Date No.

3.5.21 Since we have beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

When we take $t = \sin^2\theta$, $\theta \in [0, \frac{\pi}{2}]$,

we have $B(x,y) = \int_0^{\frac{\pi}{2}} \sin^{2x-2}\theta \cdot \cos^{2y-2}\theta \cdot 2\sin\theta \cdot \cos\theta \, d\theta$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cdot \cos^{2y-1}\theta \, d\theta$$

Thus $\int_0^{\frac{\pi}{2}} \sin^{2m}x \cos^{2n}x \, dx = \frac{1}{2} B(m+\frac{1}{2}, n+\frac{1}{2})$

$$= \frac{1}{2} \frac{\Gamma(m+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{\Gamma(m+n+1)}$$

according to $B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$ where $\Gamma(x)$ is the gamma function.

Here $\Gamma(m+\frac{1}{2}) = (m-\frac{1}{2})(m-\frac{3}{2}) \cdots \frac{1}{2} \cdot \Gamma(\frac{1}{2})$

$$= \frac{1}{2^m} (2m-1)(2m-3)\cdots 1 \cdot \sqrt{\pi}$$

for m is natural number and $\Gamma(x+1) = x \cdot \Gamma(x)$.

Therefore $\int_0^{\frac{\pi}{2}} \sin^{2m}x \cos^{2n}x \, dx = \frac{1}{2} \cdot \frac{(2m-1)!!\,(2n-1)!!}{2^{m+n}\cdot(m+n)!} \cdot \pi$

$$= \frac{\pi\,(2m)!\,(2n)!}{2^{m+n+1}(2m)!!\,(2n)!!\,(m+n)!} = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1}\cdot m!\cdot n!\,(m+n)!}.$$

For $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx$, when m is even, n is even, the case is the same as above.

When m is odd, n is odd, we can make use of $\frac{1}{2}B(m+\frac{1}{2}, n+\frac{1}{2})$ to get the result.

When either m or n is odd, the other is even, (for example m even, n odd) we can make $\int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-1} x \, d\sin x = \int_0^{\frac{\pi}{2}} \sin^m x (1-\sin^2 x)^{\frac{n-1}{2}} d\sin x$.