Solutions of Equations in One Variable

Newton's Method

Numerical Analysis (9th Edition)

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Beamer Presentation Slides
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Outline

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1. Newton's Method: Derivation
2. Example using Newton's Method & Fixed-Point Iteration
3. Convergence using Newton's Method

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Newton's Method

Context

Newton's (or the *Newton-Raphson*) method is one of the most powerful and well-known numerical methods for solving a root-finding problem.

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Various ways of introducing Newton's method

- Graphically, as is often done in calculus.
- As a technique to obtain faster convergence than offered by other types of functional iteration.
- Using Taylor polynomials. We will see there that this particular derivation produces not only the method, but also a bound for the error of the approximation.

Newton's Method

Derivation

Newton's Method

Derivation

- Suppose that $f \in C^2[a, b]$. Let $p_0 \in [a, b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small."

Newton's Method

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- Consider the first Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$.

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)),$$

where $\xi(p)$ lies between p and p_0.

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where $\xi(p)$ lies between p and p_0.

- Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

Newton's Method

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- Newton's method is derived by assuming that since $|p - p_0|$ is small, the term involving $(p - p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0).$$

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- Solving for p gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

Newton's Method

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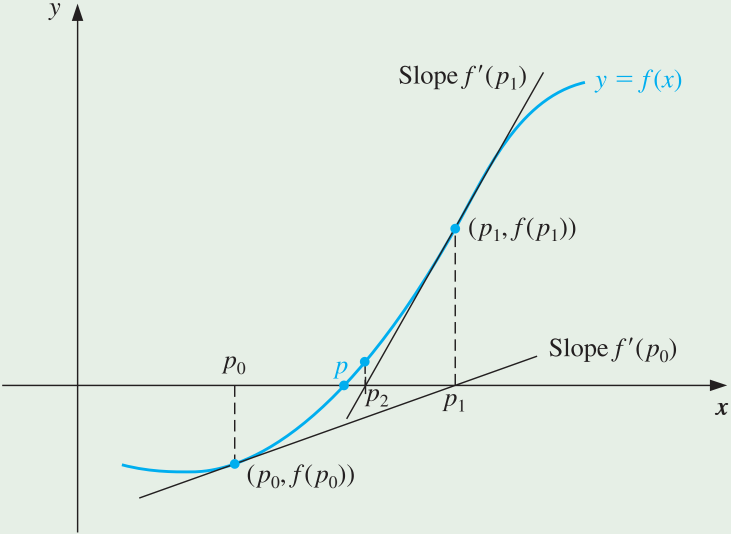
Newton's Method

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1$$

Newton's Method: Using Successive Tangents



Newton's Algorithm

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6. Output p

Newton's Method

Stopping Criteria for the Algorithm

- Various stopping procedures can be applied in Step 3.3.

Newton’s Method

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Newton's Method

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$$|p_N - p_{N-1}| < \epsilon \quad (1)$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon, \quad p_N \neq 0, \quad \text{or} \quad (2)$$

$$|f(p_N)| < \epsilon \quad (3)$$

Newton's Method

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$$\begin{aligned} |p_N - p_{N-1}| &< \epsilon \qquad (1) \\ \frac{|p_N - p_{N-1}|}{|p_N|} &< \epsilon, \quad p_N \neq 0, \quad \text{or} \qquad (2) \\ |f(p_N)| &< \epsilon \qquad (3) \end{aligned}$$

- Note that none of these inequalities give precise information about the actual error $|p_N - p|$.

Newton's Method as a Functional Iteration Technique

Functional Iteration

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Functional Iteration

- Newton's Method

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1$$

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Functional Iteration

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$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1$$

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- with

$$g\,(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \qquad \text{for } n \geq 1$$

Outline

Newton's Method

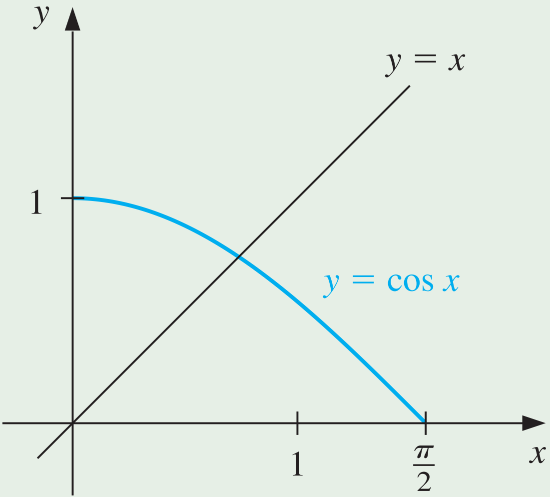
Example: Fixed-Point Iteration & Newton's Method

Consider the function

$$f(x) = \cos x - x = 0$$

Approximate a root of f using (a) a fixed-point method, and (b) Newton's Method

Newton's Method & Fixed-Point Iteration



Newton’s Method & Fixed-Point Iteration

(a) Fixed-Point Iteration for $f(x) = \cos x - x$

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- A solution to this root-finding problem is also a solution to the fixed-point problem

$$x = \cos x$$

and the graph implies that a single fixed-point p lies in $[0, \pi/2]$.

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- The following table shows the results of fixed-point iteration with $p_0 = \pi/4$.

Newton's Method & Fixed-Point Iteration

(a) Fixed-Point Iteration for $f(x) = \cos x - x$

- A solution to this root-finding problem is also a solution to the fixed-point problem

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 and the graph implies that a single fixed-point p lies in $[0, \pi/2]$.
- The following table shows the results of fixed-point iteration with $p_0 = \pi/4$.
- The best conclusion from these results is that $p \approx 0.74$.

Newton's Method & Fixed-Point Iteration

Fixed-Point Iteration: $x = \cos(x)$, $x_0 = \frac{\pi}{4}$

n	p_{n-1}	p_n	$\lvert p_n - p_{n-1} \rvert$	e_n/e_{n-1}
1	0.7853982	0.7071068	0.0782914	—
2	0.707107	0.760245	0.053138	0.678719
3	0.760245	0.724667	0.035577	0.669525
4	0.724667	0.748720	0.024052	0.676064
5	0.748720	0.732561	0.016159	0.671826
6	0.732561	0.743464	0.010903	0.674753
7	0.743464	0.736128	0.007336	0.672816

Newton's Method

(b) Newton's Method for $f(x) = \cos x - x$

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- This gives the approximations shown in the following table.

Newton's Method

Newton's Method for $f(x) = \cos(x) - x$, $x_0 = \frac{\pi}{4}$

n	p_{n-1}	$f(p_{n-1})$	$f'(p_{n-1})$	p_n	$\lvert p_n - p_{n-1}\rvert$
1	0.78539816	-0.078291	-1.707107	0.73953613	0.04586203
2	0.73953613	-0.000755	-1.673945	0.73908518	0.00045096
3	0.73908518	-0.000000	-1.673612	0.73908513	0.00000004
4	0.73908513	-0.000000	-1.673612	0.73908513	0.00000000

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- Because of the agreement of p_3 and p_4 we could reasonably expect this result to be accurate to the places listed.

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Convergence using Newton's Method

Theoretical importance of the choice of p_0

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Convergence using Newton's Method

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- The crucial assumption is that the term involving $(p - p_0)^2$ is, by comparison with $|p - p_0|$, so small that it can be deleted.
- This will clearly be false unless p_0 is a good approximation to p.
- If p_0 is not sufficiently close to the actual root, there is little reason to suspect that Newton's method will converge to the root.
- However, in some instances, even poor initial approximations will produce convergence.

Convergence using Newton's Method

Convergence Theorem for Newton's Method

Convergence using Newton's Method

Convergence Theorem for Newton's Method

- Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$.

Convergence using Newton's Method

Convergence Theorem for Newton's Method

- Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$.
- Then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$, defined by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})}$$

converging to p for any initial approximation

$$p_0 \in [p - \delta, p + \delta]$$

Convergence using Newton's Method

Convergence Theorem (1/4)

Convergence using Newton's Method

Convergence Theorem (1/4)

- The proof is based on analyzing Newton's method as the functional iteration scheme $p_n = g(p_{n-1})$, for $n \geq 1$, with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

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- Let k be in $(0, 1)$. We first find an interval $[p - \delta, p + \delta]$ that g maps into itself and for which $|g'(x)| \leq k$, for all $x \in (p - \delta, p + \delta)$.

Convergence using Newton's Method

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$$g(x) = x - \frac{f(x)}{f'(x)}.$$
- Let k be in $(0, 1)$. We first find an interval $[p - \delta, p + \delta]$ that g maps into itself and for which $|g'(x)| \leq k$, for all $x \in (p - \delta, p + \delta)$.
- Since f' is continuous and $f'(p) \neq 0$, part (a) of Exercise 29 in Section 1.1 Ex 29 implies that there exists a $\delta_1 > 0$, such that $f'(x) \neq 0$ for $x \in [p - \delta_1, p + \delta_1] \subseteq [a, b]$.

Convergence using Newton's Method

Convergence Theorem (2/4)

Convergence using Newton's Method

Convergence Theorem (2/4)

- Thus g is defined and continuous on $[p - \delta_1, p + \delta_1]$. Also

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

for $x \in [p - \delta_1, p + \delta_1]$, and, since $f \in C^2[a, b]$, we have $g \in C^1[p - \delta_1, p + \delta_1]$.

Convergence using Newton's Method

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$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

  for $x \in [p - \delta_1, p + \delta_1]$, and, since $f \in C^2[a, b]$, we have $g \in C^1[p - \delta_1, p + \delta_1]$.

- By assumption, $f(p) = 0$, so

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0.$$

Convergence using Newton's Method

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Convergence using Newton's Method

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0.$$

Convergence Theorem (3/4)

- Since g' is continuous and $0 < k < 1$, part (b) of Exercise 29 in Section 1.1 ▸ Ex 29 implies that there exists a δ, with $0 < \delta < \delta_1$, and

$$|g'(x)| \leq k, \quad \text{for all } \; x \in [p-\delta, p+\delta].$$

Convergence using Newton's Method

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0.$$

Convergence Theorem (3/4)

- Since g' is continuous and $0 < k < 1$, part (b) of Exercise 29 in Section 1.1 ▸ Ex 29 implies that there exists a δ, with $0 < \delta < \delta_1$, and
$$|g'(x)| \leq k, \quad \text{for all } \; x \in [p - \delta, p + \delta].$$
- It remains to show that g maps $[p - \delta, p + \delta]$ into $[p - \delta, p + \delta]$.

Convergence using Newton's Method

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$$|g'(x)| \leq k, \quad \text{for all } x \in [p - \delta, p + \delta].$$

- It remains to show that g maps $[p - \delta, p + \delta]$ into $[p - \delta, p + \delta]$.
- If $x \in [p - \delta, p + \delta]$, the Mean Value Theorem ▸ MVT implies that for some number ξ between x and p, $|g(x) - g(p)| = |g'(\xi)||x - p|$. So

$$|g(x) - p| = |g(x) - g(p)| = |g'(\xi)||x - p| \leq k|x - p| < |x - p|.$$

Convergence using Newton's Method

Convergence Theorem (4/4)

Convergence using Newton’s Method

Convergence Theorem (4/4)

- Since $x \in [p - \delta, p + \delta]$, it follows that $|x - p| < \delta$ and that $|g(x) - p| < \delta$. Hence, g maps $[p - \delta, p + \delta]$ into $[p - \delta, p + \delta]$.

Convergence using Newton's Method

Convergence Theorem (4/4)

- Since $x \in [p - \delta, p + \delta]$, it follows that $|x - p| < \delta$ and that $|g(x) - p| < \delta$. Hence, g maps $[p - \delta, p + \delta]$ into $[p - \delta, p + \delta]$.
- All the hypotheses of the Fixed-Point Theorem ▸ Theorem 2.4 are now satisfied, so the sequence $\{p_n\}_{n=1}^{\infty}$, defined by

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1,$$

converges to p for any $p_0 \in [p - \delta, p + \delta]$.

Outline

Newton's Method in Practice

Choice of Initial Approximation

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Choice of Initial Approximation

- The convergence theorem states that, under reasonable assumptions, Newton's method converges provided a sufficiently accurate initial approximation is chosen.

Newton's Method in Practice

Choice of Initial Approximation

- The convergence theorem states that, under reasonable assumptions, Newton's method converges provided a sufficiently accurate initial approximation is chosen.
- It also implies that the constant k that bounds the derivative of g, and, consequently, indicates the speed of convergence of the method, decreases to 0 as the procedure continues.

Newton's Method in Practice

Choice of Initial Approximation

- The convergence theorem states that, under reasonable assumptions, Newton's method converges provided a sufficiently accurate initial approximation is chosen.
- It also implies that the constant k that bounds the derivative of g, and, consequently, indicates the speed of convergence of the method, decreases to 0 as the procedure continues.
- This result is important for the theory of Newton's method, but it is seldom applied in practice because it does not tell us how to determine δ.

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- and successive approximations are generated by Newton's method.
- These will generally either converge quickly to the root,
- or it will be clear that convergence is unlikely.

Questions?

Reference Material

Exercise 29, Section 1.1

Let $f \in C[a, b]$, and let p be in the open interval (a, b).

Exercise 29 (a)

Suppose $f(p) \neq 0$. Show that a $\delta > 0$ exists with $f(x) \neq 0$, for all x in $[p - \delta, p + \delta]$, with $[p - \delta, p + \delta]$ a subset of $[a, b]$.

Return to Newton's Convergence Theorem (1 of 4)

Exercise 29 (b)

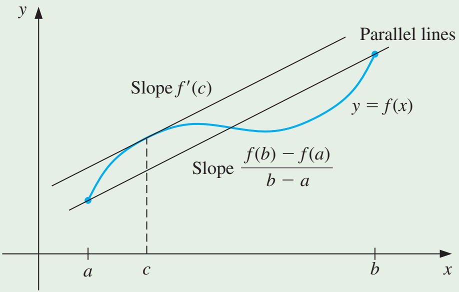
Suppose $f(p) = 0$ and $k > 0$ is given. Show that a $\delta > 0$ exists with $|f(x)| \leq k$, for all x in $[p - \delta, p + \delta]$, with $[p - \delta, p + \delta]$ a subset of $[a, b]$.

Return to Newton's Convergence Theorem (3 of 4)

Mean Value Theorem

If $f \in C[a, b]$ and f is differentiable on (a, b), then a number c exists such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Return to Newton's Convergence Theorem (3 of 4)

Fixed-Point Theorem

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$. Suppose, in addition, that g' exists on (a,b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a,b).$$

Then for any number p_0 in $[a,b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a,b]$. □

◂ Return to Newton's Convergence Theorem (4 of 4)