Introduction to Numerical Integration
Outline

1. Introduction to Numerical Integration
2. The Trapezoidal Rule
Outline

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2. The Trapezoidal Rule
3. Simpson’s Rule
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4. Comparing the Trapezoidal Rule with Simpson’s Rule
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Introduction to Numerical Integration

The Trapezoidal Rule

Simpson’s Rule

Comparing the Trapezoidal Rule with Simpson’s Rule

Measuring Precision
Introduction to Numerical Integration

Numerical Quadrature
Numerical Quadrature

- The need often arises for evaluating the definite integral of a function that has no explicit antiderivative or whose antiderivative is not easy to obtain.
Numerical Quadrature

- The need often arises for evaluating the definite integral of a function that has no explicit antiderivative or whose antiderivative is not easy to obtain.

- The basic method involved in approximating \( \int_{a}^{b} f(x) \, dx \) is called **numerical quadrature**. It uses a sum \( \sum_{i=0}^{n} a_i f(x_i) \) to approximate \( \int_{a}^{b} f(x) \, dx \).
Quadrature based on interpolation polynomials

- The methods of quadrature in this section are based on the interpolation polynomials.
Introduction to Numerical Integration

Quadrature based on interpolation polynomials

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- The basic idea is to select a set of distinct nodes \( \{x_0, \ldots, x_n\} \) from the interval \([a, b]\).
Quadrature based on interpolation polynomials

- The methods of quadrature in this section are based on the interpolation polynomials.
- The basic idea is to select a set of distinct nodes \( \{x_0, \ldots, x_n\} \) from the interval \([a, b]\).
- Then integrate the Lagrange interpolating polynomial

\[
P_n(x) = \sum_{i=0}^{n} f(x_i)L_i(x)
\]

and its truncation error term over \([a, b]\) to obtain:
Introduction to Numerical Integration

Quadrature based on interpolation polynomials (Cont’d)

\[
\int_a^b f(x) \, dx = \int_a^b \sum_{i=0}^{n} f(x_i)L_i(x) \, dx + \int_a^b \prod_{i=0}^{n} (x - x_i) \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \, dx
\]

\[
= \sum_{i=0}^{n} a_if(x_i) + \frac{1}{(n+1)!} \int_a^b \prod_{i=0}^{n} (x - x_i) f^{(n+1)}(\xi(x)) \, dx
\]

where \( \xi(x) \) is in \([a, b]\) for each \( x \) and

\[
a_i = \int_a^b L_i(x) \, dx, \quad \text{for each } i = 0, 1, \ldots, n
\]
The quadrature formula is, therefore,

\[ \int_{a}^{b} f(x) \, dx \approx \sum_{i=0}^{n} a_i f(x_i) \]
Introduction to Numerical Integration

Quadrature based on interpolation polynomials (Cont’d)

The quadrature formula is, therefore,

\[
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\]

where

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a_i = \int_{a}^{b} L_i(x) \, dx, \quad \text{for each } i = 0, 1, \ldots, n
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Quadrature based on interpolation polynomials (Cont’d)

The quadrature formula is, therefore,

\[ \int_{a}^{b} f(x) \, dx \approx \sum_{i=0}^{n} a_i f(x_i) \]

where

\[ a_i = \int_{a}^{b} L_i(x) \, dx, \quad \text{for each } i = 0, 1, \ldots, n \]

and with error given by

\[ E(f) = \frac{1}{(n+1)!} \int_{a}^{b} \prod_{i=0}^{n} (x - x_i) f^{(n+1)}(\xi(x)) \, dx \]
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5. Measuring Precision
Numerical Integration: Trapezoidal Rule

Derivation (1/3)

To derive the Trapezoidal rule for approximating $\int_{a}^{b} f(x) \, dx$, let $x_0 = a$, $x_1 = b$, $h = b - a$
To derive the Trapezoidal rule for approximating $\int_{a}^{b} f(x) \, dx$, let $x_0 = a$, $x_1 = b$, $h = b - a$ and use the linear Lagrange polynomial:

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$
Numerical Integration: Trapezoidal Rule

Derivation (1/3)

To derive the Trapezoidal rule for approximating \( \int_{a}^{b} f(x) \, dx \), let \( x_0 = a \), \( x_1 = b \), \( h = b - a \) and use the linear Lagrange polynomial:

\[
P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)
\]

Then

\[
\int_{a}^{b} f(x) \, dx = \int_{x_0}^{x_1} \left[ \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1) \right] \, dx + \frac{1}{2} \int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1) \, dx.
\]
Numerical Integration: Trapezoidal Rule

Derivation (2/3)

The product \((x - x_0)(x - x_1)\) does not change sign on \([x_0, x_1]\), so the Weighted Mean Value Theorem for Integrals can be applied to the error term.
Numerical Integration: Trapezoidal Rule

Derivation (2/3)

The product \((x - x_0)(x - x_1)\) does not change sign on \([x_0, x_1]\), so the Weighted Mean Value Theorem for Integrals can be applied to the error term to give, for some \(\xi\) in \((x_0, x_1)\),

\[
\int_{x_0}^{x_1} f''(\xi(x)) (x - x_0)(x - x_1) \, dx
\]

\[
= f''(\xi) \int_{x_0}^{x_1} (x - x_0)(x - x_1) \, dx
\]
Numerical Integration: Trapezoidal Rule

Derivation (2/3)

The product \((x - x_0)(x - x_1)\) does not change sign on \([x_0, x_1]\), so the Weighted Mean Value Theorem for Integrals can be applied to the error term to give, for some \(\xi\) in \((x_0, x_1)\),

\[
\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1) \, dx
\]

\[
= f''(\xi) \int_{x_0}^{x_1} (x - x_0)(x - x_1) \, dx
\]

\[
= f''(\xi) \left[ \frac{x^3}{3} - \frac{(x_1 + x_0)}{2} x^2 + x_0 x_1 x \right]_{x_0}^{x_1}
\]
The product \((x - x_0)(x - x_1)\) does not change sign on \([x_0, x_1]\), so the Weighted Mean Value Theorem for Integrals can be applied to the error term to give, for some \(\xi\) in \((x_0, x_1)\),

\[
\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1) \, dx
\]

\[
= f''(\xi) \int_{x_0}^{x_1} (x - x_0)(x - x_1) \, dx
\]

\[
= f''(\xi) \left[ \frac{x^3}{3} - \frac{(x_1 + x_0)}{2} x^2 + x_0 x_1 x \right]_{x_0}^{x_1}
\]

\[
= -\frac{h^3}{6} f''(\xi)
\]
Numerical Integration: Trapezoidal Rule

Derivation (3/3)

Consequently, the last equation, namely

\[ \int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1) \, dx = \frac{-h^3}{6} f''(\xi) \]
Numerical Integration: Trapezoidal Rule

Derivation (3/3)

Consequently, the last equation, namely

$$\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1) \, dx = -\frac{h^3}{6} f''(\xi)$$

implies that

$$\int_{a}^{b} f(x) \, dx = \left[ \frac{(x - x_1)^2}{2(x_0 - x_1)} f(x_0) + \frac{(x - x_0)^2}{2(x_1 - x_0)} f(x_1) \right]_{x_0}^{x_1} - \frac{h^3}{12} f''(\xi)$$
Numerical Integration: Trapezoidal Rule

**Derivation (3/3)**

Consequently, the last equation, namely

\[
\int_{x_0}^{x_1} f''(\xi(x))(x - x_0)(x - x_1) \, dx = -\frac{h^3}{6} f''(\xi)
\]

implies that

\[
\int_a^b f(x) \, dx = \left[ \frac{(x - x_1)^2}{2(x_0 - x_1)} f(x_0) + \frac{(x - x_0)^2}{2(x_1 - x_0)} f(x_1) \right]_{x_0}^{x_1} - \frac{h^3}{12} f''(\xi)
\]

\[
= \frac{(x_1 - x_0)}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)
\]
Numerical Integration: Trapezoidal Rule

Using the notation $h = x_1 - x_0$ gives the following rule:

**The Trapezoidal Rule**

$$\int_a^b f(x) \, dx = \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)$$
Numerical Integration: Trapezoidal Rule

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$$\int_a^b f(x) \, dx = \frac{h}{2}[f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)$$

Note:

- The error term for the Trapezoidal rule involves $f''$, so the rule gives the exact result when applied to any function whose second derivative is identically zero, that is, any polynomial of degree one or less.
Numerical Integration: Trapezoidal Rule

Using the notation $h = x_1 - x_0$ gives the following rule:

**The Trapezoidal Rule**

\[
\int_{a}^{b} f(x) \, dx = \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)
\]

**Note:**

- The error term for the Trapezoidal rule involves $f''$, so the rule gives the exact result when applied to any function whose second derivative is identically zero, that is, any polynomial of degree one or less.
- The method is called the Trapezoidal rule because, when $f$ is a function with positive values, $\int_{a}^{b} f(x) \, dx$ is approximated by the area in a trapezoid, as shown in the following diagram.
Trapezoidal Rule: The Area in a Trapezoid

\[ y = f(x) \]

\[ y = P_1(x) \]

\[ a = x_0 \]

\[ x_1 = b \]
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Simpson’s rule results from integrating over \([a, b]\) the second Lagrange polynomial with equally-spaced nodes \(x_0 = a, x_2 = b,\) and \(x_1 = a + h,\) where \(h = (b - a)/2:\)
Numerical Integration: Simpson’s Rule

Naive Derivation
Numerical Integration: Simpson’s Rule

Naive Derivation

Therefore

\[
\int_{a}^{b} f(x) \, dx = \int_{x_0}^{x_2} \left[ \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \right. \\
+ \left. \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] \, dx \\
+ \int_{x_0}^{x_2} \frac{(x - x_0)(x - x_1)(x - x_2)}{6} f^{(3)}(\xi(x)) \, dx.
\]
Numerical Integration: Simpson’s Rule

Naive Derivation

Therefore

\[
\int_a^b f(x) \, dx = \int_{x_0}^{x_2} \left[ \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \\
+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] \, dx \\
+ \int_{x_0}^{x_2} \frac{(x - x_0)(x - x_1)(x - x_2)}{6} f^{(3)}(\xi(x)) \, dx.
\]

Deriving Simpson’s rule in this manner, however, provides only an \(O(h^4)\) error term involving \(f^{(3)}\).
Numerical Integration: Simpson’s Rule

**Naive Derivation**

Therefore

\[
\int_a^b f(x) \, dx = \int_{x_0}^{x_2} \left[ \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \right. \\
\left. + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] \, dx \\
+ \int_{x_0}^{x_2} \frac{(x - x_0)(x - x_1)(x - x_2)}{6} f^{(3)}(\xi(x)) \, dx.
\]

Deriving Simpson’s rule in this manner, however, provides only an \(O(h^4)\) error term involving \(f^{(3)}\). By approaching the problem in another way, a higher-order term involving \(f^{(4)}\) can be derived.
Numerical Integration: Simpson’s Rule

Alternative Derivation (1/5)
Numerical Integration: Simpson’s Rule

Alternative Derivation (1/5)

Suppose that $f$ is expanded in the third Taylor polynomial about $x_1$. 
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$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2}(x - x_1)^2 + \frac{f'''(x_1)}{6}(x - x_1)^3 + \frac{f^{(4)}(\xi(x))}{24}(x - x_1)^4$$
Suppose that $f$ is expanded in the third Taylor polynomial about $x_1$. Then for each $x$ in $[x_0, x_2]$, a number $\xi(x)$ in $(x_0, x_2)$ exists with

$$
f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2}(x - x_1)^2 + \frac{f'''(x_1)}{6}(x - x_1)^3 + \frac{f^{(4)}(\xi(x))}{24}(x - x_1)^4
$$

and

$$
\int_{x_0}^{x_2} f(x) \, dx = \left[ f(x_1)(x - x_1) + \frac{f'(x_1)}{2}(x - x_1)^2 + \frac{f''(x_1)}{6}(x - x_1)^3 \right]_{x_0}^{x_2} + \frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4 \, dx
$$
Numerical Integration: Simpson’s Rule

Alternative Derivation (2/5)

Because $(x - x_1)^4$ is never negative on $[x_0, x_2]$, the Weighted Mean Value Theorem for Integrals

See Theorem
Numerical Integration: Simpson’s Rule

Alternative Derivation (2/5)

Because \((x - x_1)^4\) is never negative on \([x_0, x_2]\), the Weighted Mean Value Theorem for Integrals implies that

\[
\frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4 \, dx = \frac{f^{(4)}(\xi_1)}{24} \int_{x_0}^{x_2} (x - x_1)^4 \, dx
\]
Because \((x - x_1)^4\) is never negative on \([x_0, x_2]\), the Weighted Mean Value Theorem for Integrals implies that

\[
\frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4 \, dx = \frac{f^{(4)}(\xi_1)}{24} \int_{x_0}^{x_2} (x - x_1)^4 \, dx = \frac{f^{(4)}(\xi_1)}{120} (x - x_1)^5 \bigg|_{x_0}^{x_2}
\]

for some number \(\xi_1\) in \((x_0, x_2)\).
Numerical Integration: Simpson’s Rule

\[ \frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4 \, dx = \frac{f^{(4)}(\xi_1)}{120} (x - x_1)^5 \bigg|_{x_0}^{x_2} \]

Alternative Derivation (3/5)
Numerical Integration: Simpson’s Rule

\[ \frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4 \, dx = \frac{f^{(4)}(\xi_1)}{120}(x - x_1)^5 \bigg|_{x_0}^{x_2} \]

Alternative Derivation (3/5)

However, \( h = x_2 - x_1 = x_1 - x_0 \), so

\[(x_2 - x_1)^2 - (x_0 - x_1)^2 = (x_2 - x_1)^4 - (x_0 - x_1)^4 = 0\]
Numerical Integration: Simpson’s Rule

\[
\frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4 \, dx = \frac{f^{(4)}(\xi_1)}{120}(x - x_1)^5 \bigg|_{x_0}^{x_2}
\]

Alternative Derivation (3/5)

However, \( h = x_2 - x_1 = x_1 - x_0 \), so

\[
(x_2 - x_1)^2 - (x_0 - x_1)^2 = (x_2 - x_1)^4 - (x_0 - x_1)^4 = 0
\]

whereas

\[
(x_2 - x_1)^3 - (x_0 - x_1)^3 = 2h^3 \text{ and } (x_2 - x_1)^5 - (x_0 - x_1)^5 = 2h^5
\]
Numerical Integration: Simpson’s Rule

Alternative Derivation (4/5)
Numerical Integration: Simpson’s Rule

Alternative Derivation (4/5)

Consequently,

\[
\int_{x_0}^{x_2} f(x) \, dx = \left[ f(x_1)(x - x_1) + \frac{f'(x_1)}{2}(x - x_1)^2 + \frac{f''(x_1)}{6}(x - x_1)^3 
\right.
\]
\[
\left. + \frac{f'''(x_1)}{24}(x - x_1)^4 \right]_{x_0}^{x_2} + \frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4 \, dx
\]
Consequently,\
\[
\int_{x_0}^{x_2} f(x) \, dx = \left[ f(x_1)(x - x_1) + \frac{f'(x_1)}{2}(x - x_1)^2 + \frac{f''(x_1)}{6}(x - x_1)^3 \\
+ \frac{f'''}(x_1)}{24}(x - x_1)^4 \right]_{x_0}^{x_2} + \frac{1}{24} \int_{x_0}^{x_2} f^{(4)}(\xi(x))(x - x_1)^4 \, dx
\]
can be re-written as
\[
\int_{x_0}^{x_2} f(x) \, dx = 2hf(x_1) + \frac{h^3}{3}f''(x_1) + \frac{f^{(4)}(\xi_1)}{60}h^5
\]
Numerical Integration: Simpson’s Rule

Alternative Derivation (5/5)

If we now replace $f''(x_1)$ by the approximation given by the Second Derivative Midpoint Formula ▶ See Formula
Numerical Integration: Simpson’s Rule

Alternative Derivation (5/5)

If we now replace \( f''(x_1) \) by the approximation given by the Second Derivative Midpoint Formula, we obtain

\[
\int_{x_0}^{x_2} f(x) \, dx = 2hf(x_1) + \frac{h^3}{3} \left\{ \frac{1}{h^2} [f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^2}{12} f^{(4)}(\xi_2) \right\} + \frac{f^{(4)}(\xi_1)}{60} h^5
\]
Numerical Integration: Simpson’s Rule

Alternative Derivation (5/5)

If we now replace $f''(x_1)$ by the approximation given by the Second Derivative Midpoint Formula we obtain

$$\int_{x_0}^{x_2} f(x) \, dx = 2hf(x_1) + \frac{h^3}{3} \left\{ \frac{1}{h^2} [f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^2}{12} f^{(4)}(\xi_2) \right\}$$

$$+ \frac{f^{(4)}(\xi_1)}{60} h^5$$

$$= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{12} \left[ \frac{1}{3} f^{(4)}(\xi_2) - \frac{1}{5} f^{(4)}(\xi_1) \right]$$
Numerical Integration: Simpson’s Rule

Alternative Derivation (5/5)

If we now replace $f''(x_1)$ by the approximation given by the Second Derivative Midpoint Formula, we obtain

$$\int_{x_0}^{x_2} f(x) \, dx = 2hf(x_1) + \frac{h^3}{3} \left\{ \frac{1}{h^2} [f(x_0) - 2f(x_1) + f(x_2)] - \frac{h^2}{12} f^{(4)}(\xi_2) \right\} + \frac{f^{(4)}(\xi_1)}{60} h^5$$

$$= \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{12} \left[ \frac{1}{3} f^{(4)}(\xi_2) - \frac{1}{5} f^{(4)}(\xi_1) \right]$$

It can be shown by alternative methods that the values $\xi_1$ and $\xi_2$ in this expression can be replaced by a common value $\xi$ in $(x_0, x_2)$. This gives Simpson’s rule.
Numerical Integration: Simpson’s Rule

Simpson’s Rule

\[ \int_{x_0}^{x_2} f(x) \, dx = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi) \]

The error term in Simpson’s rule involves the fourth derivative of \( f \), so it gives exact results when applied to any polynomial of degree three or less.
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Example

Compare the Trapezoidal rule and Simpson’s rule approximations to \( \int_{0}^{2} f(x) \, dx \) when \( f(x) \) is

(a) \( x^2 \)  
(b) \( x^4 \)  
(c) \( (x + 1)^{-1} \)

(d) \( \sqrt{1 + x^2} \)  
(e) \( \sin x \)  
(f) \( e^x \)
Trapezoidal Rule .v. Simpson’s Rule

Solution (1/3)

On $[0, 2]$, the Trapezoidal and Simpson’s rule have the forms

Trapezoidal: $\int_0^2 f(x) \, dx \approx f(0) + f(2)$

Simpson’s: $\int_0^2 f(x) \, dx \approx \frac{1}{3}[f(0) + 4f(1) + f(2)]$
Trapezoidal Rule .v. Simpson’s Rule

Solution (1/3)

On $[0, 2]$, the Trapezoidal and Simpson’s rule have the forms

Trapezoidal: $\int_{0}^{2} f(x) \, dx \approx f(0) + f(2)$

Simpson’s: $\int_{0}^{2} f(x) \, dx \approx \frac{1}{3} [f(0) + 4f(1) + f(2)]$

When $f(x) = x^2$ they give

Trapezoidal: $\int_{0}^{2} f(x) \, dx \approx 0^2 + 2^2 = 4$

Simpson’s: $\int_{0}^{2} f(x) \, dx \approx \frac{1}{3} [(0^2) + 4 \cdot 1^2 + 2^2] = \frac{8}{3}$
Solution (2/3)
The approximation from Simpson’s rule is exact because its truncation error involves $f^{(4)}$, which is identically 0 when $f(x) = x^2$. 
The approximation from Simpson’s rule is exact because its truncation error involves $f^{(4)}$, which is identically 0 when $f(x) = x^2$.

The results to three places for the functions are summarized in the following table.
### Solution (3/3): Summary Results

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>(a) $x^2$</th>
<th>(b) $x^4$</th>
<th>(c) $(x + 1)^{-1}$</th>
<th>(d) $\sqrt{1 + x^2}$</th>
<th>(e) $\sin x$</th>
<th>(f) $e^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact value</td>
<td>2.667</td>
<td>6.400</td>
<td>1.099</td>
<td>2.958</td>
<td>1.416</td>
<td>6.389</td>
</tr>
<tr>
<td>Trapezoidal</td>
<td>4.000</td>
<td>16.000</td>
<td>1.333</td>
<td>3.326</td>
<td>0.909</td>
<td>8.389</td>
</tr>
<tr>
<td>Simpson’s</td>
<td>2.667</td>
<td>6.667</td>
<td>1.111</td>
<td>2.964</td>
<td>1.425</td>
<td>6.421</td>
</tr>
</tbody>
</table>

Notice that, in each instance, Simpson’s Rule is significantly superior.
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Rationale
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The degree of accuracy or precision, of a quadrature formula is the largest positive integer $n$ such that the formula is exact for $x^k$, for each $k = 0, 1, \ldots, n$. 
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Definition

The **degree of accuracy** or **precision**, of a quadrature formula is the largest positive integer $n$ such that the formula is exact for $x^k$, for each $k = 0, 1, \ldots, n$.

This implies that the Trapezoidal and Simpson’s rules have degrees of precision one and three, respectively.
Establishing the Degree of Precision
Integration and summation are linear operations; that is,

\[
\int_a^b (\alpha f(x) + \beta g(x)) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx
\]

and

\[
\sum_{i=0}^{n} (\alpha f(x_i) + \beta g(x_i)) = \alpha \sum_{i=0}^{n} f(x_i) + \beta \sum_{i=0}^{n} g(x_i),
\]

for each pair of integrable functions \( f \) and \( g \) and each pair of real constants \( \alpha \) and \( \beta \).
Numerical Integration: Measuring Precision

Establishing the Degree of Precision

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for each pair of integrable functions \( f \) and \( g \) and each pair of real constants \( \alpha \) and \( \beta \). This implies the following:
Degree of Precision

The degree of precision of a quadrature formula is \( n \) if and only if the error is zero for all polynomials of degree \( k = 0, 1, \ldots, n \), but is not zero for some polynomial of degree \( n + 1 \).
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Footnote

The Trapezoidal and Simpson’s rules are examples of a class of methods known as Newton-Cotes formulas.
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The degree of precision of a quadrature formula is $n$ if and only if the error is zero for all polynomials of degree $k = 0, 1, \ldots, n$, but is not zero for some polynomial of degree $n + 1$.

Footnote

- The Trapezoidal and Simpson’s rules are examples of a class of methods known as Newton-Cotes formulas.
- There are two types of Newton-Cotes formulas, open and closed.
Questions?
Reference Material
Suppose \( f \in C[a, b] \), the Riemann integral of \( g \) exists on \([a, b]\), and \( g(x) \) does not change sign on \([a, b]\). Then there exists a number \( c \) in \((a, b)\) with

\[
\int_a^b f(x)g(x) \, dx = f(c) \int_a^b g(x) \, dx.
\]

When \( g(x) \equiv 1 \), this result is the usual Mean Value Theorem for Integrals. It gives the average value of the function \( f \) over the interval \([a, b]\) as

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]
The Mean Value Theorem for Integrals

\[ f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx. \]
Second Derivative Midpoint Formula

\[ f''(x_0) = \frac{1}{h^2} [f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12} f^{(4)}(\xi) \]

for some \( \xi \), where \( x_0 - h < \xi < x_0 + h \).