7.3 GREEN'S FUNCTIONS

We now use Green's identities to study the Dirichlet problem. The representation formula (7.2.1) used exactly two properties of the function $v(\mathbf{x}) = (-4\pi|\mathbf{x} - \mathbf{x}_0|)^{-1}$: that it is harmonic except at $\mathbf{x}_0$ and that it has a certain singularity there. Our goal is to modify this function so that one of the terms in (7.2.1) disappears. The modified function is called the Green's function for D.

Definition. The *Green's function* $G(\mathbf{x})$ for the operator $-\Delta$ and the domain D at the point $\mathbf{x}_0 \in D$ is a function defined for $\mathbf{x} \in D$ such that:

(i) $G(\mathbf{x})$ possesses continuous second derivatives and $\Delta G = 0$ in D, except at the point $\mathbf{x} = \mathbf{x}_0$.
(ii) $G(\mathbf{x}) = 0$ for $x \in \text{bdy } D$.
(iii) The function $G(\mathbf{x}) + 1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ is finite at $\mathbf{x}_0$ and has continuous second derivatives everywhere and is harmonic at $\mathbf{x}_0$.

It can be shown that a Green's function exists. Also, it is unique by Exercise 1. The usual notation for the Green's function is $G(\mathbf{x}, \mathbf{x}_0)$.

Theorem 1. If $G(\mathbf{x}, \mathbf{x}_0)$ is the Green's function, then the solution of the Dirichlet problem is given by the formula

$$u(\mathbf{x}_0) = \iint_{\text{bdy } D} u(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} \, dS. \tag{1}$$

Proof. Let us go back to the representation formula (7.2.1):

$$u(\mathbf{x}_0) = \iint_{\text{bdy } D} \left(u \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v \right) dS, \tag{2}$$

where $v(\mathbf{x}) = -(4\pi|\mathbf{x} - \mathbf{x}_0|)^{-1}$, as before. Now let's write $G(\mathbf{x}, \mathbf{x}_0) = v(\mathbf{x}) + H(\mathbf{x})$. [This is the definition of $H(\mathbf{x})$.] Then $H(\mathbf{x})$ is a harmonic function throughout the domain D [by (iii) and (i)]. We apply Green's second identity (G2) to the pair of harmonic functions $u(\mathbf{x})$ and $H(\mathbf{x})$:

$$0 = \iint_{\text{bdy } D} \left(u \frac{\partial H}{\partial n} - \frac{\partial u}{\partial n} H \right) dS. \tag{3}$$

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formula (7.2.1):

$v \Big) dS,$ (2)

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Adding (2) and (3), we get

$$u(\mathbf{x}_0) = \iint_{\text{bdy } D} \left(u \frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G \right) dS.$$

But by (ii), G vanishes on bdy D, so the last term vanishes and we end up with formula (1). □

The only thing wrong with this beautiful formula is that it is not usually easy to find G explicitly. Nevertheless, in the next section we'll see how to use the reflection method to find G in some situations and thereby solve the Dirichlet problem for some special geometries.

SYMMETRY OF THE GREEN'S FUNCTION

For any region D we have a Green's function $G(\mathbf{x}, \mathbf{x}_0)$. It is always symmetric:

$$G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x}) \qquad \text{for } \mathbf{x} \neq \mathbf{x}_0. \tag{4}$$

In order to prove (4), we apply Green's second identity (G2) to the pair of functions $u(\mathbf{x}) = G(\mathbf{x}, \mathbf{a})$ and $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{b})$ and to the domain D_ϵ. By D_ϵ we denote the domain D with two little spheres of radii ϵ cut out around the points $\mathbf{a}$ and $\mathbf{b}$ (see Figure 1). So the boundary of D_ϵ consists of three parts: the original boundary bdy D and the two spheres $|\mathbf{x} - \mathbf{a}| = \epsilon$ and $|\mathbf{x} - \mathbf{b}| = \epsilon$. Thus

$$\iiint_{D_\epsilon} (u \Delta v - v \Delta u)\, d\mathbf{x} = \iint_{\text{bdy } D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS + A_\epsilon + B_\epsilon, \tag{5}$$

where

$$A_\epsilon = \iint_{|\mathbf{x}-\mathbf{a}|=\epsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

and B_ϵ is given by the same formula at $\mathbf{b}$. Because both u and v are harmonic in D_ϵ, the left side of (5) vanishes. Since both u and v vanish on bdy D, the integral over bdy D also vanishes. Therefore,

$$A_\epsilon + B_\epsilon = 0 \quad \text{for each } \epsilon.$$

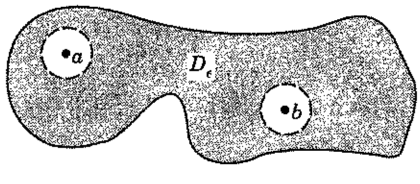

Figure 1