Exercise 3.5.19 Let $A_p = \int x^p e^{ax} \cos bx \, dx$, $B_p = \int x^p \cdot e^{ax} \sin bx \, dx$.

Then

$$A_p = \frac{1}{a} \int x^p \cdot \cos bx \, d e^{ax}$$

$$= \frac{1}{a} x^p \cdot \cos bx \cdot e^{ax} - \frac{1}{a} \int (p x^{p-1} \cos bx - b x^p \sin bx) e^{ax} \, dx$$

$$= \frac{1}{a} x^p \cos bx \cdot e^{ax} - \frac{p}{a} A_{p-1} + \frac{b}{a} B_p$$

$$B_p = \frac{1}{a} \int x^p \sin bx \, d e^{ax}$$

$$= \frac{1}{a} x^p \sin bx \cdot e^{ax} - \frac{1}{a} \int (p x^{p-1} \sin bx + b x^p \cos bx) e^{ax} \, dx$$

$$= \frac{1}{a} x^p \sin bx \cdot e^{ax} - \frac{p}{a} B_{p-1} - \frac{b}{a} A_p$$

So we can get $A_p = \frac{1}{b} x^p \sin bx \cdot e^{ax} - \frac{a}{b} B_p - \frac{p}{b} B_{p-1}$,

$$B_p = -\frac{1}{b} x^p \cos bx \cdot e^{ax} + \frac{a}{b} A_p + \frac{p}{b} A_{p-1}.$$

$$\therefore A_{p-1} = \frac{1}{b} x^{p-1} \sin bx \cdot e^{ax} - \frac{a}{b} B_{p-1} - \frac{p-1}{b} B_{p-2}.$$

$$B_p = -\frac{1}{b} x^p \cos bx \cdot e^{ax} + \frac{a}{b} \left(\frac{1}{b} x^p \sin bx \cdot e^{ax} - \frac{a}{b} B_p - \frac{p}{b} B_{p-1}\right) + \frac{p}{b} \cdot$$

$$\left(\frac{1}{b} x^{p-1} \sin bx \cdot e^{ax} - \frac{a}{b} B_{p-1} - \frac{p-1}{b} B_{p-2}\right),$$

After some computation, we could get

$$B_p = \frac{x^{p-1} \cdot e^{ax}}{a^2 + b^2} (ax \sin bx + p \sin bx - bx \cos bx) - \frac{2ap}{a^2+b^2} B_{p-1} - \frac{p^2 - p}{a^2+b^2} B_{p-2}.$$

Similarly, from $A_p = \frac{1}{b} x^p \sin bx \cdot e^{ax} - \frac{a}{b}\left(-\frac{1}{b} x^p \cos bx \cdot e^{ax} + \frac{a}{b} A_p + \frac{p}{b} A_{p-1}\right)$

$$- \frac{p}{b}\left(-\frac{1}{b} x^{p-1} \cos bx \cdot e^{ax} + \frac{a}{b} A_{p-1} + \frac{p-1}{b} A_{p-2}\right),$$

we have $A_p = \frac{x^{p-1} e^{ax}}{a^2 + b^2} (bx \sin bx + ax \cos bx + p \cos bx) - \frac{2ap}{a^2+b^2} A_{p-1} - \frac{p^2-p}{a^2+b^2} A_{p-2}.$

$\int x^2 e^{-x} \sin 3x \, dx$ is B_2 with $a = -1$, $b = 3$.

Then according to the recursive relation for B_p,

we have $\int x^2 e^{-x} \sin 3x \, dx$

$$= \frac{x \cdot e^{-x}}{10}(-x \sin 3x + 2 \sin 3x - 3x \cdot \cos 3x) + \frac{4}{10} \int x e^{-x} \sin 3x \, dx - \frac{2}{10} \int e^{-x} \sin 3x \, dx$$

$$= \frac{x \cdot e^{-x}}{10}(-x \sin 3x + 2 \sin 3x - 3x \cdot \cos 3x) + \frac{4}{10}\left(-x \sin 3x \, e^{-x} + \int \sin 3x \cdot e^{-x} \, dx + 3 \int x \cos 3x \, e^{-x} \, dx\right) - \frac{2}{10} \int e^{-x} \sin 3x \, dx$$

$$= \frac{x \cdot e^{-x}}{10}(-x \sin 3x - 2 \sin 3x - 3x \cdot \cos 3x) + \frac{2}{10} \int \sin 3x \cdot e^{-x} \, dx + \frac{12}{10} \cdot \frac{1}{10}\left(-x \cos 3x \cdot e^{-x} + \int \cos 3x \cdot e^{-x} \, dx + 3x \sin 3x \cdot e^{-x} - 3 \int \sin 3x \cdot e^{-x} \, dx\right)$$

Take $\int e^{-x} \sin 3x \, dx = -\frac{1}{10} \sin 3x \cdot e^{-x} - \frac{3}{10} \cos 3x \cdot e^{-x}$

and $\int e^{-x} \cos 3x \, dx = \frac{-1}{8} \cos 3x \cdot e^{-x} + \frac{3}{8} \sin 3x \cdot e^{-x}$ in,

we could get the answer for $\int x^2 \cdot e^{-x} \sin 3x \, dx$.