Math1024 Answer to some exercises of Fourier series

Exercise 4.5.1 (1)

Since the period is 2π, $\sin^2 x = \dfrac{1}{2} - \dfrac{1}{2}\cos 2x$ is the Fourier series.

Exercise 4.5.1 (2)

We have

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}\int_0^\pi \sin x dx = \frac{2}{\pi}, \\
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos 2nx dx = \frac{2}{\pi}\int_0^\pi \sin x \cos 2nx dx \\
&= \frac{1}{\pi}\int_0^\pi (\sin(2n+1)x - \sin(2n-1)x)dx \\
&= \frac{1}{\pi}\left(-\frac{(-1)^{2n+1}-1}{2n+1} + \frac{(-1)^{2n-1}-1}{2n-1}\right) = -\frac{4}{(4n^2-1)\pi}, \\
b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin 2nx dx = \frac{2}{\pi}\int_0^\pi \sin x \sin 2nx dx = 0.
\end{aligned}$$

The Fourier series is

$$\sin x \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx, \quad x \in (0, \pi).$$

Exercise 4.5.1 (6)

By Example 4.5.2, we have the Fourier series

$$|x| \sim \frac{1}{2} + \sum \frac{4}{(2n+1)^2\pi^2}\cos(2n+1)\pi x, \quad |x| < 1$$

for the periodic function of period 2 that extends $|x|$ on $(-1, 1)$. This gives the Fourier series

$$\frac{|x|}{p} \sim \frac{1}{2} + \sum \frac{4}{(2n+1)^2\pi^2}\cos\frac{(2n+1)\pi x}{p}, \quad |x| < p$$

for the periodic function of period $2p$, and further gives

$$|x| \sim \frac{p}{2} + \sum \frac{4p}{(2n+1)^2\pi^2}\cos\frac{(2n+1)\pi x}{p}, \quad |x| < p.$$

Exercise 4.5.1 (8)

The function is even, so that $b_n = 0$. Since the period is the standard 2π, we have

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}\int_0^\pi x\sin x dx = \pi,\\
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx dx\\
&= \frac{1}{\pi}\int_0^\pi x(\sin(n+1)x - \sin(n-1)x)dx = \frac{1}{\pi}\int_0^\pi x d\left(\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right)\\
&= (-1)^{n+1}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) - \frac{1}{\pi}\int_0^\pi \left(\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right) dx\\
&= (-1)^{n+1}\frac{2}{n^2-1}.
\end{aligned}
$$

Therefore the Fourier series is

$$
x\sin x \sim \pi - \sum_{n=1}^{+\infty}(-1)^n\frac{2}{n^2-1}\cos nx, \quad x \in (-\pi, \pi).
$$

Exercise 4.5.2

For even extension, we have (note the period is $2p$)

$$
\begin{aligned}
a_0 &= \frac{1}{2p}\int_{-p}^p f(x)dx = \frac{2}{2p}\int_0^p f(x)dx = \frac{1}{p}\int_0^p f(x)dx,\\
a_n &= \frac{2}{2p}\int_{-p}^p f(x)\cos\frac{2n\pi}{2p}xdx = \frac{2}{2p}2\int_0^p f(x)\cos\frac{2n\pi}{2p}xdx = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx,\\
b_n &= 0.
\end{aligned}
$$

For odd extension, we have

$$
\begin{aligned}
a_n &= 0,\\
b_n &= \frac{2}{2p}\int_{-p}^p f(x)\sin\frac{2n\pi}{2p}xdx = \frac{2}{2p}2\int_0^p f(x)\sin\frac{2n\pi}{2p}xdx = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}xdx.
\end{aligned}
$$

Exercise 4.5.3 (1)

The even function is the periodic extension of x^2 on $(-1,1)$ with period 2. We have

$$
\begin{aligned}
a_0 &= \frac{1}{1}\int_0^1 x^2 dx = \frac{1}{3},\\
a_n &= \frac{2}{1}\int_0^1 x^2\cos n\pi x dx = \frac{2}{n\pi}\int_0^1 x^2 d\sin n\pi x = -\frac{4}{n\pi}\int_0^1 x\sin n\pi x dx\\
&= \frac{4}{n^2\pi^2}\int_0^1 x d\cos n\pi x = \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\int_0^1 \cos n\pi x dx = (-1)^n\frac{4}{n^2\pi^2},
\end{aligned}
$$

and get

$$x^2 \sim \frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2\pi^2}\cos n\pi x, \quad |x| < 1.$$

The odd function is the periodic extension of $x|x| = \text{sign}(x)x^2$ on $(-1,1)$ with period 2. We have

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 x^2 \sin n\pi x dx = -\frac{2}{n\pi}\int_0^1 x^2 d\cos n\pi x \\ &= -\frac{2}{n\pi}\cos n\pi + \frac{4}{n\pi}\int_0^1 x\cos n\pi x dx = (-1)^{n+1}\frac{2}{n\pi} + \frac{4}{n^2\pi^2}\int_0^1 x d\sin n\pi x \\ &= (-1)^{n+1}\frac{2}{n\pi} - \frac{4}{n^2\pi^2}\int_0^1 \sin n\pi x dx = (-1)^{n+1}\frac{2}{n\pi} + \frac{4}{n^3\pi^3}((-1)^n - 1). \end{aligned}$$

and get

$$x|x| \sim \sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{2}{n\pi} + \frac{4}{n^3\pi^3}((-1)^n - 1)\right]\sin n\pi x, \quad |x| < 1.$$

Exercise 4.5.4

Suppose

$$f(x) \sim a_0 + \sum\left(a_n \cos\frac{2n\pi}{p}x + b_n \sin\frac{2n\pi}{p}x\right), \quad x \in (0,p).$$

(2) We have

$$af(x) \sim aa_0 + \sum\left(aa_n \cos\frac{2n\pi}{p}x + ab_n \sin\frac{2n\pi}{p}x\right), \quad x \in (0,p).$$

(3) We have

$$f(ax) \sim a_0 + \sum\left(a_n \cos\frac{2n\pi}{p}ax + b_n \sin\frac{2n\pi}{p}ax\right), \quad ax \in (0,p).$$

This is the same as

$$f(ax) \sim a_0 + \sum\left(a_n \cos\frac{2na\pi}{p}x + b_n \sin\frac{2na\pi}{p}x\right), \quad x \in \left(0,\frac{p}{a}\right).$$

(4) Let $\theta = \frac{2a\pi}{p}$. Then we have (again for $x \in (0,p)$)

$$\begin{aligned} f(x+a) &\sim a_0 + \sum\left(a_n \cos\frac{2n\pi}{p}(x+a) + b_n \sin\frac{2n\pi}{p}(x+a)\right) \\ &\sim a_0 + \sum\Bigg(a_n\left[\cos\frac{2n\pi}{p}x\cos n\theta - \sin\frac{2n\pi}{p}x\sin n\theta\right] \\ &\quad + b_n\left[\sin\frac{2n\pi}{p}x\cos n\theta + \cos\frac{2n\pi}{p}x\sin n\theta\right]\Bigg) \\ &\sim a_0 + \sum\left((a_n\cos n\theta + b_n\sin n\theta)\cos\frac{2n\pi}{p}x + (b_n\cos n\theta - a_n\sin n\theta)\sin\frac{2n\pi}{p}x\right). \end{aligned}$$

Exercise 4.5.5 (1)

By

$$x \sim \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi x, \quad x \in (0,1),$$

we have

$$\frac{x}{p} \sim \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi\frac{x}{p}, \quad \frac{x}{p} \in (0,1).$$

Therefore

$$x \sim \frac{p}{2} - \frac{p}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi}{n}x, \quad x \in (0,p).$$

Exercise 4.5.8

We have

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_0^{2\pi} e^{\frac{ix}{2}}e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i\left(\frac{1}{2}-n\right)x}dx = \frac{1}{2\pi i\left(\frac{1}{2}-n\right)}\ e^{i\left(\frac{1}{2}-n\right)x}\Big|_0^{2\pi} \\ &= \frac{1}{\pi i(1-2n)}(e^{i(1-2n)\pi}-e^0) = \frac{1}{\pi i(1-2n)}(-1-1) = \frac{-2i}{\pi(2n-1)}. \end{aligned}$$

Therefore

$$e^{\frac{ix}{2}} \sim \sum_{n=-\infty}^{\infty}\frac{-2i}{\pi(2n-1)}e^{inx} = \sum_{n=-\infty}^{\infty}\frac{2}{\pi(2n-1)}(-i\cos nx + \sin nx), \quad x \in (0,2\pi).$$

Taking the real and imaginary parts, we get (for $x \in (0,2\pi)$)

$$\begin{aligned} \cos\frac{x}{2} &\sim \sum_{n=-\infty}^{\infty}\frac{2}{\pi(2n-1)}\sin nx = \sum_{n=1}^{\infty}\left(\frac{2}{\pi(2n-1)}\sin nx + \frac{2}{\pi(2(-n)-1)}\sin(-n)x\right) \\ &= \sum_{n=1}^{\infty}\frac{8n}{\pi(4n^2-1)}\sin nx, \\ \sin\frac{x}{2} &\sim \sum_{n=-\infty}^{\infty}\frac{-2}{\pi(2n-1)}\cos nx \\ &= \frac{-2}{\pi} + \sum_{n=1}^{\infty}\left(\frac{-2}{\pi(2n-1)}\cos nx + \frac{-2}{\pi(2(-n)-1)}\cos(-n)x\right) \\ &= \frac{-2}{\pi} + \sum_{n=1}^{\infty}\frac{-4}{\pi(4n^2-1)}\cos nx. \end{aligned}$$

Substituting $2x$ for x, we get

$$\cos x \sim \sum_{n=1}^{\infty}\frac{8n}{\pi(4n^2-1)}\sin 2nx, \quad \sin x \sim -\frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{-4}{\pi(4n^2-1)}\cos 2nx, \quad x \in (0,\pi).$$

Exercise 4.5.11 (3)

Using $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, we get

$$\begin{aligned}\log(1-2a\cos x+a^2) &= \log(1-a(e^{ix}+e^{-ix})+a^2) = \log(1-ae^{ix})(1-ae^{-ix})\\ &= \log(1-ae^{ix})+\log(1-ae^{-ix})\\ &= -\sum_{n=1}^{\infty}\frac{(ae^{ix})^n}{n} - \sum_{n=1}^{\infty}\frac{(ae^{-ix})^n}{n}\\ &= -\sum_{n=1}^{\infty}\frac{a^n}{n}(e^{inx}+e^{-inx}) = -\sum_{n=1}^{\infty}\frac{2a^n}{n}\cos nx.\end{aligned}$$

Exercise 4.5.15

From Example 4.5.3, we have

$$x^2 \sim \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}\cos 2n\pi x - \frac{1}{n\pi}\sin 2n\pi x\right), \quad x \in (0,1).$$

Evaluating at $x = 0$, we get

$$\frac{1}{2}(0^2+1^2) = \frac{1}{3} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2}1 - \frac{1}{n\pi}0\right).$$

This gives $\sum_{n=1}^{\infty}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.