Math 1024 Final

Math 1024 Final, Spring 2011

(1) Compute the sum of convergent infinite series.

1. $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n(n+1)}(\sqrt{n+1}+\sqrt{n})}$.

2. $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n(n+1)}}$.

3. $\sum_{n=2}^{\infty} \log\left(1-\dfrac{1}{n^2}\right)$.

4. $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^2 2^n}{(n+1)3^{n+1}}$.

(2) Study the absolute and conditional convergence. Assume $p, q > 0$.

1. $\displaystyle\int_0^{2\pi} \frac{dx}{(1-\cos x)^p}$.

2. $\displaystyle\int_0^{+\infty} \frac{\sin \pi x}{\log x} dx$.

3. $\displaystyle\int_0^{+\infty} (x^{\frac{1}{x}} - 1) dx$.

4. $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)$.

5. $\sum (-1)^n \dfrac{(n+a)^p}{(n+b)^q}$.

(3) Find the center of mass of the part of the cardioid $r = 1 + \sin\theta$ outside the unit circle $x^2 + y^2 = 1$.

(4) Find the Fourier series of extension of $\sin x$ on $[0, p]$ to a periodic function of period p. Then use the evaluation of the series at 0 to find the sum of the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2 - a^2}$.
(5) Let $f(x)$ be an increasing function on $[a, b]$ satisfying $f(x) \geq x$, $f(a) = a$, $f(b) = b$. Let A be the area between $f(x)$ and the diagonal. Let B be the volume of the solid of revolution of $f(x)$ with respect to the x-axis. Let C be the volume of the solid of revolution of $f(x)$ with respect to the y-axis.

1. Find the volume of the solid of revolution of $f(x)$ with respect to the line $y = c$.
2. Find the volume of the solid of revolution of $f(x)$ with respect to the line $x = c$.
3. Find the volume of the solid of revolution of $f(x)$ with respect to the diagonal.

Answer to Math 1024 Final, Spring 2011

(1.1) By

$$\frac{1}{n\sqrt{n+1} + (n+1)\sqrt{n}} = \frac{1}{\sqrt{n(n+1)}(\sqrt{n+1} + \sqrt{n})} = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n(n+1)}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}},$$

we get the partial sum

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k(k+1)}(\sqrt{k+1} + \sqrt{k})} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{n+1}}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1} + (n+1)\sqrt{n}} = 1.$$

(1.2) By $\lim \dfrac{n}{\sqrt{n(n+1)}} = 1$ and the divergence of $\sum \dfrac{1}{n}$, the series diverges.
(1.3) The partial sum is

$$\sum_{k=2}^{n} \log\left(1 - \frac{1}{k^2}\right) = \log \prod_{k=2}^{n} \frac{k^2 - 1}{k^2} = \log\left(\frac{1 \cdot 3}{2^2} \frac{2 \cdot 4}{3^2} \frac{3 \cdot 5}{4^2} \cdots \frac{(n-1)(n+1)}{n^2}\right) = \log \frac{n+1}{2n}.$$

Therefore

$$\sum_{n=2}^{\infty} \log\left(1 - \frac{1}{n^2}\right) = -\log 2.$$

(1.4) We have

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n, \quad |x| < 1.$$

Taking derivative, we get

$$\frac{1}{(x-1)^2} = \sum_{n=1}^{+\infty} n x^{n-1}, \quad |x| < 1.$$

Multiplying x and taking derivative, we get

$$\left(\frac{x}{(x-1)^2}\right)' = \sum_{n=1}^{+\infty} n^2 x^{n-1}, \quad |x| < 1.$$

Multiplying x and integrating, we get

$$\int_0^x t\left(\frac{t}{(t-1)^2}\right)' dt = \sum_{n=1}^{+\infty} \frac{n^2}{n+1} x^{n+1}, \quad |x| < 1.$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^n \frac{n^2 2^n}{(n+1)3^{n+1}} &= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{n^2}{n+1}\left(-\frac{2}{3}\right)^{n+1} = -\frac{1}{2}\int_0^{-\frac{2}{3}} t\left(\frac{t}{(t-1)^2}\right)' dt \\
&= \frac{1}{2}\int_{-\frac{2}{3}}^{0} td\left(\frac{t}{(t-1)^2}\right) = \frac{1}{2}\left(-\frac{\left(-\frac{2}{3}\right)^2}{\left(-\frac{2}{3}-1\right)^2} - \int_{-\frac{2}{3}}^{0}\frac{t}{(t-1)^2}dt\right) \\
&= \frac{1}{2}\left(-\frac{2^2}{5^2} - \int_{-\frac{5}{3}}^{-1}\frac{t+1}{t^2}dt\right) = \frac{1}{2}\left(-\frac{2^2}{5^2} + \log\frac{5}{3} - 1 + \frac{3}{5}\right) \\
&= \frac{1}{2}\log\frac{5}{3} - \frac{7}{25}.
\end{aligned}$$

(2.1) The integral is improper at 0^+ and $2\pi^-$. Since the integrand is positive, the convergence is the same as absolute convergence.

We have $\lim_{x\to 0^+}\frac{x^{2p}}{(1-\cos x)^p} = 2^p$ and $\lim_{x\to 2\pi^-}\frac{(2\pi - x)^{2p}}{(1-\cos x)^p} = 2^p$. Therefore the convergence is the same as the convergence of $\int_0^1 \frac{dx}{x^{2p}}$ and $\int_{2\pi-1}^{2\pi}\frac{dx}{(2\pi-x)^{2p}}$. This is true if and only if $2p < 1$.

(2.2) The integral is improper at $+\infty$. It is proper at 1 since $\lim_{x\to 1}\frac{\sin \pi x}{\log x} = \pi$. It is also proper at 0 because $\lim_{x\to 0}\frac{\sin\pi x}{\log x} = 0$.

The integral $\int_a^b \sin\pi x dx$ is bounded, and $\frac{1}{\log x}$ is decreasing and converges to 0 as $x \to +\infty$. By the Dirichlet test, the improper integral converges.

If we take the absolute value, then $\left|\frac{\sin \pi x}{\log x}\right| \geq \left|\frac{\sin\pi x}{x}\right|$ for $x > 1$. By the argument of Example 6.1.3, the integral $\int_1^{+\infty}\left|\frac{\sin\pi x}{x}\right| dx$ diverges. Therefore by the comparison test, the integral $\int_0^{+\infty}\left|\frac{\sin\pi x}{\log x}\right| dx$ diverges.

We conclude that $\int_0^{+\infty}\frac{\sin\pi x}{\log x}dx$ converges conditionally.

(2.3) The integral is improper at $+\infty$. Moreover, by $x^{\frac{1}{x}} > 1$ for $x > 1$, the convergence is the same as absolute convergence.

For big x, we have $x^{\frac{1}{x}} - 1 = e^{\frac{\log x}{x}} - 1 = \dfrac{\log x}{x} + o\left(\dfrac{\log x}{x}\right)$. Therefore the convergence of $\displaystyle\int_0^{+\infty} (x^{\frac{1}{x}} - 1)dx$ is the same as the convergence of $\displaystyle\int_0^{+\infty} \frac{\log x}{x} dx$. Since the later one diverges, we see that $\displaystyle\int_0^{+\infty} (x^{\frac{1}{x}} - 1)dx$ diverges.

(2.4). Since $\frac{\log x}{x}$ is decreasing and converges to 0 as $x \to +\infty$, we see that $x^{\frac{1}{x}} - 1 = e^{\frac{\log x}{x}} - 1$ is also decreasing and converges to 0 as $x \to +\infty$. By integral comparison test, $\sum_{n=1}^{\infty}(\sqrt[n]{n} - 1)$ converges if and only if $\displaystyle\int_0^{+\infty} (x^{\frac{1}{x}} - 1)dx$ converges. By (2.3), we see that $\sum_{n=1}^{\infty}(\sqrt[n]{n} - 1)$ diverges.

(2.5) For the series to converge, we must have $\lim_{n\to+\infty} \dfrac{(n+a)^p}{(n+b)^q} = 0$. This means $p < q$. In this case, we also know that $\dfrac{(n+a)^p}{(n+b)^q}$ is decreasing for big n. By the Leibniz test, the series $\sum (-1)^n \dfrac{(n+a)^p}{(n+b)^q}$ converges for $p < q$.

For the absolute series, we have

$$\lim \frac{\frac{(n+a)^p}{(n+b)^q}}{\frac{1}{n^{q-p}}} = 1.$$

By the comparison test, the absolute series converges if and only if $\sum \dfrac{1}{n^{q-p}}$ converges, which means $q - p > 1$.

We conclude that $\sum (-1)^n \dfrac{(n+a)^p}{(n+b)^q}$ absolutely converges when $q - p > 1$ and conditionally converges when $1 \geq q - p > 0$.

(3) The part of the cardioid corresponds to $\theta \in [0, \pi]$ and is symmetric with respect to the y axis. Therefore the center of mass lies on the y-axis (or has x-coordinate zero).

We have

$$ds = \sqrt{r^2 + r'^2} d\theta = \sqrt{2(1 + \sin\theta)} d\theta.$$

The length of the part of the cardioid is

$$\int_0^{\pi} ds = \int_0^{\pi} \sqrt{2(1 + \sin\theta)} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1 + \cos\theta)} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\frac{\theta}{2} d\theta = 4\sqrt{2}.$$

On the other hand,

$$
\begin{aligned}
\int_0^\pi y ds &= \int_0^\pi (1+\sin\theta)\sin\theta\sqrt{2(1+\sin\theta)}d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos\theta)\cos\theta\sqrt{2(1+\cos\theta)}d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\left(\cos\theta+\frac{1+\cos 2\theta}{2}\right)\cos\frac{\theta}{2}d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos\frac{3}{2}\theta+\cos\frac{1}{2}\theta+\cos\frac{1}{2}\theta+\frac{1}{2}\cos\frac{5}{2}\theta+\frac{1}{2}\cos\frac{3}{2}\theta\right)d\theta \\
&= \left(4\sin\frac{1}{2}\theta+\sin\frac{3}{2}\theta+\frac{1}{5}\sin\frac{5}{2}\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{24}{5}\sqrt{2}.
\end{aligned}
$$

Therefore the y-axis of the center of mass is

$$
\bar{y} = \frac{\frac{24}{5}\sqrt{2}}{4\sqrt{2}} = \frac{6}{5}.
$$

(4) If p is not numtiple of 2π, the coefficients are

$$
\begin{aligned}
a_0 &= \frac{1}{p}\int_0^p \sin x dx = \frac{1-\cos p}{p}, \\
a_n &= \frac{2}{p}\int_0^p \sin x\cos\frac{2n\pi}{p}x dx = \frac{1}{p}\int_0^p \left(\sin\left(\frac{2n\pi}{p}+1\right)x-\sin\left(\frac{2n\pi}{p}-1\right)x\right)dx \\
&= \frac{1}{p}\left(\frac{1-\cos\left(\frac{2n\pi}{p}+1\right)p}{\frac{2n\pi}{p}+1}-\frac{1-\cos\left(\frac{2n\pi}{p}-1\right)p}{\frac{2n\pi}{p}-1}\right) \\
&= \frac{1-\cos p}{2n\pi+p}-\frac{1-\cos p}{2n\pi-p} = \frac{-2p(1-\cos p)}{4n^2\pi^2-p^2}, \\
b_n &= \frac{2}{p}\int_0^p \sin x\sin\frac{2n\pi}{p}x dx = \frac{1}{p}\int_0^p \left(\cos\left(\frac{2n\pi}{p}-1\right)x-\cos\left(\frac{2n\pi}{p}+1\right)x\right)dx \\
&= \frac{1}{p}\left(\frac{\sin\left(\frac{2n\pi}{p}-1\right)p}{\frac{2n\pi}{p}-1}-\frac{\sin\left(\frac{2n\pi}{p}+1\right)p}{\frac{2n\pi}{p}+1}\right) \\
&= -\frac{\sin p}{2n\pi-p}-\frac{\sin p}{2n\pi+p} = -\frac{4n\pi\sin p}{4n^2\pi^2-p^2}.
\end{aligned}
$$

The Fourier series is

$$
\sin x \sim \frac{1-\cos p}{p}+\sum_{n=1}^{\infty}\frac{1}{4n^2\pi^2-p^2}\left(-2p(1-\cos p)\cos\frac{2n\pi}{p}x-4n\pi\sin p\sin\frac{2n\pi}{p}x\right), \quad 0<x<p.
$$

If $p=2k\pi$ is a multiple of 2π, then $\sin x = \sin\frac{2k\pi}{p}x$ is the Fourier series of $\sin x$ on $[0,p]$.

The function satisfies the condition of Theorem 8.21 everywhere. Taking the value of $x = 0$, we get $f(0^+) = \sin 0 = 0$, $f(0^-) = \sin p$. Therefore

$$\frac{\sin p}{2} = \frac{1 - \cos p}{p} + \sum_{n=1}^{\infty} \frac{-2p(1 - \cos p)}{4n^2\pi^2 - p^2}.$$

By

$$\frac{\sin p}{1 - \cos p} = \frac{2\sin\frac{p}{2}\cos\frac{p}{2}}{2\sin^2\frac{p}{2}} = \cot\frac{p}{2},$$

we get

$$\sum_{n=1}^{\infty} \frac{2p}{4n^2\pi^2 - p^2} = \frac{1}{p} - \frac{1}{2}\cot\frac{p}{2}.$$

Letting $p = 2\pi a$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2a}\cot\pi a = \frac{1 - \pi a\cot\pi a}{2a^2}.$$

(5) We have

$$A = \int_a^b (f(x) - x)dx, \quad B = \int_a^b \pi f(x)^2 dx, \quad C = \int_a^b \pi x^2 df(x).$$

Thus

$$\int_a^b f(x)dx = A + \frac{b^2 - a^2}{2},$$

and

$$C = \pi b^2 f(b) - \pi a^2 f(a) - \int_a^b 2\pi x f(x)dx = \pi(b^3 - a^3) - \int_a^b 2\pi x f(x)dx.$$

The volume of the solid of revolution of $f(x)$ with respect to the line $y = c$ is

$$\begin{aligned}
\int_a^b \pi(f(x) - c)^2 dx &= \pi\int_a^b f(x)^2 dx - 2\pi c\int_a^b f(x)dx + \int_a^b \pi c^2 dx \\
&= B - 2\pi c\left(A + \frac{b^2 - a^2}{2}\right) + \pi c^2(b - a) \\
&= B - 2\pi c A - \pi c(b^2 - a^2) + \pi c^2(b - a).
\end{aligned}$$

The volume of the solid of revolution of $f(x)$ with respect to the line $x = c$ is

$$\begin{aligned}
\int_a^b \pi(x - c)^2 df(x) &= \int_a^b \pi x^2 df(x) + \int_a^b \pi(-2cx + c^2)df(x) \\
&= C + \pi(-2cx + c^2)f(x)\Big|_{x=a}^{x=b} + \int_a^b 2\pi c f(x)dx \\
&= C + \pi(-2cb + c^2)b - \pi(-2ca + c^2)a + 2\pi c\left(A + \frac{b^2 - a^2}{2}\right) \\
&= C + 2\pi c A - \pi c(b^2 - a^2) + \pi c^2(b - a).
\end{aligned}$$

The volume of the solid of revolution of $f(x)$ with respect to the diagonal is

$$\int_a^b \pi \left(\frac{f(x)-x}{\sqrt{2}}\right)^2 \frac{1+f'(x)}{\sqrt{2}}dx = \frac{\pi}{2\sqrt{2}}\left(\int_a^b (f(x)-x)^2dx + \int_a^b (f(x)-x)^2 df(x)\right)$$

For the first part,

$$\begin{aligned}\pi\int_a^b (f(x)-x)^2dx &= \pi\int_a^b (f(x)^2-2xf(x)+x^2)dx \\ &= B+C-\pi(b^3-a^3)+\frac{1}{3}\pi(b^3-a^3) = B+C-\frac{2}{3}\pi(b^3-a^3).\end{aligned}$$

For the second part,

$$\begin{aligned}\pi\int_a^b (f(x)-x)^2df(x) &= \pi\int_a^b (f(x)^2-2xf(x)+x^2)df(x) \\ &= \pi\int_a^b f(x)^2df(x) - \pi\int_a^b xdf(x)^2 + \pi\int_a^b x^2df(x) \\ &= \frac{1}{3}\pi f(x)^3|_a^b - \pi xf(x)^2|_a^b + \pi\int_a^b f(x)^2dx + \pi\int_a^b x^2df(x) \\ &= \frac{1}{3}\pi(b^3-a^3) - \pi(b^3-a^3) + B + C. = B+C-\frac{2}{3}\pi(b^3-a^3).\end{aligned}$$

Therefore the volume is

$$\int_a^b \pi \left(\frac{f(x)-x}{\sqrt{2}}\right)^2 \frac{1+f'(x)}{\sqrt{2}}dx = \frac{1}{\sqrt{2}}\left(B+C-\frac{2}{3}\pi(b^3-a^3)\right) = \frac{B+C}{\sqrt{2}} - \frac{\sqrt{2}}{3}\pi(b^3-a^3).$$

Math 1024 Final, Spring 2013

(1) Determine convergence.

1. $\displaystyle\int_0^{+\infty} (\pi - 2\arctan x)^p dx.$

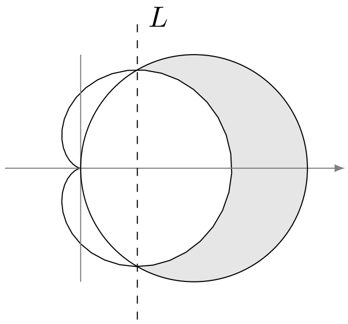
2. $\displaystyle\int_0^{+\infty} \sin^2\left(x+\frac{1}{x}\right) dx.$

(2) Determine convergence ($p, q > 0$). For the convergent ones, specify absolute or conditional.

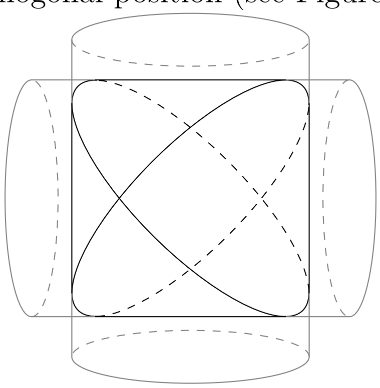
1. $\displaystyle\sum \frac{1}{(\log n)^{p\log n}}.$

2. $\displaystyle\sum \frac{p(p+1)(p+2)\cdots(p+n)}{q(q+1)(q+2)\cdots(q+n)}r^n.$

(3) Consider the region R outside the cardioid $r = 1+\cos\theta$ and inside the circle $r = 3\cos\theta$. Let L be the line connecting the intersection of the two curves. Write the integral that computes the volume of the solid obtained by revolving R around L. You do not need to calculate the integral.



(4) Find the surface area of the intersection of two round solid cylinders of radius 1 in orthogonal position (see Figure 3.33 on page 239).



(5) For $0 < a < \pi$, consider

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x < a, \\ 0, & \text{if } a \le x \le \pi; \end{cases} \qquad g(x) = \begin{cases} 1, & \text{if } 0 \le x < a, \\ 0, & \text{if } a \le x \le 2\pi. \end{cases}$$

The function $g(x)$ appeared in Example 4.5.1.

1. Directly compute the Fourier coefficients of the even extension of period 2π of $f(x)$.
2. What do you get if you evaluate the Fourier series of the first part at $x = \pi$?
3. What do you get by applying Parsival's identity to the first part?
4. Use the first part to derive the Fourier series of $g(x)$.

Answer to Math 1024 Final, Spring 2013

(1.1) Let $y = \frac{\pi}{2} - \arctan x$. Then

$$\int_0^{+\infty} (\pi - 2\arctan x)^p dx = \int_{\frac{\pi}{2}}^0 (2y)^p d\cot y = 2^p \int_0^{\frac{\pi}{2}} \frac{y^p}{\sin^2 y} dy.$$

The original integral is improper at $+\infty$, which translates into the new integral being improper at 0^+. By

$$\lim_{y\to 0^+} \frac{\frac{y^p}{\sin^2 y}}{y^{p-2}} = 1.$$

the convergence is the same as the convergence of $\int_0^{\frac{\pi}{2}} y^{p-2}dx$, which is the same as $p > 1$.

(1.2) The integral is improper at $+\infty$.

We may prove the divergence of $\int_0^{+\infty} \sin^2 xdx$ by Cauchy criterion:

$$\int_{n\pi+\frac{\pi}{4}}^{(n+1)\pi-\frac{\pi}{4}} \sin^2 xdx > \int_{n\pi+\frac{\pi}{4}}^{(n+1)\pi-\frac{\pi}{4}} \left(\frac{1}{\sqrt{2}}\right)^2 dx = \frac{\pi}{4}.$$

We just need to slightly modify the idea. The key is to find a (slight smaller) interval on which the following is satisfied

$$n\pi + \frac{\pi}{4} < x + \frac{1}{x} < (n+1)\pi - \frac{\pi}{4}.$$

For $n\pi + \frac{\pi}{3} < x < (n+1)\pi - \frac{\pi}{3}$, we have

$$n\pi + \frac{\pi}{4} < n\pi + \frac{\pi}{3} + \frac{1}{x} < x + \frac{1}{x} < (n+1)\pi - \frac{\pi}{3} + \frac{1}{x}.$$

So it is sufficient to have $-\frac{\pi}{3} + \frac{1}{x} < -\frac{\pi}{4}$, which is the same as $x > \frac{12}{\pi}$. By $x > n\pi$, it is sufficient to have $n\pi > \frac{12}{\pi}$, or $n \geq 2$.

So for $n \geq 2$, we have $n\pi+\frac{\pi}{3} < x < (n+1)\pi-\frac{\pi}{3}$ implying $n\pi+\frac{\pi}{4} < x+\frac{1}{x} < (n+1)\pi-\frac{\pi}{4}$, so that

$$\int_{n\pi+\frac{\pi}{3}}^{(n+1)\pi-\frac{\pi}{3}} \sin^2\left(x + \frac{1}{x}\right) dx > \int_{n\pi+\frac{\pi}{3}}^{(n+1)\pi-\frac{\pi}{3}} \left(\frac{1}{\sqrt{2}}\right)^2 dx = \frac{\pi}{6}.$$

The Cauchy criterion fails and the integral diverges.

(2.1) We claim that $\frac{1}{(\log n)^{p\log n}} < \frac{1}{n^2}$. The inequality is the same as $n^2 < (\log n)^{p\log n}$. By taking log, the inequality is the same as $2\log n < p(\log n)\log(\log n)$, or the same as $2 < p\log(\log n)$. The last inequality holds for big n.

Therefore $\frac{1}{(\log n)^{p\log n}} < \frac{1}{n^2}$ for big n. By the comparison test and the convergence of $\sum \frac{1}{n^2}$, the series $\sum \frac{1}{(\log n)^{p\log n}}$ converges.

The convergence is absolute because the terms are non-negative.

(2.2) We have $\frac{a_n}{a_{n-1}} = \frac{p+n}{q+n}r$ converging to r. By the ratio test, the series absolutely converges for $|r| < 1$ and diverges for $|r| > 1$.

Now suppose $r = 1$. Then

$$\frac{a_n}{a_{n-1}} = \frac{p+n}{q+n} = 1 - \frac{q-p}{n} + o\left(\frac{1}{n}\right).$$

Because all terms are non-negative, there is no conditional convergence. [*The following is similar to Example 4.2.8*]

If $q - p > 1$, then pick t satisfying $q - p > t > 1$, and we have

$$\frac{a_n}{a_{n-1}} = 1 - \frac{q-p}{n} + o\left(\frac{1}{n}\right) < 1 - \frac{t}{n} + o\left(\frac{1}{n}\right) = \frac{\frac{1}{n^t}}{\frac{1}{(n-1)^t}}$$

for big n. By the ratio test and the convergence of $\sum \frac{1}{n^t}$, we conclude that $\sum a_n$ converges.

If $q - p < 1$, then we have

$$\frac{a_n}{a_{n-1}} = 1 - \frac{q-p}{n} + o\left(\frac{1}{n}\right) > 1 - \frac{1}{n} = \frac{\frac{1}{n}}{\frac{1}{n-1}}$$

for big n. By the ratio test and the divergence of $\sum \frac{1}{n}$, we conclude that $\sum a_n$ diverges.

If $q - p = 1$, then the series is $\sum \frac{q-1}{q+n} r^n$ and diverges for $r = 1$.

Now consider $r = -1$. The series is alternating and the absolute convergence is the same as $r = 1$. [*The following is similar to Example 4.3.4*]

If $q - p > 0$, then $|a_n|$ is decreasing. Like before, pick t satisfying $q - p > t > 0$, and we have

$$\frac{|a_n|}{|a_{n-1}|} < \frac{\frac{1}{n^t}}{\frac{1}{(n-1)^t}}$$

for big n. Since $\frac{1}{n^t}$ converges to 0, this implies that $|a_n|$ converges to 0. By the Leibniz test, the series $\sum a_n$ converges.

If $q - p \le 0$, then $|a_n|$ is increasing and therefore does not converge to 0. Therefore $\sum a_n$ diverges.

We conclude that the series absolutely converges for $|r| < 1$, or $|r| = 1$ and $q - p > 1$. The series conditionally converges for $r = -1$ and $1 \ge q - p > 0$. Otherwise the series diverges.

(3) The line L is $x = \frac{3}{4}$. The circle on the right of L is

$$x = 3\cos^2\theta, \quad y = 3\cos\theta\sin\theta, \quad \theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right].$$

The cardioid on the right of L is

$$x = (1+\cos\theta)\cos\theta, \quad y = (1+\cos\theta)\sin\theta, \quad \theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right].$$

We may use the shell method for calculating the volume of revolution. This means that a section is the cylinder of distance $x - \frac{3}{4}$ from L, with height $2|y|$. Note that each cylinder is parameterized by $\theta \in \left[0, \frac{\pi}{3}\right]$. (We will have double counting if negative θ is included.) Moreover, the distance between the cylinders is $|\Delta x|$. However, as θ goes from 0 to $\frac{\pi}{3}$, x is decreasing. Therefore we should really use $-\Delta x$ for the distance between the cylinders.

We conclude the formula for the volume of the solid of revolution is

$$\int_0^{\frac{\pi}{3}} 2\pi \left(x - \frac{3}{4}\right) 2y(-dx)$$

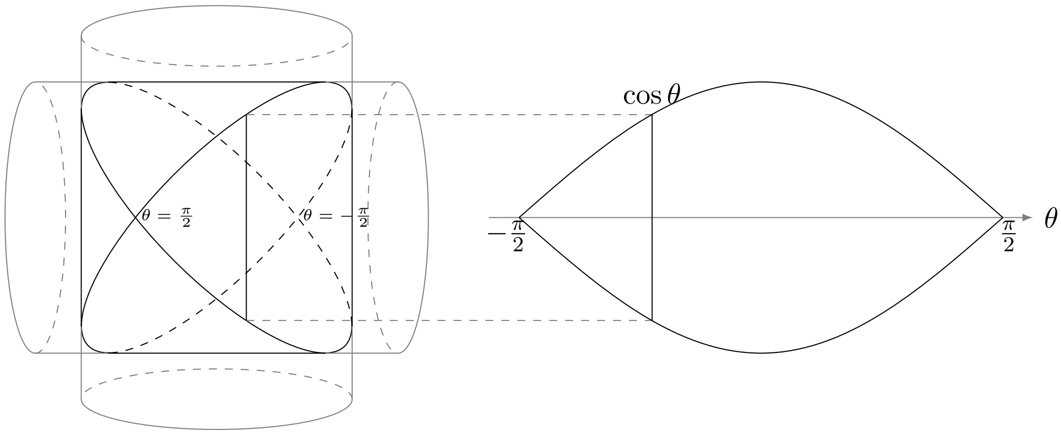
for the revolutions of both the circle and the cardioid. The volume of the difference is

$$\begin{aligned}
&\int_0^{\frac{\pi}{3}} 2\pi \left(3\cos^2\theta - \frac{3}{4}\right) 2(3\cos\theta\sin\theta)(-d(3\cos^2\theta)) \\
&- \int_0^{\frac{\pi}{3}} 2\pi \left((1+\cos\theta)\cos\theta - \frac{3}{4}\right) 2((1+\cos\theta)\sin\theta)(-d((1+\cos\theta)\cos\theta)) \\
&= \pi \int_0^{\frac{\pi}{3}} [2\cdot 3^3(4\cos^2\theta - 1)\cos^2\theta\sin^2\theta - (4\cos^2\theta + 4\cos\theta - 3)(1+\cos\theta)(1+2\cos\theta)\sin^2\theta]d\theta \\
&= \pi \int_0^{\frac{\pi}{3}} [2\cdot 3^3(4\cos^2\theta - 1)\cos^2\theta - (2\cos\theta+3)(2\cos\theta-1)(1+\cos\theta)(1+2\cos\theta)]\sin^2\theta d\theta \\
&= \pi \int_0^{\frac{\pi}{3}} [2\cdot 3^3\cos^2\theta - (2\cos\theta+3)(1+\cos\theta)](4\cos^2\theta-1)\sin^2\theta d\theta \\
&= \pi \int_0^{\frac{\pi}{3}} [2\cdot 3^3\cos^2\theta - (2\cos\theta+3)(1+\cos\theta)](4\cos^2\theta-1)\sin^2\theta d\theta \\
&= \pi \int_0^{\frac{\pi}{3}} (52\cos^2\theta - 5\cos\theta - 3)(4\cos^2\theta - 1)\sin^2\theta d\theta.
\end{aligned}$$

(4) The surface consists of four identical pieces. Each piece is part of the cylinder. Consider the pieces on the right, which we may unwrap to become a region in the plane. Let θ be the angle in the circular direction, such that the piece ranges from $\theta = -\dfrac{\pi}{2}$ to $\theta = \dfrac{\pi}{2}$. Then the straight line (section) at angle θ is from $-\cos\theta$ to $\cos\theta$. (In the picture in the lecture note on page 239, $x = \sin\theta$, and the "height" $\sqrt{1-x^2} = \cos\theta$.) Moreover, the distance between the straight lines is given by $\Delta\theta$. Therefore the area of one piece is

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta = 4.$$

The total area is $4 \times 4 = 16$.



(5.1) Let $h(x)$ be the even extension. Its Fourier coefficients are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{a}{\pi},$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos nx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_0^{a} \cos nx dx = \frac{2}{\pi n}\sin na,$$
$$b_n = 0.$$

We get

$$h(x) \sim \frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi n}\sin na \cos nx.$$

(5.2) The function $h(x)$ is continuous at π and satisfies the condition of Theorem 4.5.4, with $h(\pi) = 0$. Therefore evaluation gives us

$$0 = \frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi n}\sin na \cos n\pi = \frac{a}{\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{2}{\pi n}\sin na.$$

This implies

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin na}{n} = -\frac{a}{2}.$$

(5.3) The Parseval's identity tells us

$$\frac{2a}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} |h(x)|^2 dx = 2\frac{a^2}{\pi^2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\sin^2 na.$$

This means

$$\sum_{n=1}^{\infty} \frac{\sin^2 na}{n^2} = \frac{a(\pi - a)}{2}.$$

(5.4) By shifting $h(x)$ to the right by distance a, we get

$$h(x-a) = \begin{cases} 1, & \text{if } 0 \le x < 2a, \\ 0, & \text{if } 2a \le x \le 2\pi. \end{cases}$$

We also have

$$\begin{aligned} h(x-a) &\sim \frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin na \cos n(x-a) \\ &= \frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin na (\cos na \cos nx + \sin na \sin nx) \\ &= \frac{a}{\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi n} (\sin 2na \cos nx + (1-\cos 2na) \sin nx). \end{aligned}$$

On the other hand, $g(x)$ is obtained from $h(x-a)$ by substituting $2a$ by a. Therefore

$$g(x) \sim \frac{a}{2\pi} + \sum_{n=1}^{\infty} \frac{1}{\pi n} (\sin na \cos nx + (1-\cos na) \sin nx).$$

This is the Fourier series in Example 4.5.1.

Math 1024 Final, Spring 2015

(1) Determine the absolute convergence, conditional convergence, or divergence.

1. $\sum \left(1 - p\dfrac{\log n}{n}\right)^{qn}$, $p, q > 0$.

2. $\sum \dfrac{(-1)^{\frac{n(n-1)}{2}}}{n^p + (-1)^n}$, $p > 0$.

(2) Determine the absolute convergence, conditional convergence, or divergence.

1. $\displaystyle\int_0^{+\infty} \sin\left(x + \frac{1}{x}\right) dx$.

2. $\displaystyle\int_0^{+\infty} \frac{1}{x} \sin\left(x + \frac{1}{x}\right) dx$.

(3) Use power series to argue that $\displaystyle\int_0^1 \frac{\log(1-x)}{x} dx = -\frac{\pi^2}{6}$. Then further explain that $\displaystyle\int_0^{+\infty} \frac{y dy}{e^y - 1} = \frac{\pi^2}{6}$.

(4) Extend the function $\sin x$ on interval $(0, p)$ to even and odd functions of period $2p$. Find the Fourier series of the two extensions. Then use the two Fourier series to derive the Fourier series of the following functions

1. $f(x) = \sin x$ on $(0, p)$, $f(x) = 0$ on $(-p, 0)$, f has period $2p$.

2. $g(x) = \sin x$ on $(-p, 0)$, $g(x) = 0$ on $(0, p)$, g has period $2p$.

Answer to Math 1024 Final, Spring 2015

(1.1) By

$$\left(1-p\frac{\log n}{n}\right)^{qn}=e^{qn\log\left(1-p\frac{\log n}{n}\right)}=e^{-qn\left(p\frac{\log n}{n}+p^2\frac{(\log n)^2}{n^2}+o\left(\frac{(\log n)^2}{n^2}\right)\right)}$$
$$=e^{-qp\log n-qp^2\frac{(\log n)^2}{n}+o\left(\frac{(\log n)^2}{n}\right)}=\frac{1}{n^{pq}}e^{-qp^2\frac{(\log n)^2}{n}+o\left(\frac{(\log n)^2}{n}\right)},$$

and

$$\lim e^{-qp^2\frac{(\log n)^2}{n}+o\left(\frac{(\log n)^2}{n}\right)}=1,$$

the series $\sum\left(1-p\dfrac{\log n}{n}\right)^{qn}$ converges if and only if $\sum\dfrac{1}{n^{pq}}$ converges. This means $pq>1$. Because all terms are positive, the convergence is absolute.

(1.2) The series has two positive terms followed by two negative terms, and the terms converge to 0. Therefore the (absolute or conditional) convergence of the series is the same as the series

$$\sum(-1)^k a_k=\sum(-1)^k\left(\frac{1}{(2k)^p+1}+\frac{1}{(2k+1)^p-1}\right)$$

obtained by combining two consecutive terms of the same sign. Since a_k is decreasing and converges to 0, the series converges by the Leibniz rule.

For the absolute value series $\sum a_k$, by $\lim\dfrac{a_k}{\frac{1}{n^p}}=2^{p-1}$, the series $\sum a_k$ converges if and only if $\sum\dfrac{1}{n^p}$ converges, which means $p>1$.

We conclude that the series converges absolutely for $p>1$ and conditionally for $0<p\le 1$.

(2.1) The integral is improper at $+\infty$.

For $x>\frac{20}{\pi}$ and $x\in\left[2n\pi,2n\pi+\frac{\pi}{5}\right]$, we have $0<\frac{1}{x}<\frac{\pi}{20}$ and

$$x+\frac{1}{x}\in\left[2n\pi,2n\pi+\frac{\pi}{5}+\frac{\pi}{20}\right]=\left[2n\pi,2n\pi+\frac{\pi}{4}\right],\quad \sin\left(x+\frac{1}{x}\right)\ge\frac{1}{\sqrt{2}}.$$

By taking sufficiently large n, we get the following: For any B, there are $a=2n\pi>B$ and $b=2n\pi+\frac{\pi}{5}>B$, such that

$$\int_a^b\sin\left(x+\frac{1}{x}\right)dx\ge\int_{2n\pi}^{2n\pi+\frac{\pi}{5}}\frac{1}{\sqrt{2}}dx=\frac{\pi}{5\sqrt{2}}.$$

This shows that the integral fails the Cauchy criterion and diverges.

Alternative 1. For big $b > 0$ and $a = b^{-1}$, we have

$$\begin{aligned}
\int_1^b \sin\left(x + \frac{1}{x}\right) dx &= \int_1^a \sin\left(\frac{1}{y} + y\right) d\frac{1}{y} = \int_a^1 \frac{1}{y^2} \sin\left(y + \frac{1}{y}\right) dy \\
&= \int_a^1 \sin\left(y + \frac{1}{y}\right) dy - \int_a^1 \left(1 - \frac{1}{y^2}\right) \sin\left(y - \frac{1}{y}\right) dy \\
&= \int_a^1 \sin\left(y + \frac{1}{y}\right) dy - \int_a^1 \sin\left(y - \frac{1}{y}\right) d\left(y - \frac{1}{y}\right) \\
&= \int_a^1 \sin\left(y + \frac{1}{y}\right) dy + \cos\left(1 - \frac{1}{1}\right) - \cos\left(a - \frac{1}{a}\right).
\end{aligned}$$

For $b \to +\infty$, we have $a \to 0^+$. Then

$$\lim_{b \to +\infty} \int_1^b \sin\left(x + \frac{1}{x}\right) dx = \int_0^1 \sin\left(y + \frac{1}{y}\right) dy + 1 - \lim_{a \to 0^+} \cos\left(a - \frac{1}{a}\right).$$

Since $\lim_{a \to 0^+} \cos\left(a - \frac{1}{a}\right)$ diverges, we conclude that $\int_1^{+\infty} \sin\left(x + \frac{1}{x}\right) dx$ diverges.

Alternative 2. We have

$$\begin{aligned}
\int \sin\left(x + \frac{1}{x}\right) dx &= \int \sin x \cos\frac{1}{x} dx + \int \cos x \sin\frac{1}{x} dx \\
&= -\int \cos\frac{1}{x} d\cos x + \int \sin\frac{1}{x} d\sin x \\
&= -\cos\frac{1}{x}\cos x + \int \frac{1}{x^2}\cos x \sin\frac{1}{x} dx + \sin\frac{1}{x}\sin x + \int \frac{1}{x^2}\sin x \cos\frac{1}{x} dx \\
&= -\cos\left(x + \frac{1}{x}\right) + \int \frac{1}{x^2}\sin\left(x + \frac{1}{x}\right) dx.
\end{aligned}$$

By $\left|\frac{1}{x^2}\sin\left(x + \frac{1}{x}\right)\right| \le \frac{1}{x^2}$, the convergence of $\int_1^{+\infty} \frac{1}{x^2} dx$ and the comparison test, $\int_1^{+\infty} \frac{1}{x^2}\sin\left(x + \frac{1}{x}\right) dx$ converges. Since $\lim_{x \to +\infty} \cos\left(x + \frac{1}{x}\right)$ diverges, we conclude that $\int_1^{+\infty} \sin\left(x + \frac{1}{x}\right) dx$ diverges.

(2.2) The integral is improper at 0^+ and $+\infty$.

By

$$\left|\frac{1}{x}\sin\left(x + \frac{1}{x}\right) - \frac{1}{x}\sin x\right| = \left|\frac{1}{x} 2\cos\left(x + \frac{1}{2x}\right)\sin\frac{1}{2x}\right| \le \frac{1}{x^2},$$

and the absolute convergence of $\int_1^{+\infty} \frac{1}{x^2} dx$, the improper integral $\int_1^{\infty} \frac{1}{x}\sin\left(x + \frac{1}{x}\right) dx$ (absolutely or conditionally) converges if and only if $\int_1^{\infty} \frac{1}{x}\sin x dx$ (absolutely or conditionally) converges. Since we know $\int_1^{\infty} \frac{1}{x}\sin x dx$ conditionally converges, we conclude that the improper integral $\int_1^{\infty} \frac{1}{x}\sin\left(x + \frac{1}{x}\right) dx$ also conditionally converges.

For the integral $\int_0^1 \frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx$ that is improper at 0^+, we use the change of variable $y=\frac{1}{x}$

$$\int_0^1 \frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx = \int_{+\infty}^1 y\sin\left(\frac{1}{y}+y\right)\frac{-1}{y^2}dy = \int_1^{+\infty}\frac{1}{y}\sin\left(y+\frac{1}{y}\right)dy,$$

$$\int_0^1 \left|\frac{1}{x}\sin\left(x+\frac{1}{x}\right)\right|dx = \int_{+\infty}^1 \left|y\sin\left(\frac{1}{y}+y\right)\right|\frac{-1}{y^2}dy = \int_1^{+\infty}\left|\frac{1}{y}\sin\left(y+\frac{1}{y}\right)\right|dy.$$

So the (absolute or conditional) convergence of $\int_0^1 \frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx$ is the same as the (absolute or conditional) convergence of $\int_1^{+\infty}\frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx$. By what we just argued, $\int_0^1 \frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx$ conditionally converges.

We conclude that $\int_0^{\infty}\frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx$ conditionally converges.

Alternative.

$$\begin{aligned}\int_2^{+\infty}\frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx &= -\int_2^{+\infty}\frac{1}{x\left(1-\frac{1}{x^2}\right)}d\cos\left(x+\frac{1}{x}\right) = -\int_2^{+\infty}\frac{x}{x^2-1}d\cos\left(x+\frac{1}{x}\right)\\ &= -\frac{x}{x^2-1}\cos\left(x+\frac{1}{x}\right)\Big|_2^{+\infty} + \int_2^{+\infty}\cos\left(x+\frac{1}{x}\right)d\frac{x}{x^2-1}\\ &= \frac{2}{2^2-1}\cos\left(2+\frac{1}{2}\right) - \int_2^{+\infty}\frac{x^2+1}{(x^2-1)^2}\cos\left(x+\frac{1}{x}\right)dx.\end{aligned}$$

By $\left|\frac{x^2+1}{(x^2-1)^2}\cos\left(x+\frac{1}{x}\right)\right| \le \frac{x^2+1}{(x^2-1)^2}$ and the convergence of $\int_2^{+\infty}\frac{x^2+1}{(x^2-1)^2}dx$, we conclude that $\int_2^{+\infty}\frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx$ converges. Similar argument can be applied to the convergence of $\int_0^{\frac{1}{2}}\frac{1}{x}\sin\left(x+\frac{1}{x}\right)dx$.

(3) Integrating the geometric series $\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$, we get

$$\log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, \quad |x|<1.$$

This implies

$$\frac{\log(1-x)}{x} = -\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}, \quad |x|<1.$$

Integrating again, we get

$$\int_0^x \frac{\log(1-t)}{t}dt = -\sum_{n=1}^{\infty}\frac{x^n}{n^2}, \quad |x|<1.$$

Since the right side converges at $x = 1$, by Theorem 4.4.4, we get

$$-\sum_{n=1}^{\infty}\frac{1}{n^2} = -\lim_{x\to 1^-}\sum_{n=1}^{\infty}\frac{x^n}{n^2} = -\lim_{x\to 1^-}\int_0^x \frac{\log(1-t)}{t}dt = \int_0^1 \frac{\log(1-t)}{t}dt.$$

On the other hand, in Examples 4.5.9 and 4.5.10, the left side is $-\frac{\pi^2}{6}$. Therefore $\int_0^1 \frac{\log(1-x)}{x}dx = -\frac{\pi^2}{6}$.

Let $y = -\log(1-x)$. Then $x = 1 - e^{-y}$ and

$$\int_0^1 \frac{\log(1-x)}{x}dx = \int_{-\log 1}^{-\log 0^+} \frac{-y}{1-e^{-y}}d(1-e^{-y}) = -\int_0^{+\infty}\frac{y}{1-e^{-y}}e^{-y}dy = -\int_0^{+\infty}\frac{ydy}{e^y-1}.$$

Since the left side is $-\frac{\pi^2}{6}$, we conclude that $\int_0^{+\infty}\frac{ydy}{e^y-1} = \frac{\pi^2}{6}$.

(4) The even and odd extensions are

$$F_e(x) = \begin{cases}\sin x, & \text{on } (0,p) \\ -\sin x, & \text{on } (-p,0)\end{cases}, \quad F_o(x) = \begin{cases}\sin x, & \text{on } (0,p) \\ \sin x, & \text{on } (-p,0)\end{cases} = \sin x \text{ on } (-p,p).$$

The even extension has

$$\begin{aligned}
a_0 &= \frac{1}{2p}\int_{-p}^{p} F_e(x)dx = \frac{1}{p}\int_0^p F_e(x)dx = \frac{1}{p}\int_0^p \sin x dx = \frac{1-\cos p}{p}, \\
a_n &= \frac{2}{2p}\int_{-p}^{p} F_e(x)\cos\frac{2n\pi}{2p}dx = \frac{2}{p}\int_0^p \sin x\cos\frac{2n\pi}{2p}dx \\
&= \frac{1}{p}\int_0^p \left[\sin\left(1+\frac{n\pi}{p}\right)x + \sin\left(1-\frac{n\pi}{p}\right)x\right]dx \\
&= \frac{1}{p}\left[\frac{1-\cos\left(1+\frac{n\pi}{p}\right)p}{1+\frac{n\pi}{p}} + \frac{1-\cos\left(1-\frac{n\pi}{p}\right)p}{1-\frac{n\pi}{p}}\right] \\
&= \frac{1-(-1)^n\cos p}{p+n\pi} + \frac{1-(-1)^n\cos p}{p-n\pi} = \frac{2p(1-(-1)^n\cos p)}{p^2-n^2\pi^2}.
\end{aligned}$$

We get the Fourier series

$$F_e(x) \sim \frac{1-\cos p}{p} + 2p\sum_{n=1}^{\infty}\frac{1-(-1)^n\cos p}{p^2-n^2\pi^2}\cos\frac{n\pi}{p}x, \quad x \in (-p,p).$$

The odd extension has

$$\begin{aligned}
b_n &= \frac{2}{2p}\int_{-p}^{p} F_o(x)\sin\frac{2n\pi}{2p}dx = \frac{2}{p}\int_0^p \sin x\sin\frac{2n\pi}{2p}dx \\
&= \frac{1}{p}\int_0^p \left[\cos\left(1-\frac{n\pi}{p}\right)x - \cos\left(1+\frac{n\pi}{p}\right)x\right]dx \\
&= \frac{1}{p}\left[\frac{\sin\left(1-\frac{n\pi}{p}\right)p}{1-\frac{n\pi}{p}} - \frac{\sin\left(1+\frac{n\pi}{p}\right)p}{1+\frac{n\pi}{p}}\right] = (-1)^n\frac{2n\pi\sin p}{p^2-n^2\pi^2}.
\end{aligned}$$

We get the Fourier series

$$F_o(x) \sim 2\pi \sin p \sum_{n=1}^{\infty} \frac{(-1)^n n}{p^2 - n^2\pi^2} \sin \frac{n\pi}{p} x, \quad x \in (-p, p).$$

Then we further get Fourier series

$$\begin{aligned} f(x) &= \frac{1}{2}(F_e(x) + F_o(x)) \\ &\sim \frac{1 - \cos p}{2p} + \sum_{n=1}^{\infty} \frac{1}{p^2 - n^2\pi^2} \left(p(1 - (-1)^n \cos p) \cos \frac{n\pi}{p} x + (-1)^n n\pi \sin p \sin \frac{n\pi}{p} x \right), \\ g(x) &= \frac{1}{2}(-F_e(x) + F_o(x)) \\ &\sim -\frac{1 - \cos p}{2p} + \sum_{n=1}^{\infty} \frac{1}{p^2 - n^2\pi^2} \left(p(-1 + (-1)^n \cos p) \cos \frac{n\pi}{p} x + (-1)^n n\pi \sin p \sin \frac{n\pi}{p} x \right). \end{aligned}$$