mathbb{F}_q$ and $\mathbf{w_1}, \mathbf{w_2} \in \mathcal{H}_q$, define

$$T_\psi\left(\mathbf{w_1}, \mathbf{w_2}\right) = \sum_{z\in\mathcal{H}_q} \psi\left(\delta(\mathbf{w_1}, \mathbf{z}) - \delta(\mathbf{w_2}, \mathbf{z})\right).$$

Lemma 2. *For a non-trivial additive character ψ of $\mathbb{F}_q$ and two vectors $\mathbf{w_1}, \mathbf{w_2} \in \mathcal{H}_q$ with $\mathbf{w_1} \ne \mathbf{w_2}$, we have*

$$T_\psi\left(\mathbf{w_1}, \mathbf{w_2}\right) = \begin{cases} -q(\psi(\tau) + \psi(-\tau)), & \textit{where } \tau^2 = 4\alpha\delta(\mathbf{w_1}, \mathbf{w_2}), \tau \in \mathbb{F}_q^*, \\ 0, & \textit{if } \delta(\mathbf{w_1}, \mathbf{w_2}) \textit{ is a square in } \mathbb{F}_q^*. \end{cases}$$

Proof. First we rewrite $T_\psi(\mathbf{w_1}, \mathbf{w_2})$ as

$$T_\psi(\mathbf{w_1}, \mathbf{w_2}) = \sum_{t \in \mathbb{F}_q} \psi(t) N_\psi(\mathbf{w_1}, \mathbf{w_2}; t),$$

where

$$N_\psi(\mathbf{w_1}, \mathbf{w_2}; t) = \#\{\mathbf{z} \in \mathcal{H}_q : \delta(\mathbf{w_1}, \mathbf{z}) - \delta(\mathbf{w_2}, \mathbf{z}) = t\}.$$

Let

$$\mathbf{w_1} = (a, b), \mathbf{w_2} = (c, d) \in \mathcal{H}_q$$

with $\mathbf{w_1} \neq \mathbf{w_2}$ and $\mathbf{z} = (x, y) \in \mathcal{H}_q$. Then $\delta(\mathbf{w_1}, \mathbf{z}) - \delta(\mathbf{w_2}, \mathbf{z}) = t$ if and only if

$$d((x-a)^2 - \alpha(y-b)^2) - b((x-c)^2 - \alpha(y-d)^2) = bdty.$$

Recall the orthogonality property of additive characters

$$\sum_{u \in \mathbb{F}_q} \psi(\gamma u) = \begin{cases} 0, & \text{if } \gamma \in \mathbb{F}_q^*, \\ q, & \text{if } \gamma = 0. \end{cases} \tag{1}$$

Then we write

$$N_\psi(\mathbf{w_1}, \mathbf{w_2}; t) = \sum_{\substack{y \in \mathbb{F}_q^* \\ x \in \mathbb{F}_q}} \frac{1}{q} \sum_{u \in \mathbb{F}_q} \psi\left(u \Psi_{\mathbf{w_1}, \mathbf{w_2}, t}(x, y)\right),$$

where

$$\Psi_{\mathbf{w_1}, \mathbf{w_2}, t}(x, y) = d(x-a)^2 - d\alpha(y-b)^2 + b(x-c)^2 - b\alpha(y-d)^2 - bdty.$$

To simplify, we isolate the cases for $u = 0$ and $y = 0$ and define the functions

$$\begin{aligned} G_\psi(\mathbf{w_1}, \mathbf{w_2}; u) &= \sum_{x \in \mathbb{F}_q} \psi\left(u\{d(x-a)^2 - b(x-c)^2\}\right), \\ H_\psi(\mathbf{w_1}, \mathbf{w_2}; u, t) &= \sum_{y \in \mathbb{F}_q} \psi\left(u\{b\alpha(y-d)^2 - d\alpha(y-b)^2 - bdyt\}\right), \\ J_\psi(\mathbf{w_1}, \mathbf{w_2}) &= q - 1 - \frac{1}{q} \sum_{u \in \mathbb{F}_q^*} G_\psi(\mathbf{w_1}, \mathbf{w_2}; u) \psi\left(u\{b\alpha d^2 - d\alpha b^2\}\right). \end{aligned}$$

Then $N_\psi(\mathbf{w_1}, \mathbf{w_2}; t)$ can be written as

$$N_\psi(\mathbf{w_1}, \mathbf{w_2}; t) = J_\psi(\mathbf{w_1}, \mathbf{w_2}) + \frac{1}{q} \sum_{u \in \mathbb{F}_q^*} G_\psi(\mathbf{w_1}, \mathbf{w_2}; u) H_{\mathbf{w_1}, \mathbf{w_2}}(u, t).$$

By using the orthogonality property of additive characters (1) and noticing that $J_\psi(\mathbf{w_1}, \mathbf{w_2})$ is independent of t, we obtain

$$T_\psi\left(\mathbf{w_1}, \mathbf{w_2}\right) = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \psi(t) \sum_{u \in \mathbb{F}_q^*} G_\psi(\mathbf{w_1}, \mathbf{w_2}; u) H_{\mathbf{w_1}, \mathbf{w_2}}(u, t).$$

Since $\mathbf{w_1} \neq \mathbf{w_2}$, if $b = d$, then $a \neq c$ and

$$d(x-a)^2 - b(x-c)^2 = 2b(c-a)\left(x - \frac{a+c}{2}\right).$$

Hence, by (1) again we have $G_\psi(\mathbf{w_1}, \mathbf{w_2}; u) = 0$ for any $u \neq 0$. Therefore, if $b = d$ then

$$T_\psi\left(\mathbf{w_1}, \mathbf{w_2}\right) = 0.$$

If $b \neq d$, then

$$d(x-a)^2 - b(x-c)^2 = (d-b)\left(x - \frac{ad-bc}{d-b}\right)^2 - \frac{bd(a-c)^2}{d-b}.$$

To evaluate $G_\psi(\mathbf{w_1}, \mathbf{w_2}; u)$, we recall some standard properties of Gauss sums over $\mathbb{F}_q$ (we refer to [14] for details). The classical Gauss sum $G(\psi)$ is defined as

$$G(\psi) = \sum_{z \in \mathbb{F}_q} \psi(z^2).$$

It is easy to see that

$$G(\psi) = \sum_{z \in \mathbb{F}_q} \eta(z)\psi(z),$$

where η is the quadratic character of $\mathbb{F}_q$. We know that

$$G(\psi)^2 = \eta(-1)q,$$

and

$$\sum_{z \in \mathbb{F}_q} \psi(tz^2) = \eta(t) G(\psi). \tag{2}$$

If $b \neq d$, then by using (2) we see that

$$G_\psi(\mathbf{w_1}, \mathbf{w_2}; u) = G(\psi)\eta(u(d-b))\psi\left(-\frac{bd(a-c)^2u}{d-b}\right). \tag{3}$$

For $H_\psi(\mathbf{w_1}, \mathbf{w_2}; u, t)$, noticing that

$$\begin{aligned} b\alpha(y-d)^2 - d\alpha(y-b)^2 - bdyt& \\ &= \alpha(b-d)\left(y - \frac{bdt}{2\alpha(b-d)}\right)^2 + bd\alpha(d-b) + \frac{b^2d^2t^2}{4\alpha(d-b)}, \end{aligned}$$

we obtain again from (2) that

$$H_\psi(\mathbf{w_1}, \mathbf{w_2}; u, t) = G(\psi)\eta(u\alpha(b-d))\psi\left(ubd\alpha(d-b) + \frac{b^2d^2t^2u}{4\alpha(d-b)}\right). \tag{4}$$

Since $G(\psi)^2 = \eta(-1)q$ and $\eta(\alpha) = -1$, using (3), (4) and (1), we may rewrite $T_\psi(\mathbf{w_1}, \mathbf{w_2})$ as

$$T_\psi(\mathbf{w_1}, \mathbf{w_2}) = -\sum_{t \in \mathbb{F}_q} \psi(t) \sum_{u \in \mathbb{F}_q} \psi\left(u f_{\mathbf{w_1}, \mathbf{w_2}}(t)\right), \tag{5}$$

where

$$f_{\mathbf{w_1}, \mathbf{w_2}}(t) = bd\alpha(d-b) + \frac{b^2d^2t^2}{4\alpha(d-b)} - \frac{bd(a-c)^2}{d-b}.$$

Recalling (1) one sees that in the identity (5) there is no contribution from the terms $t \in \mathbb{F}_q$ with $f_{\mathbf{w_1}, \mathbf{w_2}}(t) \neq 0$. One derives that $f_{\mathbf{w_1}, \mathbf{w_2}}(t) = 0$ if and only if

$$t^2 = 4\alpha\frac{(a-c)^2 - \alpha(d-b)^2}{bd} = 4\alpha\delta(\mathbf{w_1}, \mathbf{w_2}).$$

Therefore if $\delta(\mathbf{w_1}, \mathbf{w_2})$ is a square in $\mathbb{F}_q^*$, then $T_\psi(\mathbf{w_1}, \mathbf{w_2}) = 0$, otherwise, let

$$\tau^2 = 4\alpha\delta(\mathbf{w_1}, \mathbf{w_2})$$

for some $\tau \in \mathbb{F}_q^*$, then

$$T_\psi(\mathbf{w_1}, \mathbf{w_2}) = -q\left(\psi(\tau) + \psi(-\tau)\right),$$

which completes the proof. $\square$

2.2 Some Graph Theory Tools

We now recall some graph theory results similar to those of L. A. Vinh [19].

First, for any fixed $a \in \mathbb{F}_q$, the finite non-Euclidean graph $V_q(\alpha, a)$ has vertices as the points in $\mathcal{H}_q$ and edges between vertices $\mathbf{z}, \mathbf{w} \in \mathcal{H}_q$ if and only if $\delta(\mathbf{z}, \mathbf{w}) = a$. It is known that, except for $a = 0$ or 4α, the graph $V_q(\alpha, a)$ is a $(q^2 - q, q+1, 2\sqrt{q})$-regular graph, that is, the graph $V_q(\alpha, a)$ is $(q+1)$-regular on $(q^2 - q)$ vertices with $|\lambda| \le 2\sqrt{q}$ for any non-trivial eigenvalue λ of the adjacency matrix of the graph $V_q(\alpha, a)$, see [5, 10, 11, 12, 13, 19] and references therein.

Next, we use the following well-known result, see [1, Chapter 9].

Lemma 3. *Let $\mathfrak{G}$ be an (n, d, λ)-regular graph. For every set of vertices $\mathcal{A}$ of $\mathfrak{G}$, denote by $e(\mathcal{A})$ the number of pairs $(u, v) \in \mathcal{A} \times \mathcal{A}$ such that uv is an edge of $\mathfrak{G}$. Then*

$$\left| e(\mathcal{A}) - \frac{d}{n} (\#\mathcal{A})^2 \right| \le \lambda \#\mathcal{A}.$$

.

3 Proof of Theorem 1

By the Cauchy inequality,

$$|S_\psi(\mathcal{E}, \mathcal{F})|^2 = \left| \sum_{\mathbf{w} \in \mathcal{E}} \sum_{\mathbf{z} \in \mathcal{F}} \psi(\delta(\mathbf{w}, \mathbf{z})) \right|^2 \le FW, \qquad (6)$$

where

$$W = \sum_{\mathbf{z} \in \mathcal{F}} \left| \sum_{\mathbf{w} \in \mathcal{E}} \psi(\delta(\mathbf{w}, \mathbf{z})) \right|^2.$$

Extending the range of summation over $\mathbf{z} \in \mathcal{F}$ to the whole plane $\mathcal{H}_q$, squaring out and changing the order of summation, we obtain

$$\begin{aligned} W \le \sum_{\mathbf{z} \in \mathcal{H}_q} \left| \sum_{\mathbf{w} \in \mathcal{E}} \psi(\delta(\mathbf{w}, \mathbf{z})) \right|^2 &= \sum_{\mathbf{w_1}, \mathbf{w_2} \in \mathcal{E}} \sum_{\mathbf{z} \in \mathcal{H}_q} \psi(\delta(\mathbf{w_1}, \mathbf{z}) - \delta(\mathbf{w_2}, \mathbf{z})) \\ &= q(q-1)E + \sum_{\mathbf{w_1} \ne \mathbf{w_2} \in \mathcal{E}} T_\psi(\mathbf{w_1}, \mathbf{w_2}). \end{aligned} \qquad (7)$$

By Lemma 2

$$\sum_{\mathbf{w_1}\neq\mathbf{w_2}\in\mathcal{E}} T_\psi\left(\mathbf{w_1},\mathbf{w_2}\right) = -q \sum_{\substack{\mathbf{w_1}\neq\mathbf{w_2}\in\mathcal{E}\\ 4\alpha\delta(\mathbf{w_1},\mathbf{w_2})=\tau^2\neq 0}} \left(\psi(\tau)+\psi(-\tau)\right). \tag{8}$$

We now estimate the above sum trivially as

$$\left|\sum_{\mathbf{w_1}\neq\mathbf{w_2}\in\mathcal{E}} T_\psi\left(\mathbf{w_1},\mathbf{w_2}\right)\right| \leq 2qE^2.$$

Therefore, we see from (7) that

$$W \leq q^2E + 2qE^2,$$

and using (6), we obtain

$$S_\psi\left(\mathcal{E},\mathcal{F}\right) \leq \left(q+\sqrt{2qE}\right)\sqrt{E\cdot F}.$$

On the other hand, the right hand side of (8) can be rewritten as

$$\sum_{\mathbf{w_1}\neq\mathbf{w_2}\in\mathcal{E}} T_\psi\left(\mathbf{w_1},\mathbf{w_2}\right) = -q\sum_{\tau\in\mathbb{F}_q^*}\psi(\tau) \sum_{\substack{\mathbf{w_1}\neq\mathbf{w_2}\in\mathcal{E}\\ 4\alpha\delta(\mathbf{w_1},\mathbf{w_2})=\tau^2}} 1. \tag{9}$$

To estimate the second sum on the right hand side, we use Lemma 3. As we have mentioned in Section 2.2, for $\tau\neq 0, 4\alpha$ each graph $V_q(\alpha,\tau^2/(4\alpha))$ is a $(q^2-q, q+1, 2\sqrt{q})$-regular graph. So, by Lemma 3, whenever $\tau\neq 0,4\alpha$, we have

$$\left|\sum_{\substack{\mathbf{w_1}\neq\mathbf{w_2}\in\mathcal{E}\\ 4\alpha\delta(\mathbf{w_1},\mathbf{w_2})=\tau^2}} 1 - \frac{q+1}{q^2-q}E^2\right| = \left|e(\mathcal{E}) - \frac{q+1}{q^2-q}E^2\right| \leq 2\sqrt{q}E. \tag{10}$$

Also, trivially for $\tau = 4\alpha$ the solutions of the equation $\delta(\mathbf{w_1},\mathbf{w_2}) = 4\alpha$ are of the forms $\mathbf{w_1} = (a,b), \mathbf{w_2} = (a,-b)$. Hence for $\tau = 4\alpha$,

$$\sum_{\substack{\mathbf{w_1}\neq\mathbf{w_2}\in\mathcal{E}\\ 4\alpha\delta(\mathbf{w_1},\mathbf{w_2})=\tau^2}} 1 \leq E. \tag{11}$$

Using (10) for $\tau \neq 4\alpha$ and (11) for $\tau = 4\alpha$, we derive from (9) that

$$\sum_{\mathbf{w_1} \neq \mathbf{w_2} \in \mathcal{E}} T_\psi(\mathbf{w_1}, \mathbf{w_2}) = -\frac{q(q+1)E^2}{q^2-q} \sum_{\substack{\tau \in \mathbb{F}_q^* \\ \tau \neq 4\alpha}} \psi(\tau) + R = \frac{2(q+1)E^2}{q-1} + R$$

where

$$|R| \leq 2q(q-2)\sqrt{q}E + qE.$$

Therefore

$$\begin{aligned} \left| \sum_{\mathbf{w_1} \neq \mathbf{w_2} \in \mathcal{E}} T_\psi(\mathbf{w_1}, \mathbf{w_2}) \right| &\leq \frac{2(q+1)E^2}{q-1} + 2q(q-2)\sqrt{q}E + qE \\ &\leq 2q(q+1)E + 2q(q-2)\sqrt{q}E + qE \end{aligned}$$

since trivially $E \leq q(q-1)$. Inserting this bound in (7), and we obtain

$$\begin{aligned} W &\leq q(q-1)E + 2q(q+1)E + 2q^{3/2}(q-2)E + qE \\ &= 2qE\left(q^{3/2} + \frac{3q}{2} - 2\sqrt{q} + 1\right) < 3q^{5/2}E, \end{aligned}$$

since

$$\frac{3z}{2} - 2z^{1/2} + 1 \leq \frac{1}{2}z^{3/2}$$

for $z \geq 1$. Now, recalling (6) we finally derive

$$S_\psi(\mathcal{E}, \mathcal{F}) \leq q^{5/4}\sqrt{3E \cdot F},$$

which completes the proof of Theorem 1.

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