STATISTICS OF THE ZEROS OF ZETA FUNCTIONS IN A FAMILY OF CURVES OVER A FINITE FIELD

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ABSTRACT. Let $\mathbb{F}_q$ be a finite field of cardinality q and $l \geq 2$ be a prime number such that $q \equiv 1 \pmod{l}$. Extending the work of Faifman and Rudnick [6] on hyperelliptic curves, we study the distribution of zeros of zeta functions of curves over $\mathbb{F}_q$ varying over the moduli spaces of cyclic l-fold covers of $\mathbb{P}^1(\mathbb{F}_q)$, in the limit of large genus. The zeros all lie on a circle, according to the Riemann Hypothesis for curves, and their angles are uniformly distributed. Moreover, the number of angles inside a fixed symmetric interval $\mathbf{I}$ is asymptotically a sum of $\frac{l-1}{2}$ identical independent Gaussian random variables, each of which comes naturally from a Dirichlet character and has mean $4g|\mathbf{I}|/(l-1)$ and variance $\frac{4}{\pi^2}\log(2g|\mathbf{I}|)$. These results continue to hold for shrinking intervals as long as the expected number of angles $2g|\mathbf{I}|$ tends to infinity.

1. INTRODUCTION

Let C be a smooth projective curve of genus $g \geq 1$ over a finite field $\mathbb{F}_q$ of cardinality q. Its zeta function $Z_C(u)$ is a rational function of the form

$$Z_C(u) = \frac{P_C(u)}{(1-u)(1-qu)},$$

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where $P_C(u) \in \mathbb{Z}[u]$ is a polynomial of degree $2g$ with $P_C(0) = 1$, satisfying the functional equation

$$P_C(u) = \left(qu^2\right)^g P_C\left(\frac{1}{qu}\right),$$

and having all its zeros on the circle $|u| = 1/\sqrt{q}$ (this is the Riemann Hypothesis for curves [19]). There is a unitary symplectic matrix $\Theta_C \in \mathrm{USp}(2g)$, defined up to conjugacy, such that

$$P_C(u) = \det\left(I - u\sqrt{q}\Theta_C\right).$$

The eigenvalues of Θ_C are of the form $e(\theta_{C,j}), j = 1, \ldots, 2g$, where $e(\theta) = e^{2\pi i\theta}$. We may assume that $\{\theta_{C,j}\} \subset \left(-\frac{1}{2}, \frac{1}{2}\right]$.

In a recent beautiful paper [6] Faifman and Rudnick studied the statistics of the set of angles $\{\theta_{C,j}\}$ as C is drawn at random from a family of hyperelliptic curves of genus g defined over $\mathbb{F}_q$ (q is odd). More precisely, denote by $\mathcal{H}_{2g+2}$ the family of curves having an affine equation of the form $Y^2 = F(X)$, with $F \in \mathbb{F}_q[X]$ a monic square-free polynomial of degree $2g+2$ and assign the uniform probability measure on $\mathcal{H}_{2g+2}$. Given a subinterval $\mathbf{I} \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$, let

$$N_{\mathbf{I}}(C) = \#\left\{j : \theta_{C,j} \in \mathbf{I}\right\}, \quad C \in \mathcal{H}_{2g+2}.$$

They proved that if $2g|\mathbf{I}| \to \infty$ as $g \to \infty$, then

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathcal{H}_{2g+2}}\left(a < \frac{N_{\mathbf{I}} - 2g|\mathbf{I}|}{\sqrt{\frac{2}{\pi^2}\log(2g|\mathbf{I}|)}} < b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,\mathrm{d}x\,.$$

This result is in analogy to the work of Selberg ([15],[16],[17]), who studied the fluctuations in the number $N(t)$ of zeros of the Riemann zeta function $\zeta(s)$ up to height t. It also complements the well-known work of Katz and Sarnak [9], who proved

that when the genus g is fixed and $q \to \infty$, the conjugacy classes $\{\Theta_C : C \in \mathcal{H}_{2g+2}\}$ become uniformly distributed in $\mathrm{USp}(2g)$, hence the statistics of $N_{\mathbf{I}}$ are the same as those of the corresponding quantity for a random matrix in $\mathrm{USp}(2g)$. Showing consistency with this theory, moreover, in the limit of large matrix size, i.e. $g \to \infty$, the statistics of this and other related quantities, such as the logarithm of the characteristic polynomial of a random matrix, have been found to have Gaussian fluctuations in various ensembles of random matrices ([14],[4],[1],[8],[10],[18], [5],[7],[20]). This implies, in particular, that if $2g|\mathbf{I}| \to \infty$ as $g \to \infty$, then

$$\lim_{g\to\infty}\left(\lim_{q\to\infty}\mathrm{Prob}_{\mathcal{H}_{2g+2}}\left(a < \frac{N_{\mathbf{I}} - 2g|\mathbf{I}|}{\sqrt{\frac{2}{\pi^2}\log(2g|\mathbf{I}|)}} < b\right)\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,\mathrm{d}x\,.$$

However, in the above approach of Katz and Sarnak it is crucial to take $q \to \infty$ first. The result of Faifman and Rudnick reveals what happens if q is fixed and $g \to \infty$ instead.

Instead of averaging over the family of hyperelliptic curves arising from monic square-free polynomials, the proof of Faifman and Rudnick can be easily adapted to the moduli space of hyperelliptic curves of a fixed genus. Let $l \geq 2$ be a fixed prime number such that $q \equiv 1 \pmod{l}$. Extending the work of Faifman and Rudnick, in this paper we study the statistics of the set of angles $\{\theta_{C,j}\}$ when the curve C varies over the moduli spaces of cyclic l-fold covers of $\mathbb{P}^1(\mathbb{F}_q)$, in the limit of large genus. More precisely, denote by $\mathcal{H}_{g,l}$ the moduli space of cyclic l-fold covers of $\mathbb{P}^1(\mathbb{F}_q)$ of genus g. The moduli space $\mathcal{H}_{g,l}$ is not irreducible for $l \geq 3$, so we break it into a disjoint union of irreducible components $\mathcal{H}^{(d_1,\ldots,d_{l-1})}$ indexed by the inertia type of

the branch points, and

$$\mathcal{H}_{g,l} = \bigcup_{\substack{d_1+2d_2+\cdots+(l-1)d_{l-1}\equiv 0 \pmod{l} \\ g=\frac{l-1}{2}(d_1+\cdots+d_{l-1}-2)}} \mathcal{H}^{(d_1,\ldots,d_{l-1})}.$$

For any symmetric subinterval $\mathbf{I} \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ and any curve $C \in \mathcal{H}_{g,l}$, denote

$$N_{\mathbf{I}}(C) = \#\left\{j : \theta_{C,j} \in \mathbf{I}\right\},$$

we will study the distribution of the quantity $N_{\mathbf{I}}(C)$ as C varies over any irreducible component $\mathcal{H}^{(d_1,\ldots,d_{l-1})}$ of the moduli space $\mathcal{H}_{g,l}$ in the limit $g \to \infty$. It turns out there is a more subtle structure which we would like to stress as follows.

Each curve $C \in \mathcal{H}_{g,l}$ is equipped with affine model

$$C : Y^l = F(X)$$

for some l-th power-free polynomial $F(X) \in \mathbb{F}_q[X]$. The Zeta function of the curve C can be decomposed as

$$Z_C(u) = Z_{\mathbb{P}^1(\mathbb{F}_q)}(u) \prod_{j=1}^{l-1} L\left(\left(\frac{F}{\cdot}\right)_l^j, u\right),$$

where $\left(\frac{F}{\cdot}\right)_l$ is the l-th power residue symbol and $L\left(\left(\frac{F}{\cdot}\right)_l^j, u\right)$ is the L-function attached to the character $\left(\frac{F}{\cdot}\right)_l^j$ (see Section 2 for more details). We may choose a different affine model hence a different l-th power free polynomial $F(X)$ for the same curve C, however, a little thought reveals that the L-function $L\left(\left(\frac{F}{\cdot}\right)_l^j, u\right)$ is independent of the choice of $F(X)$. For each j, it is known that $L\left(\left(\frac{F}{\cdot}\right)_l^j, u\right)$ is a polynomial of degree $\widetilde{g} = \frac{2g}{l-1}$ and the Riemann hypothesis holds true, that is, we

can factor it as

$$L\left(\left(\frac{F}{\cdot}\right)_l^j, u\right) = \prod_{i=1}^{\widetilde{g}} \left(1 - u\sqrt{q}e\left(\theta_{C,j,i}\right)\right).$$

The collection of all the angles $\{\theta_{C,j,i} : 1 \le j \le l-1, 1 \le i \le \widetilde{g}\}$ gives the set of angles $\{\theta_{C,j}\}$ for the Zeta function $Z_C(u)$. Denote for each j

$$N_{j,\mathbf{I}}(C) = \#\{1 \le i \le \widetilde{g} : \theta_{C,j,i} \in \mathbf{I}\},$$

we will investigate the more subtle question, the joint distribution of the quantities $N_{j,\mathbf{I}}(C)$ for all j when C runs over any irreducible component $\mathcal{H}^{(d_1,\dots,d_{l-1})}$ of $\mathcal{H}_{g,l}$ as $g \to \infty$. Since $\mathbf{I}$ is symmetric we have

$$N_{j,\mathbf{I}}(C) = N_{l-j,\mathbf{I}}(C),$$

it is enough to consider $N_{j,\mathbf{I}}(C)$ for $1 \le j \le \frac{l-1}{2}$. Assigning the uniform probability measure on each irreducible component $\mathcal{H}^{(d_1,\dots,d_{l-1})}$ so that each curve C in the component is counted with weight $1/|\mathrm{Aut}(C)|$, we prove the following result.

Theorem 1. *Let $\mathbf{I} \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ be a symmetric subinterval and $|\mathbf{I}|$ be the length of $\mathbf{I}$. Assume that $g|\mathbf{I}| \to \infty$ as $g \to \infty$. Then*

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathcal{H}^{(d_1,\dots,d_{l-1})}}\left(a_j < \frac{N_{j,\mathbf{I}} - \widetilde{g}|\mathbf{I}|}{\sqrt{\frac{2}{\pi^2}\log(\widetilde{g}|\mathbf{I}|)}} < b_j, 1 \le j \le \frac{l-1}{2}\right) = \prod_{j=1}^{\frac{l-1}{2}} \frac{1}{\sqrt{2\pi}} \int_{a_j}^{b_j} e^{-\frac{x^2}{2}}\,\mathrm{d}x\,,$$

where

$$\widetilde{g} = \frac{2g}{l-1} = d_1 + d_2 + \cdots + d_{l-1} - 2,$$

and $\mathcal{H}^{(d_1,\dots,d_{l-1})}$ *is any irreducible component of the moduli space* $\mathcal{H}_{g,l}$ *with non-negative integers* $d_1,\dots,d_{l-1}$ *satisfying the conditions* $\sum_{i=1}^{l-1} id_i \equiv 0 \pmod{l}$ *and* $g = \frac{l-1}{2}\left(\sum_{i=1}^{l-1} d_i - 2\right)$.

This result implies that as C varies over any irreducible component of the moduli space $\mathcal{H}_{g,l}$, asymptotically in the limit of $g \to \infty$, the number of angles inside a symmetric interval $\mathbf{I}$ of the zeros of the Zeta function $Z_C(u)$ arising from different characters are independent identical Gaussian distribution with mean $\widetilde{g}|\mathbf{I}|$ and variance $\frac{2}{\pi^2}\log(\widetilde{g}|\mathbf{I}|)$. This holds true as long as $g|\mathbf{I}| \to \infty$ when $g \to \infty$. Secondly, since

$$N_{\mathbf{I}}(C) = \sum_{j=1}^{l-1} N_{j,\mathbf{I}}(C) = 2\sum_{j=1}^{\frac{l-1}{2}} N_{j,\mathbf{I}}(C),$$

$N_{\mathbf{I}}$ is asymptotically Gaussian distribution with mean value $2g|\mathbf{I}|$ and variance $(l-1)\frac{2}{\pi^2}\log(g|\mathbf{I}|)$. Finally, when $l = 2$, there is only one quadratic character, this reduces to the result of Faifman and Rudnick.

This paper is organized as follows. In Section 2 we collect several results which will be used later. In Section 3–5 we actually prove a more general result. Applying it in Section 6 we complete the proof of Theorem 1.

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2. Preliminaries

In this section we collect several results which will be used later. Interested readers can refer to [13] for more details.

2.1. **The Zeta functions of function fields.** $\mathbb{F}_q$ is a finite field of cardinality q, $l \geq 2$ is a prime number such that $q \equiv 1 \pmod{l}$ and $F(X) \in \mathbb{F}_q[X]$ is an l-th power-free polynomial of degree $d \geq 1$. Let $K = \mathbb{F}_q(X)$ be the rational function field over $\mathbb{F}_q$ and let $L = K(Y)$ be a finite extension of K, where Y satisfies the equation

$$C: \quad Y^l = F(X). \tag{1}$$

We list several facts about the extension L/K (see [13, Chapter 9 and 10] for more details and more general situation of Galois extensions of function fields).

First, denote by $\zeta_L(s)$ the zeta function of the function field L given by

$$\zeta_L(s) = \prod_{P \in \mathcal{S}_L} \left(1 - NP^{-s}\right)^{-1},$$

where the product is over $\mathcal{S}_L$, the set of all primes of L. For the rational function field K we have

$$\zeta_K(s) = \left(1 - q^{-s}\right)^{-1} \left(1 - q^{1-s}\right)^{-1}. \tag{2}$$

The curve given by affine model (1) is singular in general, however, there is a complete non-singular curve defined over $\mathbb{F}_q$ which gives the function field L. This is what we mean for the curve C. For such a curve, $Z_C\left(q^{-s}\right) = \zeta_L(s)$, i.e., the zeta function of the curve C coincides with the zeta function of the function field L (see [13, pp. 57, Chap 5] for details).

Next, since $q \equiv 1 \pmod{l}$, $\mathbb{F}_q$ contains a primitive l-th root of unity, hence L/K is a geometric Abelian extension of function fields over $\mathbb{F}_q$ with Galois group $G = \mathrm{Gal}(L/K) \simeq \mathbb{Z}/l\mathbb{Z}$. Let $\chi : G \to \mathbb{C}^*$ be a character, we define the Artin L-function

$L(s,\chi)$ as follows. For a prime P of K which is unramified in L, denote $(P, L/K) \in G$ to be the Frobenius automorphism at P, we define the local factor $L_P(s,\chi)$ as

$$L_P(s,\chi) = \left(1 - \chi\left((P, L/K)\right) NP^{-s}\right)^{-1} .$$

If P is ramified in L, then $\chi\left((P, L/K)\right) = 0$, and the local factor in this case is $L_P(s,\chi) = 1$. The Artin L-function $L(s,\chi)$ is given by

$$L(s,\chi) = \prod_{P \in \mathcal{S}_K} \left(1 - \chi\left((P, L/K)\right) NP^{-s}\right)^{-1},$$

where the product is over $\mathcal{S}_K$, the set of all primes of K, which consists of monic irreducible polynomials $P \in \mathbb{F}_q[X]$ with $NP = q^{\deg P}$ and $P = \infty$ with $N(\infty) = q$. If $\chi = \chi_0$, the trivial character, then $L(s,\chi_0) = \zeta_K(s)$ given by (2). We have the relation

$$\zeta_L(s) = \zeta_K(s) \prod_{\chi_0 \neq \chi \in \widehat{G}} L(s,\chi)\,. \tag{3}$$

Each function $L(s,\chi)$ is a polynomial of degree d_χ of which by the Riemann hypothesis for algebraic curves over finite fields ([19]) all inverse roots have absolute value $\sqrt{q}$. The degree d_χ can also be determined explicitly ([13, pp. 131]).

Lastly, let $\left(\frac{F}{\cdot}\right)_l$ be the l-th power residue symbol constructed from a fixed character $\eta : \mathbb{F}_q^* \to \mathbb{C}^*$ of order l (see exercises 3–6a, p.p. 85–86, [12]), then all the non-trivial characters of G are given by $\left(\frac{F}{\cdot}\right)_l^j, 1 \leq j \leq l-1$. We write $\chi = \left(\frac{F}{\cdot}\right)_l$. The relation (3) can be written explicitly as

$$\zeta_L(s) = \zeta_K(s) \prod_{j=1}^{l-1} L\left(s, \chi^j\right) .$$

Here

$$L\left(s,\chi^{j}\right)=\prod_{P\in\mathcal{S}_{K}}\left(1-\chi\left(P\right)^{j}q^{-s\deg P}\right)^{-1},1\leq j\leq l-1\,, \tag{4}$$

where the set $\mathcal{S}_K$ consists of all monic irreducible polynomials $P\in\mathbb{F}_q[X]$ and $P=\infty$.

For each j, the degree of the polynomials $L(s,\chi^j)$ is given by

$$d_{j}=\deg_{K}F\left(\chi^{j}\right)-2, \tag{5}$$

where in our special case $F(\chi^j)$ is defined to be the minimal effective divisor F such that χ^j is trivial on the ray module F, that is, the group of principal divisors generated by elements $a\in K^*$ such that $\mathrm{ord}_P(a-1)\geq \mathrm{ord}_P F$ for all primes P in the support of F ([13, pp. 253]). Since χ is of order l, l is a prime number, we are contended with the fact that

$$d_1=d_2=\cdots=d_{l-1}.$$

We also remark that $\left(\frac{F}{\cdot}\right)_l:\mathbb{F}_q[X]\to\mathbb{C}$ is a Dirichlet character modulo $F(X)$ satisfying the following property: for $D\in\mathbb{F}_q[X]$ with $D=\prod_P P^{m_P}$, then

$$\left(\frac{F}{D}\right)_{l}:=\prod_{P}\left(\frac{F}{P}\right)_{l}^{m_{P}}.$$

2.2. **Beurling-Selberg functions.** Let $\mathbf{I}=[-\beta/2,\beta/2]$ be an interval, symmetric about the origin, of length $0<\beta<1$, and $K\geq 1$ an integer. Beurling-Selberg polynomials $I_K^{\pm}$ are trigonometric polynomials approximating the indicator function $\mathbf{1_I}$ satisfying (see the exposition in [11, Chapter 1.2]):

- $I_K^{\pm}$ are trigonometric polynomials of degree $\leq K$.

- Monotonicity:

$$I_K^- \le \mathbf{1}_{\mathbf{I}} \le I_K^+$$

- The integral of $I_K^\pm$ is close to the length of the interval:

$$\int_0^1 I_K^\pm(x)\,\mathrm{d}x = \int_0^1 \mathbf{1}_{\mathbf{I}}(x)\,\mathrm{d}x \pm \frac{1}{K+1} \tag{6}$$

- $I_K^\pm(x)$ are even (since the interval $\mathbf{I}$ is symmetric about the origin).

  As a consequence of (6), the non-zero Fourier coefficients of $I^\pm$ satisfy

$$\left|\widehat{I}_K^\pm(k) - \widehat{\mathbf{1}}_{\mathbf{I}}(k)\right| \le \frac{1}{K+1}$$

  and in particular

$$\left|\widehat{I}_K^\pm(k)\right| \le \frac{1}{K+1} + \min\left(\beta, \frac{\pi}{|k|}\right), \quad 0 < |k| \le K.$$

  This implies

$$\left|\widehat{I}_K^\pm(k)k\right| \ll 1, \quad k \in \mathbb{Z}.$$

- If $K\beta > 1$, then ([6, Propsition 4.1])

$$\sum_{n\ge 1} nI_K^\pm(n)^2 = \frac{1}{2\pi^2}\log K\beta + O(1). \tag{7}$$

All the implied constants above are independent of K and β. We consider

$$\sum_P \widehat{I}_K^\pm(\deg P)^2(\deg P)^2|P|^{-1},$$

where the sum is over monic irreducible polynomials $P \in \mathbb{F}_q[X]$. The prime number theorem $\#\{P : \deg P = n\} = q^n/n + O(q^n/2/n)$ gives

$$\sum_P \widehat{I}_K^{\pm}(\deg P)^2(\deg P)^2|P|^{-1} \quad = \quad \sum_{1\le n\le K} \widehat{I}_K^{\pm}(\deg P)^2 \left(n + O\left(nq^{-n/2}\right)\right).$$

Using (7) we obtain (this is equation (7.4) in [6])

$$\sum_P \widehat{I}_K^{\pm}(\deg P)^2(\deg P)^2|P|^{-1} = \frac{1}{2\pi^2}\log K\beta + O(1).$$

2.3. **Arithmetic of polynomials over $\mathbb{F}_q$.** We collect several lemmas which are only used in Section 6. Let $\chi : \mathbb{F}_q[X] \to \mathbb{C}$ be a non-trivial Dirichlet character modulo Q with $\deg Q = m$. All polynomials $F \in \mathbb{F}_q[X]$ appearing in this subsection are monic.

Lemma 1. *As $d \to \infty$,*

$$\left|\sum_{\substack{\gcd(F,G)=1,\\ \deg F=d}} \chi(F)\right| \le q^m\sigma(G),$$

where $\sigma(G)$ is the number of monic divisors of G given by $\sigma(G) = \sum_{D|G} 1$.

Proof. We use the Möbius function μ to treat the condition $\gcd(F, G) = 1$. Hence

$$\sum_{\substack{\gcd(F,G)=1,\\ \deg F=d}} \chi(F) = \sum_{\deg F=d} \chi(F) \sum_{D|F,D|G} \mu(D) = \sum_{D|G} \mu(D) \sum_{\substack{\deg F=d,\\ D|F}} \chi(F).$$

The right hand side can be written as

$$\sum_{\substack{D|G,\\ \deg D\le d}} \mu(D)\chi(D) \sum_{\deg F=d-\deg D} \chi(F).$$

Since

$$\sum_{\deg F=n} \chi(F) = 0, \quad n \geq m,$$

we have

$$\begin{aligned} \left| \sum_{\substack{\gcd(F,G)=1,\\ \deg F=d}} \chi(F) \right| &= \left| \sum_{D|G} \mu(D)\chi(D) \sum_{\deg F=d-\deg D\leq m} \chi(F) \right| \\ &\leq \sum_{D|G} q^m \leq q^m \sigma(G). \end{aligned}$$

For each nonnegative integers $d_1, \ldots, d_r$, denote by $\mathcal{G}_{d_1,\ldots,d_r}$ the set of vectors $(F_1, \ldots, F_r) \in \mathbb{F}_q[X]^r$ such that $F_1, \ldots, F_r$ are monic, square-free, relatively prime and $\deg F_i = d_i$ for $1 \leq i \leq r$.

Lemma 2. *As $d \to \infty$,*

$$\left| \sum_{\substack{\gcd(F,G)=1,\\ F\in\mathcal{G}_d}} \chi(F) \right| \ll q^{m+d/2}\sigma(G).$$

Proof. We use the Möbius function μ to pick out the monic square-free polynomials via the formula

$$\sum_{A^2|F} \mu(A) = \begin{cases} 1 & : \quad F \text{ is square-free} \\ 0 & : \quad \text{otherwise} \end{cases}$$

where we sum over all monic polynomials whose second power divides F. Hence

$$\sum_{\substack{\gcd(F,G)=1,\\ F\in\mathcal{G}_d}} \chi(F) = \sum_{\substack{\gcd(F,G)=1,\\ \deg F=d}} \chi(F) \sum_{A^2|F} \mu(A) = \sum_{\substack{\gcd(F,G)=1,\\ \deg F=d}} \chi(F) \sum_{F=A^2B} \mu(A).$$

The right hand side can be written as

$$\sum_{\substack{\gcd(A,G)=1,\\ \deg A\le d/2}} \mu(A)\chi(A)^2 \sum_{\substack{\gcd(B,G)=1,\\ \deg B=d-2\deg A}} \chi(B).$$

By Lemma 1 we find that

$$\left|\sum_{\substack{\gcd(F,G)=1,\\ F\in\mathcal{G}_d}} \chi(F)\right| \le \sum_{\substack{\gcd(A,G)=1,\\ \deg A\le d/2}} q^m\sigma(G) \le q^{m+d/2}\sigma(G).$$

Lemma 3. *Denote*

$$S(\chi,\mathcal{G}_{d_1,\dots,d_r}) = \sum_{(F_1,\dots,F_r)\in\mathcal{G}_{(d_1,\dots,d_r)}} \chi(F_1F_2^2\cdots F_r^r).$$

Assume that the order of χ is at least $r+1$. Then as $d_1+\cdots+d_r\to\infty$,

$$|S(\chi,\mathcal{G}_{d_1,\dots,d_r})| \ll (d_{\max})^r\, q^{d_1+\cdots+d_r+m-d_{\max}/2},$$

where $d_{\max}=\max\{d_i : 1\le i\le r\}$.

Proof. By symmetry, we may assume that $d_1=d_{\max}$. Hence

$$S(\chi,\mathcal{G}_{d_1,\dots,d_r}) = \sum_{(F_2,\dots,F_r)\in\mathcal{G}_{(d_2,\dots,d_r)}} \chi(F_2^2\cdots F_r^r) \sum_{\substack{\gcd(F_1,F_2\cdots F_r)=1,\\ F_1\in\mathcal{G}_{d_1}}} \chi(F_1)\,.$$

By Lemma 2

$$S(\chi,\mathcal{G}_{d_1,\dots,d_r}) \ll \sum_{(F_2,\dots,F_r)\in\mathcal{G}_{(d_2,\dots,d_r)}} q^{m+d_1/2}\sigma(F_2\cdots F_r) \le q^{m+d_1/2}\prod_{j=2}^{r}\sum_{F\in\mathcal{G}_{d_j}}\sigma(F)\,.$$

Since

$$\sum_{F\in\mathcal{G}_d}\sigma(F)=\sum_{FG\in\mathcal{G}_d}1\le 2\sum_{\deg F\le d/2}\ \sum_{\deg G=d-\deg F}1=2\sum_{\deg F\le d/2}q^{d-\deg F},$$

this implies that

$$\sum_{F\in\mathcal{G}_d}\sigma(F)\le 2\sum_{n\le d/2}q^{d-n}q^n\ll(d+1)q^d\,.$$

Therefore

$$S(\chi,\mathcal{G}_{d_1,\dots,d_r})\ll q^{m+d_1/2}\prod_{j=2}^{r}(d_j+1)q^{d_j}\ll(d_1)^r q^{\sum_{j=1}^r d_j+m-d_1/2}.$$

This completes the proof of Lemma 3.

Lemma 4. *If $d_1+\cdots+d_r\to\infty$, then*

$$\#\mathcal{G}_{d_1,\dots,d_r}\asymp q^{d_1+\cdots+d_r}\,. \tag{8}$$

Proof. If $d_1,\dots,d_r\to\infty$, Lemma 4 is implied by a simplified version of Proposition 3.1 of [3], which actually proved a more precise asymptotic formula

$$\#\mathcal{G}_{d_1,\dots,d_r}\sim Cq^{d_1+\cdots+d_r},$$

where C is a constant depending on r. The proof is based on the function field version of the Wiener-Ikehara Tauberian Theorem [13]. Our condition $d_1+\cdots+d_r\to\infty$ is slightly more general, however, it is straightforward to modify their proof accordingly to obtain a lower bound

$$\#\mathcal{G}_{d_1,\dots,d_r}\gg q^{d_1+\cdots+d_r}.$$

We omit the details here. Since trivially $\#\mathcal{G}_{d_1,\dots,d_r}\le q^{d_1+\cdots+d_r}$, this completes the proof of Lemma 4.

3. Proof of A General Result: preparation

3.1. **A general result.** Denote by $\mathcal{F}_{d,l}$ the set of l-th power free polynomials $F(X) \in \mathbb{F}_q[X]$ of degree $d \geq 2$. Given $F \in \mathcal{F}_{d,l}$, denote by C_F the curve defined by affine model

$$C_F : Y^l = F(X).$$

The genus of such curve C_F is given by the formula $g_{_F} = (l-1)(R-2)/2$, where R is the number of branch points of the cyclic cover $C_F \to \mathbb{P}^1$. In particular if we write

$$F(X) = aF_1(X)F_2^2(X)\cdots F_{l-1}^{l-1}(X),$$

where $a \in \mathbb{F}_q^*$ and $F_1, \ldots, F_{l-1}$ are monic square-free, pairwise coprime polynomials of degrees $d_1, \ldots, d_{l-1}$, respectively, then the genus of C_F is given by

$$g_{_F} = \frac{l-1}{2}\left(d_1 + \cdots + d_{l-1} - 2 + \delta_d\right),$$

here $\delta_d = 0$ if $l|d$ and $\delta_d = 1$ otherwise. We are contended with the fact that

$$g_{_F} \asymp d = \deg F, \quad \text{as } d \to \infty.$$

The zeta function of the curve C_F can be written as

$$Z_{C_F}(u) = Z_{\mathbb{P}^1(\mathbb{F}_q)}(u)\prod_{j=1}^{l-1} L\left(u, \left(\frac{F}{\cdot}\right)_l^j\right),$$

where $\left(\frac{F}{\cdot}\right)_l$ is the l-th power residue symbol and $L\left(u, \left(\frac{F}{\cdot}\right)_l^j\right)$ is the L-function attached to the character $\left(\frac{F}{\cdot}\right)_l^j$ given by

$$L\left(u, \left(\frac{F}{\cdot}\right)_l^j\right) = \prod_{P \in \mathcal{S}_K} \left(1 - \left(\frac{F}{P}\right)_l^j u^{\deg P}\right)^{-1}. \tag{9}$$

Moreover, for each j, from (5) we known that $L\left(u, \left(\frac{F}{\cdot}\right)_l^j\right)$ is a polynomial of degree

$$\widetilde{g}_{_F} = \frac{2g_{_F}}{l-1} \asymp d, \quad \text{as } d \to \infty$$

and the Riemann hypothesis holds true, that is, we can factor it as

$$L\left(u, \left(\frac{F}{\cdot}\right)_l^j\right) = \prod_{i=1}^{\widetilde{g}_{_F}} \left(1 - u\sqrt{q} e\left(\theta_{C_{_F},j,i}\right)\right). \tag{10}$$

For any symmetric subinterval $\mathbf{I} \subset \left(-\frac{1}{2}, \frac{1}{2}\right]$, denote

$$N_{j,\mathbf{I}}(C_{_F}) = \#\left\{1 \le i \le \widetilde{g}_{_F} : \theta_{C_{_F},j,i} \in \mathbf{I}\right\}, \quad 1 \le j \le l-1.$$

We are interested in the distribution of $N_{j,\mathbf{I}}(C_{_F})$'s as F varies over a subset of $\mathcal{F}_{d,l}$. We will prove the following general result.

Theorem 2. *For a sequence of $d \to \infty$, suppose that the subsets $\mathcal{X}_d \subset \mathcal{F}_{d,l}$ satisfy the following two conditions: there exist real positive numbers $\epsilon, A, C_1, C_2, C_3$ such that*

(i) *if $\chi : \mathbb{F}_q[X] \to \mathbb{C}$ is a non-trivial Dirichlet character modulo $f \in F[X]$, then*

$$\frac{1}{\#\mathcal{X}_d} \sum_{F \in \mathcal{X}_d} \chi(F) \le C_2 q^{-d\epsilon + A\deg f},$$

(ii) *if $P \in \mathbb{F}_q[X]$ is a monic irreducible polynomial, then*

$$\frac{1}{\#\mathcal{X}_d} \sum_{\substack{F \in \mathcal{X}_d \\ P|F}} 1 \leq C_3 q^{-\epsilon \deg P}.$$

We assign uniform probability measure on $\mathcal{X}_d$. Assume that $d|\mathbf{I}| \to \infty$ as $d \to \infty$. Then

$$\lim_{d\to\infty} \mathrm{Prob}_{\mathcal{X}_d} \left(a_j < \frac{N_{j,\mathbf{I}}(F) - \widetilde{g}_F|\mathbf{I}|}{\sqrt{\frac{2}{\pi^2}\log(\widetilde{g}_F|\mathbf{I}|)}} < b_j, 1 \leq j \leq \frac{l-1}{2} \right) = \prod_{j=1}^{\frac{l-1}{2}} \frac{1}{\sqrt{2\pi}} \int_{a_j}^{b_j} e^{-\frac{x^2}{2}}\, \mathrm{d}x\,.$$

3.2. **Proof of Theorem 2: Preparation.** For each j, computing $u\frac{\mathrm{d}}{\mathrm{d}u} \log L\left(u, \left(\frac{F}{\cdot}\right)_l^j\right)$ in two different ways by using the expressions of (9) and (10) and equating the coefficients, we derive for each $n \geq 1$ the identity

$$-q^{n/2} \sum_{i=1}^{\widetilde{g}_F} e\left(n\theta_{C_F,j,i}\right) = \sum_{\substack{P \\ \deg P | n}} \deg P \left(\frac{F}{P}\right)_l^{jn/\deg P} + \left(\frac{F}{\infty}\right)_l^{jn},$$

where the sum is over all monic irreducible polynomials $P \in \mathbb{F}_q[X]$ of degree dividing n. Taking complex conjugate we have for each integer $0 \neq n \in \mathbb{Z}$ the equation

$$\sum_{i=1}^{\widetilde{g}_F} e\left(n\theta_{C_F,j,i}\right) = -q^{|n|/2} \left(\sum_{\substack{P \\ \deg P | n}} \deg P \left(\frac{F}{P}\right)_l^{jn/\deg P} + \left(\frac{F}{\infty}\right)_l^{jn} \right). \tag{11}$$

For the symmetric subinterval $\mathbf{I} = [-\beta/2, \beta/2]$, $0 < \beta < 1/2$, let $I_K^{\pm}(x)$ be the trigonometric polynomials of degree K defined in Section 2 such that $I_K^- \leq \mathbf{1}_{\mathbf{I}} \leq I_K^+$. Since $d|\mathbf{I}| = d\beta \to \infty$ as $d \to \infty$, we can choose integers $K = K(d)$ in such a way

that

$$\frac{d}{K} \to \infty,\ K\beta \to \infty,\ \text{ and } \frac{d}{K} \ll (\log K\beta)^{1/4} \ \text{ as } d \to \infty. \tag{12}$$

From the monotonicity of $I_K^{\pm}$, we have

$$N_{j,K}^{-}(F) \leq N_{j,\mathbf{I}}(C_F) \leq N_{j,K}^{+}(F),$$

where

$$N_{j,K}^{\pm}(F) = \sum_{i=1}^{\widetilde{g}_F} I_K^{\pm}\left(\theta_{C_F,j,i}\right) .$$

Let $I_K(x) = \sum_n c(n)e(nx) = I_K^{\pm}(x)$. We collect several properties of the Fourier coefficients $c(n)$ from Section 2 as follows:

(i) $c(n) = 0$ if $|n| > K$.

(ii) $c(0) = \beta + O\left(K^{-1}\right)$.

(iii) $|c(n)n| \ll 1$ for any $n \in \mathbb{Z}$.

(iv) We have

$$\sum_P c(\deg P)^2(\deg P)^2|P|^{-1} = \frac{1}{2\pi^2}\log K\beta + O(1),$$

where the sum runs over monic irreducible polynomials $P \in \mathbb{F}_q[X]$.

Denote $N_{j,K}(F) = N_{j,K}^{\pm}(F)$. Then

$$N_{j,K}(F) \ = \ \sum_{i=1}^{\widetilde{g}_F} I_{j,K}\left(\theta_{C_F,j,i}\right) = \sum_{n\in\mathbb{Z}} c(n) \sum_{i=1}^{\widetilde{g}_F} e\left(n\theta_{C_F,j,i}\right) .$$

From the identity (11) we obtain

$$N_{j,K}(F) \;=\; \widetilde{g}_{_F} c(0) - \sum_{0\neq n\in\mathbb{Z}} c(n) q^{-|n|/2} \left(\sum_{\substack{P \\ \deg P | n}} \deg P \left(\frac{F}{P}\right)_l^{jn/\deg P} + \sum_{j=1}^{l-1} \left(\frac{F}{\infty}\right)_l^{jn} \right).$$

Since $|c(n)n| \ll 1$,

$$\sum_{|n|\leq K} |c(n)| q^{-|n|/2} \ll 1\,.$$

Using the fact that $\left|\left(\frac{F}{\infty}\right)_l\right| \leq 1$, and $c(n) = c(-n) \in \mathbb{R}$, we derive

$$N_{j,K}(F) = \widetilde{g}_{_F}\beta - \sum_{1\leq n\leq K} c(n) q^{-n/2} \sum_{\substack{P \\ \deg P | n}} \deg P \left(\left(\frac{F}{P}\right)_l^{jn/\deg P} + \left(\frac{F}{P}\right)_l^{-jn/\deg P} \right) + O\left(\frac{\widetilde{g}_{_F}}{K}\right).$$

We may rewrite it as

$$N_{j,K}(F) = \widetilde{g}_{_F}\beta + S_{j,K}(F) + O\left(\frac{\widetilde{g}_{_F}}{K}\right),$$

where

$$S_{j,K}(F) = -\sum_f c\left(\deg f\right) |f|^{-1/2} \Lambda(f) \left(\left(\frac{F}{f}\right)_l^{j} + \left(\frac{F}{f}\right)_l^{l-j} \right),$$

here the sum is over monic polynomials $f \in \mathbb{F}_q[X]$, $|f| := q^{\deg f}$, and $\Lambda(f) = \deg P$ if $f = P^k$ is a power of a monic irreducible polynomial $P \in \mathbb{F}_q[X]$, and $\Lambda(f) = 0$ otherwise.

Using property (iii) of $c(n)$ in the form $c(\deg f)\Lambda(f) = O(1)$, we find

$$S_{j,K}(F) \ll \sum_{\deg f \leq K} |f|^{-1/2} \ll q^{K/2}$$

and hence

$$|N_{j,K}(F) - \widetilde{g}_{_F}\beta| \ll \frac{\widetilde{g}_{_F}}{K} + q^{K/2}\,.$$

Noting $|\mathbf{I}| = \beta$, $\widetilde{g}_{_F} \asymp d$ and taking $K \asymp \log_q d - \log_q \log d$, we have deduced that the zeros are uniformly distributed:

Proposition 3. *As $d \to \infty$, for each $F \in \mathcal{F}_{d,l}$, every fixed symmetric interval $\mathbf{I} = [-\beta/2, \beta/2]$ contains asymptotically $\widetilde{g}_{_F}|\mathbf{I}|$ angles $\theta_{C_F,j,i}$ for each j. In fact*

$$N_{j,\mathbf{I}}(C_F) = \widetilde{g}_{_F}|\mathbf{I}| + O\left(\frac{\widetilde{g}_{_F}}{\log \widetilde{g}_{_F}}\right).$$

Denote

$$T_{j,K}(F) = -\sum_P c(\deg P) \deg P |P|^{-1/2} \left(\left(\frac{F}{P}\right)_l^j + \left(\frac{F}{P}\right)_l^{l-j}\right), \tag{13}$$

and

$$\triangle_{j,K}(F) = -\sum_P c(2\deg P) \deg P |P|^{-1} \left(\left(\frac{F}{P}\right)_l^{2j} + \left(\frac{F}{P}\right)_l^{l-2j}\right), \tag{14}$$

where the sums are over monic irreducible polynomials $P \in \mathbb{F}_q[X]$. Then

$$S_{j,K}(F) - T_{j,K}(F) - \triangle_{j,K}(F) \;=\; -\sum_{f=P^r, r\geq 3} c(\deg f)|f|^{-1/2}\Lambda(f)\left(\left(\frac{F}{f}\right)_l^j + \left(\frac{F}{f}\right)_l^{l-j}\right).$$

Using $c(\deg f)\Lambda(f) = O(1)$, it is easy to see that this is bounded by

$$\ll \sum_{f=P^r, r\geq 3} q^{-r\deg P/2} \leq \sum_{r\geq 3}\sum_{n\leq K/3} q^{-rn/2} q^n \ll 1.$$

Therefore

$$N_{j,K}(F) - \widetilde{g}_{_F}\beta = T_{j,K}(F) + \triangle_{j,K}(F) + O\left(\frac{\widetilde{g}_{_F}}{K}\right). \tag{15}$$

We denote by $\langle \bullet \rangle$ the mean value of any quantity defined on $\mathcal{X}_d$, that is, let $\chi : \mathcal{F}_{d,l} \to \mathbb{C}$ be a map, then

$$\langle \chi \rangle := \frac{1}{\#\mathcal{X}_d} \sum_{F \in \mathcal{X}_d} \chi(F)\,.$$

The goal is to compute for any fixed nonnegative integers $r_1, r_2, \dots, r_{(l-1)/2}$ the moment $\left\langle \prod_{j=1}^{(l-1)/2} \left(N_{j,K}(\bullet) - \widetilde{g}_\bullet \beta\right)^{r_j} \right\rangle$. For this purpose we need to compute various moments for $\triangle_{j,K}(\bullet)$ and $T_{j,K}(\bullet)$ first.

4. Moments of $\triangle_{j,K}$ and $T_{j,K}$

4.1. **Moments of $\triangle_{j,K}(F)$.** For each j and each fixed positive integer r, we have from (14)

$$\triangle_{j,K}(F)^{2r} = \sum_{P_1,\dots,P_{2r}} \prod_{i=1}^{2r} c(2\deg P_i)|P_i|^{-1} \deg P_i \sum_{\lambda_1,\dots,\lambda_{2r} \in \{2j, l-2j\}} \left(\frac{F}{P_1^{\lambda_1}\cdots P_{2r}^{\lambda_{2r}}}\right)_l .$$

Hence

$$\langle \triangle_{j,K}^{2r} \rangle = \sum_{P_1,\dots,P_{2r}} \prod_{i=1}^{2r} c(2\deg P_i)|P_i|^{-1} \deg P_i \sum_{\lambda_1,\dots,\lambda_{2r} \in \{2j, l-2j\}} \left\langle \left(\frac{\bullet}{P_1^{\lambda_1}\cdots P_{2r}^{\lambda_{2r}}}\right)_l \right\rangle .$$

If $P_1^{\lambda_1}\cdots P_{2r}^{\lambda_{2r}}$ is not an l-th power in $\mathbb{F}_q[X]$, then $\left(\frac{\bullet}{P_1^{\lambda_1}\cdots P_{2r}^{\lambda_{2r}}}\right)_l : \mathbb{F}_q[X] \to \mathbb{C}$ is a non-trivial Dirichlet character modulo a polynomial dividing $P_1\cdots P_{2r}$. By the condition (i) of Theorem 2 and using the property $|c(n)n| \ll 1$, we find that the total contribution in this case is

$$\langle \triangle_{j,K}^{2r} \rangle_1 \ll \sum_{P_1,\dots,P_{2r}} \prod_{i=1}^{2r} |P_i|^{-1} \sum_{\lambda_1,\dots,\lambda_{2r} \in \{2j, l-2j\}} q^{-d\epsilon + A(\deg P_1 + \dots + \deg P_{2r})}.$$

Noting the range of K in (12), as $d \to \infty$ we obtain that for the fixed r

$$\begin{aligned}\langle \triangle_{j,K}^{2r} \rangle_1 &\ll 2^{2r} q^{-d\epsilon} \left(\sum_{\substack{P \\ \deg P \le K}} q^{(A-1)\deg P} \right)^{2r} \\ &\ll 2^{2r} q^{-d\epsilon + 2AKr} \ll 1 .\end{aligned}$$

If $P_1^{\lambda_1} \cdots P_{2r}^{\lambda_{2r}} = a^l$ for some $a \in \mathbb{F}_q[X]$, then $\left(\frac{\bullet}{P_1^{\lambda_1} \cdots P_{2r}^{\lambda_{2r}}} \right)_l$ is a trivial Dirichlet character. We assume that $l \ge 3$ since $l = 2$ has been handled in [6]. For this to happen, in the set $\{P_1, P_2, \ldots, P_{2r}\}$, each P_i must be paired off with at least one $P_j, i \ne j$ such that $P_i = P_j$. Hence the total contribution in this case is at most

$$\langle \triangle_{j,K}^{2r} \rangle_2 \ll \left(\sum_P |P|^{-2} \right)^r \ll 1 .$$

We conclude that

$$\langle \triangle_{j,K}^{2r} \rangle = \langle \triangle_{j,K}^{2r} \rangle_1 + \langle \triangle_{j,K}^{2r} \rangle_2 \ll 1 . \tag{16}$$

4.2. **Moments of $T_{j,K}(F)$.** For the sake of clarity, we deal with the moment $\langle (T_{j,K})^r \rangle$ first. This already contains almost all the important features of the proof. Then we will compute in general the moment $\left\langle \prod_{j=1}^{(l-1)/2} (T_{j,K})^{r_j} \right\rangle$.

For each fixed positive integer r, from (13) we have

$$T_{j,K}(F)^r = (-1)^r \sum_{P_1,\ldots,P_r} \prod_{i=1}^{r} c(\deg P_i) |P_i|^{-1/2} (\deg P_i) \sum_{\lambda_1,\ldots,\lambda_r \in \{j, l-j\}} \left(\frac{F}{P_1^{\lambda_1} \cdots P_r^{\lambda_r}} \right)_l .$$

Hence

$$\langle (T_{j,K})^r \rangle = (-1)^r \sum_{P_1,\ldots,P_r} \prod_{i=1}^{r} c(\deg P_i) |P_i|^{-1/2} (\deg P_i) \sum_{\lambda_1,\ldots,\lambda_r \in \{j, l-j\}} \left\langle \left(\frac{\bullet}{P_1^{\lambda_1} \cdots P_r^{\lambda_r}} \right)_l \right\rangle .$$

If $P_1^{\lambda_1}\cdots P_r^{\lambda_r}$ is not an l-th power in $\mathbb{F}_q[X]$, then $\left(\frac{\bullet}{P_1^{\lambda_1}\cdots P_r^{\lambda_r}}\right)_l : \mathbb{F}_q[X]\to\mathbb{C}$ is a non-trivial Dirichlet character modulo a polynomial dividing $P_1\cdots P_r$. By the condition (i) of Theorem 2, and using the property that $|c(n)n|\ll 1$, the total contribution in this case is

$$\langle (T_{j,K})^r\rangle_1 \ll \sum_{P_1,\ldots,P_r}\prod_{i=1}^{r}|P_i|^{-1/2}\sum_{\lambda_1,\ldots,\lambda_r\in\{j,l-j\}} q^{-d\epsilon+A(\deg P_1+\ldots+\deg P_r)}.$$

Noting the range of K in (12), we obtain that

$$\langle (T_{j,K})^r\rangle_1 \ \ll\ 2^r q^{-d\epsilon}\left(\sum_{\substack{P\\ \deg P\le K}} q^{A\deg P}\right)^r,$$

which yields

$$\langle (T_{j,K})^r\rangle_1 \ \ll\ 2^r q^{-d\epsilon+(A+1)Kr}\ll 1. \tag{17}$$

If $P_1^{\lambda_1}\cdots P_r^{\lambda_r}=a^l$ for some $a\in\mathbb{F}_q[X]$, then $\left(\frac{\bullet}{P_1^{\lambda_1}\cdots P_r^{\lambda_r}}\right)_l$ is a trivial Dirichlet character. For this to happen, in the set $\{P_1,P_2,\ldots,P_r\}$, each P_i must be paired off with at least one $P_j, i\neq j$ such that $P_i=P_j$. There are two cases.

4.2.1. *Case one.* If each P_i is paired off with exactly one $P_j, i\neq j$ such that $P_i=P_j$, then $r=2s$ is an even number. The number of choices for such P_i's is $\frac{(2s)!}{s!2^s}$. Moreover, the exponents must satisfy the condition $\lambda_i+\lambda_j=l$, and there are exactly two choices for λ_i and λ_j. Hence the total contribution in this case is

$$\langle (T_{j,K})^r\rangle_0 = 2^s\frac{(2s)!}{s!2^s}\sum_{\substack{P_1,\ldots,P_s\\ \text{distinct}}}\prod_{i=1}^{s}c(\deg P_i)^2|P_i|^{-1}(\deg P_i)^2\left\langle\left(\frac{\bullet}{P_1\cdots P_s}\right)_l^l\right\rangle.$$

For $F \in \mathcal{X}_d$, we may write

$$\left(\frac{F}{P_1 \cdots P_s}\right)_l^l = 1 - \begin{cases} 1, & \exists P_i | F \\ 0, & \text{otherwise} \end{cases},$$

and hence

$$\left\langle \left(\frac{\bullet}{P_1 \cdots P_s}\right)_l^l \right\rangle = 1 - \frac{1}{\#\mathcal{X}_d} \sum_{\substack{F \in \mathcal{X}_d \\ \exists P_i | F}} 1 \,.$$

It is easy to see that

$$\sum_{\substack{F \in \mathcal{X}_d \\ \exists P_i | F}} 1 \leq \sum_{i=1}^{s} \sum_{\substack{F \in \mathcal{X}_d \\ P_i | F}} 1.$$

Hence by the condition (ii) of Theorem 2

$$\left\langle \left(\frac{\bullet}{P_1 \cdots P_s}\right)_l^l \right\rangle = 1 + O\left(\sum_{i=1}^{s} |P_i|^{-\epsilon}\right) .$$

The contribution from the error term $O\left(\sum_{i=1}^{s} |P_i|^{-\epsilon}\right)$ is bounded by

$$E \ll \frac{s 2^s (2s)!}{s! 2^s} \left(\sum_P c(\deg P)^2 (\deg P)^2 |P|^{-1}\right)^{s-1} \left(\sum_P |P|^{-1-\epsilon}\right).$$

Using property (iv) of $c(n)$ and the estimate $\sum_P |P|^{-1-\epsilon} \ll 1$, we find

$$E \ll (\log K\beta)^{s-1} \,.$$

The main term is

$$\frac{(2s)!}{s!} \sum_{\substack{P_1, \ldots, P_s \\ \text{distinct}}} \prod_{i=1}^{s} c(\deg P_i)^2 |P_i|^{-1} (\deg P_i)^2 \,.$$

Now we remove the restriction that $P_1, \ldots, P_s$ are distinct, introducing again an error of $O\left((\log K\beta)^{s-2}\right)$. This gives us

$$\langle (T_{j,K})^r \rangle_0 = \frac{(2s)!}{s!} \left(\sum_P c(\deg P)^2 (\deg P)^2 |P|^{-1} \right)^s + O\left((\log K\beta)^{s-1}\right).$$

Using property (iv) of $c(n)$ again we derive that

$$\langle (T_{j,K})^r \rangle_0 \;=\; \frac{(2s)!}{s!} \left(\frac{1}{2\pi^2} \log K\beta + O(1) \right)^s + O\left((\log K\beta)^{s-1}\right),$$

and from it we obtain

$$\langle (T_{j,K})^r \rangle_0 \;=\; \frac{(2s)!}{2^s \pi^{2s} s!} (\log K\beta)^s + O\left((\log K\beta)^{s-1}\right), \tag{18}$$

where $r = 2s$ is an even number.

4.2.2. *Case two.* This is the case that in the set $\{P_1, P_2, \ldots, P_r\}$, each P_i is paired off with at least one $P_j, i \neq j$ such that $P_i = P_j$, and there is one P_i which is paired off with at least two others $P_{j_1}, P_{j_2}, i \neq j_1 \neq j_2$ such that $P_i = P_{j_1} = P_{j_2}$. To describe this case we denote by σ:

$$\sigma : \{1, 2, \ldots, r\} = \bigcup_{i=1}^{t} I_i$$

a disjoint partition of the set $\{1, 2, \ldots, r\}$ such that

$$\delta_1 = |I_1| \geq 3, \quad \delta_i = |I_i| \geq 2, \quad 2 \leq i \leq t. \tag{19}$$

From

$$\sum_{i=1}^{t} \delta_i = r \geq 3 + 2(t-1),$$

we have

$$t-1 \leq \frac{r-3}{2}\,. \tag{20}$$

The contribution of this case with respect to σ is bounded by

$$E_\sigma \ll \sum_{\substack{P_1,\dots,P_t \\ \text{distinct}}} \prod_{i=1}^{t} \left|c(\deg P_i)^{\delta_i}\right| (\deg P_i)^{\delta_i} |P_i|^{-\delta_i/2}\,.$$

For each $\delta > 0$, denote

$$\alpha(\delta) = \sum_{P} \left|c(\deg P)^{\delta}\right| (\deg P)^{\delta} |P|^{-\delta/2}.$$

It is clear that

$$\alpha(\delta) \ll 1, \text{ if } \delta \geq 3,$$

and from property (iv) of $c(n)$,

$$\alpha(2) \ll \log K\beta\,.$$

Hence we obtain, using (19) and (20) that

$$E_\sigma \ll \alpha(\delta_1) \prod_{i=2}^{t} \alpha(\delta_i) \ll (\log K\beta)^{(r-3)/2}\ .$$

Since the number of such partitions σ depends only on r, which is a fixed positive integer, the total contribution of this case to $\langle (T_{j,K})^r \rangle$ is still bounded by

$$E \ll (\log K\beta)^{(r-3)/2}\ .$$

Combining this estimate with (17) and (18) we conclude that

$$\langle (T_{j,K})^r \rangle \;=\; \frac{\delta(r) r!}{2^{r/2}\pi^r\,(r/2)!}\,(\log K\beta)^{r/2} + O\left((\log K\beta)^{-1+r/2}\right),$$

where $\delta(r) = 1$ is r is even, and $\delta(r) = 0$ if r is odd.

4.3. **General moments of $T_{j,k}$.** Denote $t = \frac{l-1}{2}$. For any nonnegative integers $r_1, \dots, r_t$, let $r = \sum_{j=1}^t r_j$. We have

$$\prod_{j=1}^t T_{j,K}(F)^{r_j} = (-1)^r \sum_{P_{j,i}} \prod_{j,i} c\left(\deg P_{j,i}\right) |P_{j,i}|^{-1/2} \deg P_{j,i} \sum_{\lambda_{j,i}\in\{j,l-j\}} \left(\frac{F}{\prod_{j,i} P_{j,i}^{\lambda_{j,i}}}\right)_l,$$

where the index j, i run over the range $1 \le j \le t$ and $1 \le i \le r_j$. Hence

$$\left\langle \prod_{j=1}^t (T_{j,K})^{r_j} \right\rangle = (-1)^r \sum_{P_{j,i}} \prod_{j,i} c\left(\deg P_{j,i}\right) |P_{j,i}|^{-1/2} \deg P_{j,i} \sum_{\lambda_{j,i}\in\{j,l-j\}} \left\langle \left(\frac{\bullet}{\prod_{j,i} P_{j,i}^{\lambda_{j,i}}}\right)_l \right\rangle.$$

Similarly, if $\prod_{j,i} P_{j,i}^{\lambda_{j,i}}$ is not an l-th power in $\mathbb{F}_q[X]$, then $\left(\frac{\bullet}{\prod_{j,i} P_{j,i}^{\lambda_{j,i}}}\right)_l : \mathbb{F}_q[X] \to \mathbb{C}$ is a nontrivial Dirichlet character modulo a polynomial dividing $\prod_{j,i} P_{j,i}^{\lambda_{j,i}}$, by the condition (i) of Theorem 2, the total contribution in this case is bounded by $O(1)$. If $\prod_{j,i} P_{j,i}^{\lambda_{j,i}} = a^l$ for some $a \in \mathbb{F}_q[X]$, then $\left(\frac{\bullet}{\prod_{j,i} P_{j,i}^{\lambda_{j,i}}}\right)_l$ is a trivial Dirichlet character. For this to happen, each element of the set $\{P_{j,i}\}$ must be paired off with others. There are also two cases. If there is one element that is paired off with at least two other elements, using similar argument the total contribution in this case is bounded by $O\left((\log K\beta)^{(r-3)/2}\right)$. The main contribution comes from the remaining case, that is, in the set $\{P_{j,i}\}$, each $P_{j,i}$ is paired off with exactly one $P_{j',i'}$ such that $(j,i) \ne (j',i')$. Since $l \ge 3$, $\prod_{j,i} P_{j,i}^{\lambda_{j,i}} = a^l$, $\lambda_{j,i} \in \{j, l-j\}$ and $1 \le j \le t$, this happens if and only if for each j, $1 \le j \le t$, $r_j = 2s_j$ is even, and in the set $\{P_{j,i} : 1 \le i \le r_j\}$, each $P_{j,i}$ is paired off with exactly one $P_{j,i'}$ such that $i \ne i'$, and

$\lambda_{j,i} + \lambda_{j,i'} = l$. The number of such choices of $P_{j,i}$ and $\lambda_{j,i}$ for each j is $2^{s_j}\frac{(2s_j)!}{2^{s_j}s_j!}$. Hence the total contribution in this case is

$$\left\langle \prod_{j=1}^{t} (T_{j,K})^{r_j} \right\rangle_0 = \prod_{j=1}^{t} 2^{s_j} \frac{(2s_j)!}{s_j! 2^{s_j}} \sum_{\substack{P_{j,i} \\ \text{all distinct}}} \prod_{j,i} c(\deg P_{j,i})^2 |P_{j,i}|^{-1} (\deg P_{j,i})^2 \left\langle \left(\frac{\bullet}{\prod_{j,i} P_{j,i}} \right)^l \right\rangle_l ,$$

where the index j, i runs over the range $1 \leq j \leq t$ and $1 \leq i \leq s_j$. Since

$$\left\langle \left(\frac{\bullet}{\prod_{j,i} P_{j,i}} \right)^l \right\rangle_l = 1 + O\left(\sum_{j,i} |P_{j,i}|^{-\epsilon} \right) ,$$

the error term arising from $O\left(\sum_{j,i} |P_{j,i}|^{-\epsilon}\right)$ is $E \ll (\log K\beta)^{-1+\sum_j s_j}$, and the main term is

$$\prod_{j=1}^{t} 2^{s_j} \frac{(2s_j)!}{s_j! 2^{s_j}} \sum_{\substack{P_{j,i} \\ \text{all distinct}}} \prod_{j,i} c(\deg P_{j,i})^2 |P_{j,i}|^{-1} (\deg P_{j,i})^2 .$$

We may remove the restriction that all $P_{j,i}$'s are distinct, introducing again an error of $O\left((\log K\beta)^{-2+\sum_j s_j}\right)$. This gives us

$$\left\langle \prod_{j=1}^{t} (T_{j,K})^{r_j} \right\rangle_0 = \prod_{j=1}^{t} \frac{(2s_j)!}{s_j!} \left(\sum_{P} c(\deg P)^2 (\deg P)^2 |P|^{-1} \right)^{s_j} + O\left((\log K\beta)^{-1+\sum_j s_j} \right) .$$

From it we obtain that

$$\left\langle \prod_{j=1}^{t} (T_{j,K})^{r_j} \right\rangle_0 = \prod_{j=1}^{t} \frac{(2s_j)!}{2_j^s \pi^{2s_j} s_j!} (\log K\beta)^{s_j} + O\left((\log K\beta)^{-1+\sum_j s_j} \right) .$$

Replacing $t = (l-1)/2$ and combining the above estimates together we conclude that

$$\left\langle \prod_{j=1}^{\frac{l-1}{2}} (T_{j,K})^{r_j} \right\rangle_0 = \prod_{j=1}^{\frac{l-1}{2}} \frac{\delta(r_j) r_j!}{2^{r_j/2} \pi^{r_j} (r_j/2)!} (\log K\beta)^{r_j/2} + O\left((\log K\beta)^{-1+r/2} \right) , \tag{21}$$

where $\delta(s) = 1$ if s is even and $\delta(s) = 0$ if s is odd, and $r = \sum_{j=1}^{(l-1)/2} r_j$.

5. Proof of Theorem 2

Now we have all the ingredients to prove Theorem 2. Let $t = (l-1)/2$. For any nonnegative integers $r_1, \dots, r_t$, from (15) we have

$$\prod_{j=1}^{t} (N_{j,K}(F) - \widetilde{g}_{_F}\beta)^{r_j} = \prod_{j=1}^{t} (T_{j,K}(F))^{r_j} + E(F),$$

where

$$E(F) \ll \sum_{\substack{\forall j, u_j+v_j+w_j=r_j \\ \sum_j v_j+w_j \geq 1}} \prod_{j=1}^{t} |T_{j,K}(F)|^{u_j} \, |\triangle_{j,K}(F)|^{v_j} \left(\frac{\widetilde{g}_{_F}}{K}\right)^{w_j}.$$

Using the Cauchy-Schwartz inequality we obtain

$$\langle E(\bullet)\rangle \ll \sum_{\substack{\forall j, u_j+v_j+w_j=r_j \\ \sum_j v_j+w_j \geq 1}} \left\langle \prod_{j=1}^{t} (T_{j,K})^{2u_j} \right\rangle^{1/2} \left(\frac{\widetilde{g}_{_F}}{K}\right)^{\sum_j w_j} \left\langle \prod_{j=1}^{t} (\triangle_{j,K})^{2v_j} \right\rangle^{1/2}.$$

Since $\widetilde{g}_{_F}/K \ll (\log K\beta)^{1/4}$, by applying the estimates of $T_{j,K}$ and $\triangle_{j,K}$ in (21) and (16) respectively, we have

$$\langle E(\bullet)\rangle \ll \sum_{\substack{\forall j, u_j+v_j+w_j=r_j \\ \sum_j v_j+w_j \geq 1}} (\log K\beta)^{\sum_j w_j/4} \, (\log K\beta)^{\sum_j u_j/2}.$$

Since $\sum_j u_j + v_j + w_j = \sum_j r_j = r$ and $\sum_j v_j + w_j \geq 1$,

$$\sum_j \frac{u_j}{2} + \frac{w_j}{4} = \frac{r - \sum_j v_j + \frac{w_j}{2}}{2} \leq \frac{r - \frac{1}{2}}{2}.$$

We obtain

$$\langle E(\bullet)\rangle \ll (\log K\beta)^{\left(r-\frac{1}{2}\right)/2}.$$

Therefore

$$\left\langle \prod_{j=1}^{t} \left(\frac{N_{j,K}(\bullet) - \widetilde{g}_{\bullet}\beta}{\sqrt{\log K\beta}} \right)^{r_j} \right\rangle = \left\langle \prod_{j=1}^{t} \left(\frac{T_{j,K}}{\sqrt{\log K\beta}} \right)^{r_j} \right\rangle + O\left((\log K\beta)^{-1/4} \right) .$$

Now applying the estimate of $T_{j,K}$ in (21), we find that

$$\left\langle \prod_{j=1}^{t} \left(\frac{N_{j,K}(\bullet) - \widetilde{g}_{\bullet}\beta}{\sqrt{\log K\beta}} \right)^{r_j} \right\rangle = \prod_{j=1}^{t} \frac{\delta(r_j) r_j!}{2^{r_j/2}\pi^{r_j}\,(r_j/2)!} + O\left((\log K\beta)^{-1/4} \right) , \tag{22}$$

where the implied constant depends on the nonnegative integers $r_1, \ldots, r_t$.

Finally, for each j, since

$$N_{j,K}^{-}(F) \leq N_{j,\mathbf{I}}(C_F) \leq N_{j,K}^{+}(F) ,$$

and $N_{j,K}(F) = N_{j,K}^{\pm}(F)$, we can replace $N_{j,K}(\bullet)$ by $N_{j,\mathbf{I}}(C_{\bullet})$ and the equation (22) still holds true. Letting d, K both tend to infinity in such a way that they satisfy the condition (12) and noting that $\widetilde{g}_F \asymp d$ for $F \in \mathcal{X}_d$, we conclude that all moments of $\left(\frac{N_{1,\mathbf{I}}(C_F) - \widetilde{g}_F\beta}{\sqrt{\frac{2}{\pi^2}\log \widetilde{g}_F\beta}}, \ldots, \frac{N_{(l-1)/2,\mathbf{I}}(C_F) - \widetilde{g}_F\beta}{\sqrt{\frac{2}{\pi^2}\log \widetilde{g}_F\beta}} \right)$ as F varies in $\mathcal{X}_d$ are asymptotic to the corresponding moments of $t = (l-1)/2$ identical independent standard Gaussian distribution, where for each of the random variable the odd moments vanish and the even moments are

$$\frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} x^{2r} e^{-x^2/2} \mathrm{d}\, x = \frac{(2r)!}{2^r r!} .$$

This implies that as $d \to \infty$ and F varies in the set $\mathcal{X}_d$, $\left(\frac{N_{1,\mathbf{I}}(C_F) - \widetilde{g}_F\beta}{\sqrt{\frac{2}{\pi^2}\log \widetilde{g}_F\beta}}, \ldots, \frac{N_{t,\mathbf{I}}(C_F) - \widetilde{g}_F\beta}{\sqrt{\frac{2}{\pi^2}\log \widetilde{g}_F\beta}} \right)$ converges weakly to $(l-1)/2$ identical independent standard Gaussian variables. This completes the proof of Theorem 2. □

6. Proof of Theorem 1

6.1. **The geometric point of view.** To prove Theorem 1, we first need an explicit description of the moduli space $\mathcal{H}_{g,l}$ of cyclic l-fold covers of $\mathbb{P}^1(\mathbb{F}_q)$ of genus g. We use [2] and [3] as our references and summarize the statement as follows. Interested readers can refer to the two papers for more details.

$q \equiv 1 \pmod{l}$. For any $(l-1)$-tuples of nonnegative integers $(d_1, \ldots, d_{l-1})$, denote by $\mathcal{F}_{(d_1,\ldots,d_{l-1})}$ the subset of $\mathbb{F}_q[X]$ consisting of all polynomials of the form $F_1(X)F_2(X)^2\cdots F_{l-1}(X)^{l-1}$ such that $F_1(X), \ldots, F_{l-1}(X) \in \mathbb{F}_q[X]$ are monic, square-free, relatively prime and $\deg F_i(X) = d_i$ for $1 \le i \le l-1$. Define

$$\mathcal{F}^j_{(d_1,\ldots,d_{l-1})} = \mathcal{F}_{(d_1,\ldots,d_{j-1},d_j-1,d_{j+1},\ldots,d_{l-1})} \quad \text{for } 1 \le j \le l-1,$$

$$\mathcal{F}^0_{(d_1,\ldots,d_{l-1})} = \mathcal{F}_{(d_1,\ldots,d_{l-1})},$$

$$\mathcal{F}_{[d_1,\ldots,d_{l-1}]} = \bigcup_{j=0}^{l-1} \mathcal{F}^j_{(d_1,\ldots,d_{l-1})}.$$

For any $\mathcal{F} \subset \mathbb{F}_q[X]$, denote by $\widehat{\mathcal{F}}$ the set of polynomials αF where $\alpha \in \mathbb{F}_q^*$ and $F \in \mathcal{F}$. This defines the sets $\widehat{\mathcal{F}}_{(d_1,\ldots,d_{l-1})}, \widehat{\mathcal{F}}^j_{(d_1,\ldots,d_{l-1})}$ and $\widehat{\mathcal{F}}_{[d_1,\ldots,d_{l-1}]}$ respectively. For any $F \in \widehat{\mathcal{F}}_{[d_1,\ldots,d_{l-1}]}$ with $d_1 + 2d_2 + \cdots + (l-1)d_{l-1} \equiv 0 \pmod{l}$, it is known that the genus of the curve C_F given by affine model $Y^l = F(X)$ is always $g = \frac{l-1}{2}(d_1 + d_2 + \cdots + d_{l-1} - 2)$.

The moduli space $\mathcal{H}_{g,l}$ of cyclic l-fold covers of $\mathbb{P}^1(\mathbb{F}_q)$ of genus g splits into irreducible subspaces indexed by equivalence classes of $(l-1)$-tuples of nonnegative integers $(d_1, \ldots, d_{l-1})$ with the property that $d_1 + 2d_2 + \cdots + (l-1)d_{l-1} \equiv 0 \pmod{l}$ and $g = \frac{l-1}{2}(d_1 + \cdots + d_{l-1} - 2)$, i.e., the moduli space can be written as a disjoint

union over its connected components,

$$\mathcal{H}_{g,l} = \bigcup_{\substack{d_1+2d_2+\cdots+(l-1)d_{l-1}\equiv 0 \pmod{l} \\ g=\frac{l-1}{2}(d_1+\cdots+d_{l-1}-2)}} \mathcal{H}^{(d_1,\ldots,d_{l-1})},$$

where each component $\mathcal{H}^{(d_1,\ldots,d_{l-1})}$ is irreducible. To compute the statistics for each component, we need to count each curve, seen as a projective variety of dimension 1 up to isomorphism, with the same multiplicity. Since each curve $C \in \mathcal{H}^{(d_1,\ldots,d_{l-1})}$ has affine model $Y^l = F(X)$ for some $F(X) \in \widehat{\mathcal{F}}_{[d_1,\ldots,d_{l-1}]}$, it is enough to count its different such affine models $C' : Y^l = F(X)$.

For $g > (l-1)^2$, all curves C' isomorphic to C are obtained from the automorphisms of $\mathbb{P}^1(\mathbb{F}_q)$, namely the $q(q^2-1)$ elements of $\mathrm{PGL}_2(\mathbb{F}_q)$. By running over the elements of $\mathrm{PGL}_2(\mathbb{F}_q)$, we obtain $q(q^2-1)/|\mathrm{Aut}(C)$ different models $C' : Y^l = F(X)$ where $F \in \widehat{\mathcal{F}}_{[d_1,\ldots,d_{l-1}]}$. This shows that

$$\left|\mathcal{H}^{(d_1,\ldots,d_{l-1})}\right|' = \sum_{C\in\mathcal{H}^{(d_1,\ldots,d_{l-1})}} \frac{1}{|\mathrm{Aut}(C)|} = \frac{\#\widehat{\mathcal{F}}_{[d_1,\ldots,d_{l-1}]}}{q(q^2-1)},$$

where the $'$ notation means that curves C on the moduli spaces are counted with the usual weight $1/|\mathrm{Aut}(C)$. In conclusion counting curves $C \in \mathcal{H}^{(d_1,\ldots,d_{l-1})}$ with weight $1/|\mathrm{Aut}(C)$ is the same as counting polynomials $F \in \widehat{\mathcal{F}}_{[d_1,\ldots,d_{l-1}]}$ with weight $1/q(q^2-1)$.

6.2. **Proof of Theorem 1.** For any $(l-1)$-tuples of nonnegative integers $(d_1,\ldots,d_{l-1})$ such that $d = \sum_{i=1}^{l-1} id_i \equiv 0 \pmod{l}$ and $g = \frac{l-1}{2}\left(\sum_{i=1}^{l-1} d_i - 2\right)$, as $g \to \infty$, it is easy to see that

$$d \asymp g \asymp d_{\max} = \max\{d_i : 1 \le i \le l-1\}. \tag{23}$$

By Lemma 4 and Lemma 3, for any non-trivial Dirichlet character $\chi : \mathbb{F}_q[X] \to \mathbb{C}$ modulo Q with $\deg Q = m \geq l$, as $d \to \infty$

$$\frac{1}{\#\mathcal{F}_{(d_1,\dots,d_{l-1})}} \sum_{F \in \mathcal{F}_{(d_1,\dots,d_{l-1})}} \chi(F) \ll (d_{\max})^l q^{m-d_{\max}/2} \ll q^{-\epsilon d+m}$$

for a sufficiently small $\epsilon > 0$, and for any monic irreducible polynomial $P \in \mathbb{F}_q[X]$,

$$\frac{1}{\#\mathcal{F}_{(d_1,\dots,d_{l-1})}} \sum_{\substack{F \in \mathcal{F}_{(d_1,\dots,d_{l-1})} \\ P|F}} 1 \ll \sum_{j=1}^{l-1} \sum_{\substack{(F_1,\dots,F_{l-1}) \in \mathcal{G}_{d_1,\dots,d_{l-1}} \\ P|F_i}} 1 \ll q^{-\deg P}\,.$$

Hence as $d_1 + \cdots + d_{l-1} \to \infty$, the sets $\mathcal{X}_{(d_1,\dots,d_{l-1})} = \mathcal{F}_{(d_1,\dots,d_{l-1})}$ satisfy the conditions (i) and (ii) of Theorem 2. Moreover, since l and q are both finite, as $d \to \infty$, we can also choose $\mathcal{X}_{(d_1,\dots,d_{l-1})}$ to be the sets

$$\widehat{\mathcal{F}}_{[d_1,\dots,d_{l-1}]} = \bigcup_{j=0}^{l-1} \widehat{\mathcal{F}}^j_{(d_1,\dots,d_{l-1})} \subset \bigcup_{j=0}^{l-1} \widehat{\mathcal{F}}_{d-j,l}\,,$$

and they also satisfy the conditions (i) and (ii) of Theorem 2. Since counting curves $C \in \mathcal{H}^{(d_1,\dots,d_{l-1})}$ with weight $1/|\mathrm{Aut}(C)$ is the same as counting polynomials $F \in \widehat{\mathcal{F}}_{[d_1,\dots,d_{l-1}]}$ with weight $1/q(q^2-1)$, applying Theorem 2, we finish the proof of Theorem 1. □

References

[1] T. H. Baker, P. J. Forrester, *Finite N fluctuation formulas for random matrices*, J. Stat. Phys. **88** (1997), 1371–1385.

[2] A. Bucur, C. David, B. Feigon, M. Lalín, *Statistics for traces of cyclic trigonal curves over finite fields*, arxiv.:0907.5434, 2009. To appear in IMRN.

[3] A. Bucur, C. David, B. Feigon, M. Lalín, *Biased statistics for traces of cyclic p-fold covers over finite fields*, preprint, 2009.

[4] O. Costin, J. Lebowitz, *Gaussian Fluctuation in Random Matrices*, Physical Review Letters **75** (1995), 69–72.

[5] P. Diaconis, S. Evans, *Linear functionals of eigenvalues of random matrices*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2615–2633.

[6] D. Faifman, Z. Rudnick, *Statistics of the zeros of zeta functions in families of hyperelliptic curves over a finite field*, arXiv:0803.3534. To appear in Compositio Math.

[7] C. P. Hughes, J. P. Keating, N. O'Connell, *On the Characteristic Polynomial of a Random Unitary Matrix*, Commun. Math. Phys. **220** (2001), 429–451.

[8] K. Johansson, *On random matrices from classical compact groups*, Ann. of Math. **145** (1997), 519–545.

[9] N. M. Katz, P. Sarnak, "Random Matrices, Frobenius Eigenvalues, and Monodromy", Amer. Math. Soc. Colloq. Publ., vol. 45, American Mathematical Socitey, Providence, RI, 1999.

[10] J. P. Keating, N. Snaith, *Random Matrix Theory and* $\zeta(1/2+it)$, Commun. Math. Phys. **214** (2000), 57–89.

[11] H. L. Montgomery, "Ten lectures on the interface between analytic number theory and harmonic analysis". CBMS Regional Conference Series in Mathematics, **84**. American Mathematical Society, Providence, RI, 1994.

[12] C. Moreno, "Algebraic curves over finite fields", Cambridge Tracts in Mathematics **97**, Cambridge University Press, 1991.

[13] M. Rosen, "Number theory in function fields". Graduate Texts in Mathematics, **210**. Springer-Verlag, New York, 2002.

[14] H.D. Politzer, *Random-matrix description of the distribution of mesoscopic conductance*, Phys. Rev. B **40**. no. 17 (1989), 11917–11919.

[15] A. Selberg, *On the remainder in the formula for* $N(T)$*, the number of zeros of* $\zeta(s)$ *in the strip* $0 < t < T$, Avh. Norske Vid. Akad. Oslo. I. 1944, (1944). no. 1, 1–27.

[16] A. Selberg, *Contributions to the theory of the Riemann zeta-function*, Arch. Math. Naturvid. **48** (1946), no. 5, 89–155.

[17] A. Selberg, *Contributions to the theory of Dirichlet's L-functions*, Skr. Norske Vid. Akad. Oslo. I. 1946, (1946), no. 3, 1–62.

[18] A. Soshnikov, *The central limit theorem for local linear statistics in classical compact groups and related combinatorial identities*, Ann. Probab. **28** (2000), no. 3, 1353–1370.

[19] A. Weil, *Sur les Courbes Algébriques et les Variétés qui s'en Déduisent*, Publ. Inst. Math. Univ. Strasbourg **7** (1945), Hermann et Cie., Paris, 1948.

[20] K. Wieand, *Eigenvalue distributions of random unitary matrices*, Probab. Theory Related Fields **123** (2002), no. 2, 202–224.

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