Math1024 Answer to Homework 1

EXERCISE 2.6.1 (3)

$f'(x) = \log x + 1$, $f''(x) = \frac{1}{x}$.

Let$f'(x) = \log x + 1 = 0 \Rightarrow x = e^{-1}$ is a candidate of local extrema. $f''(e^{-1}) = e > 0$, then we could use Taylor expansion of $f(x)$ at $x = e^{-1}$ here, which is

$$f(x) = x \log x = -e^{-1} + \frac{e}{2}(x - e^{-1})^2 + o((x - e^{-1})^2)$$

From Theorem 2.6.1, we know that now $x = e^{-1}$ is a local minimum.

EXERCISE 2.6.2

Differentiate $x^3 + y^3 = 6xy$, we have $3x^2 + 3y^2y' = 6y + 6xy'$, move terms and simplify it, we can get $(y^2 - 2x)y' = 2y - x^2$.

Then the implicit function is well-defined when $y^2 - 2x \neq 0$. Under this condition, the first order derivative is

$$y' = \frac{2y - x^2}{y^2 - 2x}.$$

Combine $2y - x^2 = 0$ and $x^3 + y^3 = 6xy$, that is, by solving system

$$x^3 + y^3 = 6xy$$
$$2y - x^2 = 0$$

We can find $P = (\sqrt[3]{16}, \sqrt[3]{32})$ is the only candidate for local extrema of the implicit function. Finally, differentiate $3x^2 + 3y^2y' = 6y + 6xy'$, we have

$$y'' = \frac{2x + 2y(y') - 4y'}{2x - y^2}.$$

Since $y'(P) = 0$, $y''(P) < 0 \Rightarrow$ P is a local maximum.

EXERCISE 2.6.5 (2)

$f'(x) = x \sin x$, $f'(0) = 0$. $f''(x) = \sin x + x \cos x$, $f''(0) = 0$. $f'''(x) = 2 \cos x - x \sin x$, $f'''(0) = 2 \neq 0$, then 0 is not a local extrema.

EXERCISE 2.6.5 (4)

$f'(x) = \frac{1}{2!}x^2e^x$, $f'(0) = 0$. $f''(x) = x^2e^x + \frac{1}{2!}x^2e^x$, $f''(0) = 0$. $f'''(x) = e^2 + 2x^2e^x + \frac{1}{2!}x^2e^x$, $f'''(0) = 1 > 0$, then 0 is not a local extrema.

EXERCISE 2.6.6 (2)

For $x \neq 0$ close to 0, we have

$$
\begin{aligned}
f(x) &= \frac{1}{x+bx^3}\left(x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^6)\right)\\
&= \frac{1}{1-(-bx^2)}\left(1-\frac{x^2}{6}+\frac{x^4}{120}+o(x^5)\right)\\
&= \left(1-bx^2+b^2x^4+o(x^5)\right)\left(1-\frac{x^2}{6}+\frac{x^4}{120}+o(x^5)\right)\\
&= 1-\left(b+\frac{1}{6}\right)x^2+\left(b^2+\frac{b}{6}+\frac{1}{120}\right)x^4+o(x^5).
\end{aligned}
$$

Since $f(0)=1$, the 4-th order approximation also holds for $x=0$.

Then, if $b>-\frac{1}{6}$, 0 is a local maximum; if $b<-\frac{1}{6}$, 0 is a local minimum; if $b=-\frac{1}{6}$, the coefficient of x^4 equals $\frac{1}{120}>0$, then 0 is a local minimum.

Exercise 2.6.7 (10)

$f(x)=x^x$ is well-defined when $x>0$.

Let$F(x)=\log f(x)=x\log x$, then$F'(x)=\frac{f'(x)}{f(x)}=\log x+1$, $f'(x)=x^x(\log x+1)$.

$f''(x)=x^x(\log+1)^2+x^{x-1}>0$ and thus $f(x)$ is convex in $(0,+\infty)$.

Exercise 2.6.8

Recall that in $(0,+\infty)$, x^p is convex when $p\geqslant 1$ or $p<0$, and is concave when $0<p<1$.

Here, we find the condition on A and B such that $f(x)$ is convex on the whole real line. Then the domain of $f(x)$ should be $(-\infty,+\infty)$ and $f(x)$ is continuous in the domain. So we have $p>0$ and $q\geqslant 0$, $f(0)=0$. Note that if $f(x)$ is convex, then $cf(x)$ is concave where c is a negative constant.

(1) $p>1$, $q>1$.

If $A\geqslant 0$ or $B\geqslant 0$, $f(x)$ is differentiable and $f'(x)$ is increasing, so $f(x)$ is convex. If either $A<0$ and $B<0$, then $f(x)$ is strictly concave in a interval(concave but not a linear function), not convex.

(2) $p>1$, $q=1$.

If $A\geqslant 0$ and $B>0$, $f(x)$ is not differentiable at the origin. Since for any $x<0$, $y>0$, $f(x)>0$ and $f(y)>0$, then $L_{x,y}(0)>f(0)=0$. Thus $L_{x,y}(z)\geqslant L_{0,y}(z)$ for any $z\in[(0,+\infty)$, and $L_{x,y}(z)>L_{x,0}(z)$ for any $z\in(-\infty,0)$. But $L_{0,y}(z)>f(0)$ for any $z\in(0,+\infty)$ and $L_{x,0}(z)>f(0)$ for any $z\in(-\infty,0)$ because of the condition of A and B, so $L_{x,y}(z)\geqslant f(z)$ for any z in $\mathbb{R}$. $f(x)$ is convex.

If $A\geqslant 0$ and $B=0$, $f(x)$ is differentiable and $f'(x)$ is increasing, so $f(x)$ is convex.

If $A<0$, then f(x) is strictly concave in a interval, not convex. If $A\geqslant 0$ and $B<0$, then you can easily find two point $x<0$ and $y>0$, let $L_{x,y}$ represent the corresponding linear function, $L_{x,y}(0)<f(0)=0$, so $f(x)$ is not convex. How to find such two point? Let me just take the case $A>0$ and $B<0$ as example. Fix a point $x>0$ on the negative part of the graph of $f(x)$, and the line passing x and origin will intersect a point z on the positive part. Take a point x between origin and z. x and y are what we want.

Similarly, you can deal with case of $p = 1$, $q > 1$.

(3) $p > 1$, $0 < q < 1$.

If $A < 0$ or $B > 0$, then $f(x)$ is strictly concave in a interval,, not convex.

If $A \geqslant 0$ and $B = 0$, $f(x)$ is differentiable and $f'(x)$ is increasing, so $f(x)$ is convex.

If $A \geqslant 0$ and $B < 0$, similar argument as in (2) shows that you can easily find two point $x < 0$ and $y > 0$, $L_{x,y}(0) < f(0) = 0$ $f(x)$ is not convex.

Similarly, you can deal with case of $0 < p < 1$, $q > 1$.

(4) $p = 1$ and $q = 1$.

In this case, on the graph is that two rays are connected at the origin. A and $-B$ are the slopes. You can easily check $A + B \geqslant 0$ in order to $f(x)$ is convex.

(5) $p = 1$ and $0 < q < 1$.

If $B > 0$, then $f(x)$ is strictly concave in a interval, not convex.

If $B = 0$, it is similar to case (4), but now $B = 0$, then $A \geqslant 0$ in order to make $f(x)$ convex.

If $B < 0$, no matter for any value of A, similar argument as in (2) shows that $f(x)$ is not convex.

Similarly, you can deal with case of $0 < p < 1$, $q = 1$,

(6) $0 < p < 1$, $0 < q < 1$.

Then we have to let $A \leqslant 0$ and $B \leqslant 0$.

If either $A \neq 0$ and $B \neq 0$, similar argument as in (2) shows that $f(x)$ is not convex. so $A = B = 0$.

(7) $p > 0$, $q = 0$.

Then we have to let $B = 0$, otherwise $f(x)$ is not continue. Similar argument as in all cases above shows that: If $p \geqslant 1$, $q = 0$, then $A \geqslant 0$. If $0 < p < 1$, $q = 0$, then $A = 0$.

Exercise 2.6.9

By concavity of $\log x$, we have: for p, $q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and $\forall x$, $y > 0$,

$$\log\left(\frac{x}{p} + \frac{y}{q}\right) \geqslant \frac{1}{p}\log x + \frac{1}{q}\log y.$$

Replace x and y respectively by x^p and y^q, then

$$\log\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \geqslant \frac{1}{p}\log x^p + \frac{1}{q}\log y^q = \log xy.$$

By the fact that $\log x$ is a increasing function, we have

$$\frac{x^p}{p} + \frac{y^q}{q} \geqslant xy.$$

Exercise 2.6.13 (3)

Exercise 2.6.14 (5)

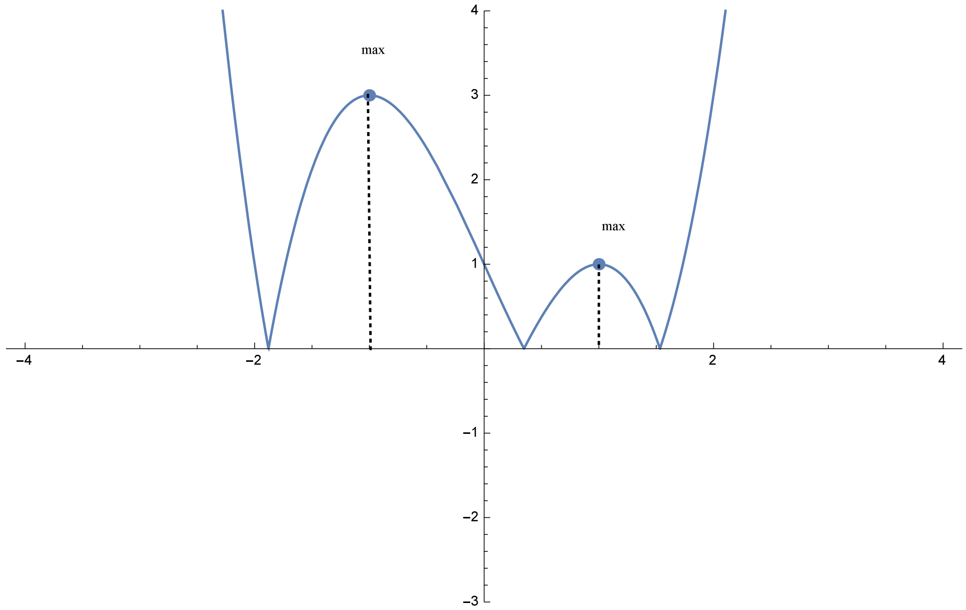


Figure 1: Exercise 2.6.13(3)

Exercise 2.6.18

When $c = 0, d \neq 0$, the function $\frac{ax+b}{cx+d}$ would become a linear function, whose graph is just a line.

when $c \neq 0, d \neq 0$,

$$\frac{ax+b}{cx+d} = \frac{\frac{a}{c}(cx+d) + b - \frac{ad}{c}}{cx+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d} = \frac{a}{c} + \frac{\frac{bc-ad}{c^2}}{x + \frac{d}{c}}.$$

From this form, the function could be seen as the shift and scaling of $\frac{1}{x}$.

Without loss of generality, we assume that $a > 0, b > 0, c > 0, d < 0$, then the graph of $\frac{ax+b}{cx+d}$ would look like Figure 5.

Exercise 2.6.19 (4)

The function $\frac{x^3}{(x-1)^2}$ is not defined at 1 and has limit

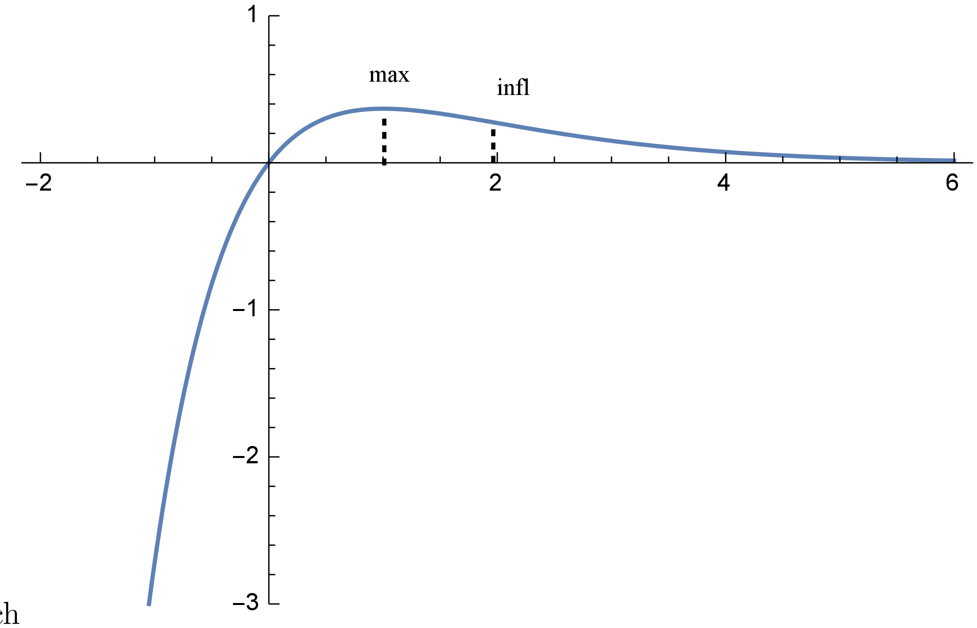
$$\lim_{x \to 1} \frac{x^3}{(x-1)^2} = \infty$$

We can know the function information as following:

Exercise 2.7.1 (3)

We know from the Taylor Expansion that we could use

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$



chch

Figure 2: Exercise 2.6.14(5), $0 < a < 1$

Table 1: Result of 2.6.19 (4)

x	$(\infty, 0)$	0	$(0,1)$	1	$(1,3)$	3	$(3,\infty)$
$f = \frac{x^3}{(x-1)^2}$	$-\infty \leftarrow$	0		∞		$\frac{27}{4}$	$\to \infty$
$f' = \frac{x^2(x-3)}{(x-1)^3}$		$+$			$-$	0	$+$
		$\nearrow$			$\searrow$	min	$\nearrow$
$f'' = \frac{6x}{(x-1)^4}$	$-$	inflection			$+$		

as n-th order approximation of a function $f(x)$ when it has n-th order derivative at x_0. We just use linear approximation of $\sqrt[n]{a^n + x}$ at $x = 0$ here, which is

$$\sqrt[n]{a^n + x} \approx \sqrt[n]{a^n + 0} + \frac{1}{n}(a^n + 0)^{\frac{1}{n}-1}(x - 0) = a + \frac{x}{na^{n-1}}$$

for $a > 0$. Thus for $\sqrt[5]{39}$, we just take $n = 5, a = 2, x = 7$. Then $\sqrt[5]{39} = 2 + \frac{7}{5 \cdot 2^4} = 2.0875$.

Exercise 2.7.3 (1)

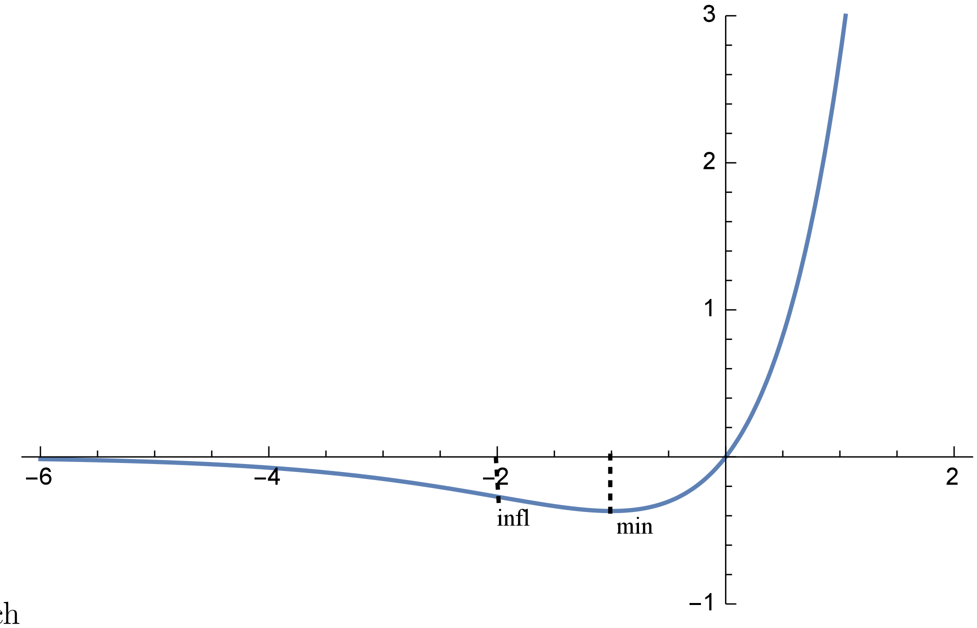
The Taylor expansion of sin$x$ tells us

$$\sin 1 = 1 - \frac{1}{3!}1^3 + R_3(1).$$

where

$$|R_3(1)| = \frac{\sin^{(4)}(c)}{4!}1^4 \leq \frac{\sin 1}{4!}, 0 < c < 1$$

Therefore $\sin 1 = 1 - \frac{1}{6} = 0.833\cdots$ and the error is $\frac{\sin 1}{4!} = 0.03506\cdots$



chch

Figure 3: Exercise 2.6.14(5), $a > 1$

EXERCISE 2.7.4 (1)

In order to find approximate values of sin1 accurate up to the 10-th digit, we would use

$$|R_n(1)| = |\frac{\sin^{(n+1)}(c)}{(n+1)!}1^{n+1}|, \text{where } 0 < c < 1$$

to control the error.

When n=13,

$$|R_{13}(1)| = |\frac{\sin^{(14)}(c)}{14!}1| \leq \frac{sin1}{14!} = 9.65 \times 10^{-12},$$

Thus

$$\sin 1 \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!} + \frac{1}{13!} = 0.8414709848$$

EXERCISE 2.7.8

For the function $f(x) = e^x - x - 2$, we know it should have the unique solution on $(1, 2)$ by $f(1) = e - 1 - 2 = e - 3 < 0$ and $f(2) = e^2 - 2 - 2 = e^2 - 4 > 0$. Then take 1 as an initial value, just keep using the iteration induced by the Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{e^{x_n} - x_n - 2}{e^{x_n} - 1}.$$

We would find the sequence just converges very close to the exact root of $f(x)$. Actually we can write some simple codes by Matlab or other math softwares to get the iterative result for these questions. Here is a sample of Matlab codes and the corresponding result(values accurate up to 8-th digit) for this question:

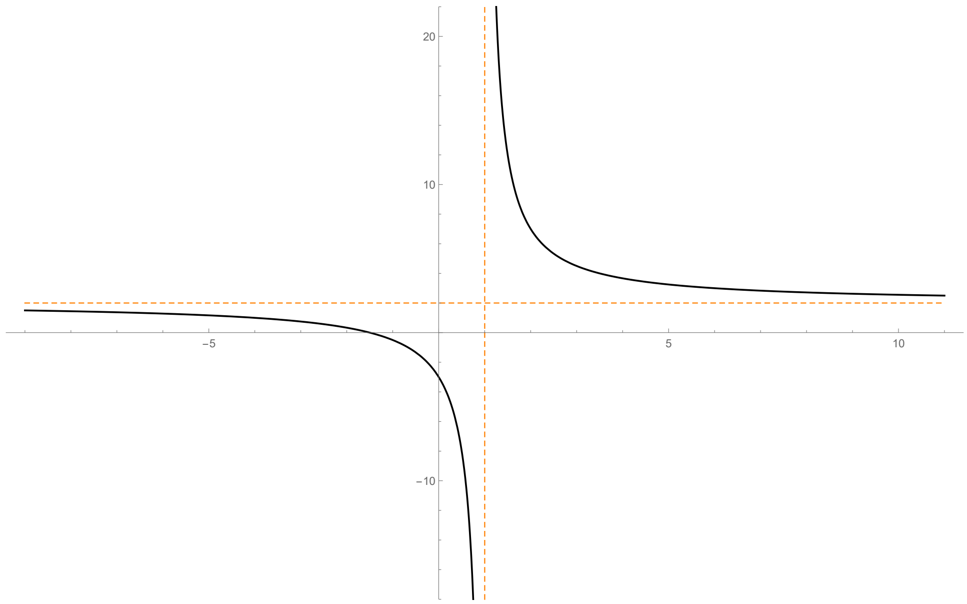


Figure 4: Graph of $\frac{ax+b}{cx+d}$ when $a > 0, b > 0, c > 0, d < 0$

```
x = 1;
err = 10^(-10); %the value of error we hope to reach
n = 1;
f = exp(1) - 1 -2;
%The next three lines are used to print result
fprintf('step      x           f(x) \n')
fprintf('----  ----------   -----------\n')
fprintf('%3i   %8.8f   %8.8f  \n', n, x, f)
while (abs(f) > err)
    n = n + 1;
    fprime = exp(x) - 1;
    y = x - f/fprime;
    x = y;
    f = exp(x) - x -2;
    fprintf('%3i   %8.8f   %8.8f  \n', n, x, f)
end
```

Exercise 2.7.10

To find the approximate value of $\sqrt{4.05}$ is equivalent to find the root of $f(x) = x^2 - 4.05$ and to find the approximate value of e^{-1} is equivalent to find the root of $g(x) = \log(x) + 1$. Similarly, we run some codes of Newton's method to get the approximate value by starting from 2 and 0.5 respectively. Here are results of iteration:

Thus the approximate value of $\sqrt{4.05}$ is 2.0124611798 and the approximate value of e^{-1} is 0.3678794412.

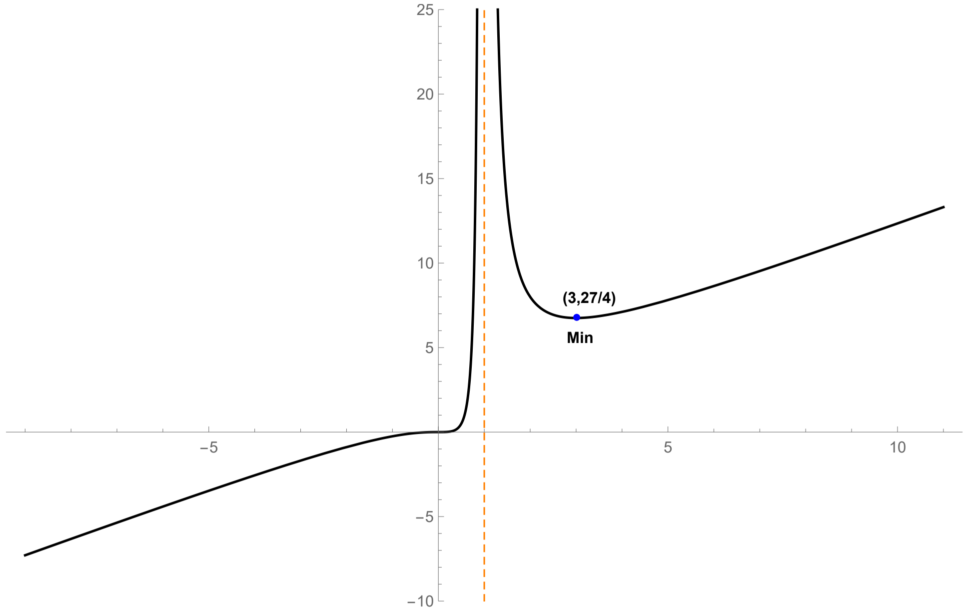


Figure 5: Graph of $\frac{x^3}{(x-1)^2}$

Table 2: Result of 2.7.8

Step	x	$f(x)$
1	1.00000000	-0.28171817
2	1.16395341	0.03861595
3	1.14642119	0.00048934
4	1.14619326	0.00000008
5	1.14619322	0.00000000

Exercise 2.7.12

This is exactly the recursive relation induced by Newton's method. Cause getting the value of $\sqrt{a}$ for $a > 0$ equals to finding the root of $f(x) = x^2 - a$.

From Newton's method, we know that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}(x_n + \frac{a}{x_n})$, which is just the recursive relation used by the ancient Babylonians to compute $\sqrt{a}$.

Exercise 2.7.15

1. For $\sqrt[3]{x} = 0$, we could get the iteration expression of Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

This means that the iterative sequence would diverge if not starting from $x_0 = 0$. Thus Newton's method would fail when starting at any $x_0 \neq 0$.

Table 3: Result of 2.7.10 for $\sqrt{4.05}$

Step	x	$f(x)$
1	2.00000000000	-0.05000000000
2	2.01250000000	0.00015625000
3	2.01246118012	0.00000000151
4	2.01246117975	0.00000000000

Table 4: Result of 2.7.10 for e^{-1}

Step	x	$g(x)$
1	0.50000000000	0.30685281944
2	0.34657359028	-0.05966010114
3	0.36725020573	-0.00171190374
4	0.36787890273	-0.00000146364
5	0.36787944117	-0.00000000000

2. For $\text{sign}(x)\sqrt{|x|} = 0$, which is

$$\begin{cases} \sqrt{x} & x \geq 0, \\ -\sqrt{-x} & x < 0. \end{cases}$$

when written in the piecewise function. Then by Newton's method,

$$\text{when} \quad x_n \geq 0, x_{n+1} = x_n - \frac{\sqrt{x_n}}{\frac{1}{2}\frac{1}{\sqrt{x_n}}} = -x_n,$$

$$\text{when} \quad x_n < 0, x_{n+1} = x_n - \frac{-\sqrt{-x_n}}{\frac{1}{2}\frac{1}{\sqrt{-x_n}}} = x_n + 2\sqrt{-x_n}\sqrt{-x_n} = -x_n.$$

This also imply that the iterative sequence would diverge if not starting from $x_0 = 0$ which actually just jumps between the starting point and the negative of it. Thus Newton's method would fail when starting at any $x_0 \neq 0$.