Math1024 Answer to Homework 3

EXERCISE 3.4.2 (2)

$\frac{d}{dx}\int_1^{x^2}\log(1+t^2)dt = 2x\log(1+x^4)$

EXERCISE 3.4.2 (6)

$$\begin{aligned}\frac{d}{dx}\int_{\tan x}^{\cot x}(1+t^2)^{\frac{3}{2}}dt &= \frac{d}{dx}\left[\int_0^{\cot x}(1+t^2)^{\frac{3}{2}}dt - \int_0^{\tan x}(1+t^2)^{\frac{3}{2}}dt\right]\\ &= -\frac{1}{\sin^2 x}(1+\cot^2 x)^{\frac{3}{2}} - \frac{1}{\cos^2 x}(1+\tan^2 x)^{\frac{3}{2}}\\ &= -\sin^{-5}x - \cos^{-5}x\end{aligned}$$

EXERCISE 3.4.3 (6)

$$\begin{aligned}\frac{d}{dx}\int_{\sin x}^{\cos x}f(t)dt &= \frac{d}{dx}\left[\int_0^{\cos x}f(t)dt - \int_0^{\sin x}f(t)dt\right]\\ &= -\sin x f(\cos x) - \cos x f(\sin x)\end{aligned}$$

EXERCISE 3.4.3 (7)

$\frac{d}{dx}\int_0^{f(x)}f(t)dt = f'(x)f[f(x)]$

EXERCISE 3.4.3 (8)

$\frac{d}{dx}\int_0^{f^{-1}(x)}f(t)dt = \frac{1}{f'(f^{-1}(x))}f[f^{-1}(x)] = \frac{x}{f'(f^{-1}(x))}$

EXERCISE 3.4.4 (3)

Let $f(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. Then

$$\begin{aligned}f'(x) &= \frac{2}{\sqrt{\pi}}e^{-x^2} > 0\\ f''(x) &= -\frac{4x}{\sqrt{\pi}}e^{-x^2}\end{aligned}$$

$f(x)$ is an increasing function, so there is no local extrema. $x=0$ is the inflection point, when $x<0$, $f(x)$ is convex and when $x>0$,$f(x)$ is concave.

EXERCISE 3.4.5 (1)

Differentiate both sides, we have $f(x) = -f(x)$, the $f(x) \equiv 0$.

Exercise 3.4.5 (2)

Differentiate both sides, we have

$$Axf(x) = xf(x) + \int_0^x f(t)dt. \qquad (*)$$

If $A = 1$, then

$$\int_0^x f(t)dt = 0.$$

Differentiate it, then $f(x) \equiv 0$.

If $A \neq 1$,

$$f(x) = \frac{\int_0^x f(t)dt}{(A-1)x},$$

so $f(x)$ is differentiable on $(0, \infty)$. Differentiate $(*)$,

$$(A-1)xf'(x) = (2-A)f(x).$$

If $A = 2$, then $f'(x) \equiv 0 \Rightarrow f(x) \equiv C$, here C is a constant. We need to plug the constant function into the equation of this question, then we have $AC = 2C$, but $A = 2$, so $f(x)$ can be an arbitrary constant function.

If $A \neq 2$, then let $k \triangleq \frac{2-A}{A-1}$,

$$\frac{f'(x)}{f(x)} = \frac{k}{x}.$$

Remember that

$$((\log f(x))' = \frac{f'(x)}{f(x)},$$

therefore

$$(\log f(x))' = k(\log x)'$$

and we get $f(x) = Cx^k$, where C is a constant. Again plug this function into the equation of the question, then C can be an arbitrary constant.

Exercise 3.4.6 (2)

By l'Hospital rule

$$\begin{aligned}
\lim_{x \to 0} \frac{1}{x^3} \int_0^x \sin t^2 dt &= \lim_{x \to 0} \frac{\sin x^2}{3x^2} \\
&= \lim_{x \to 0} \frac{2x \cos x^2}{6x} \\
&= \lim_{x \to 0} \frac{\cos x^2}{3} \\
&= \frac{1}{3}
\end{aligned}$$

EXERCISE 3.4.8 (3)

$$\lim_{x\to 0}(x^2 \sin\frac{1}{x})' = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right) = \lim_{x\to 0}\cos\frac{1}{x},$$

so the limit does not exist. $f(x)$ is not continuous at $x = 0$. And

$$\int_0^x f(t)dt = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

which is differentiable, since

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

EXERCISE 3.4.9 (1)

$$\int \sqrt[4]{1-x}dx = -\frac{4}{5}(1-x)^{\frac{5}{4}} + C$$

EXERCISE 3.1.11 (1)

The function $f(-x)$ would take opposite values of the independent variable x over [-a,-b], thus the graph of $f(-x)$ over [-a,-b] would be the same as graph of $f(x)$ over [a,b]. In other words, the area $\int_{-b}^{-a} f(-x)dx$ just equals to the area $\int_a^b f(x)dx$.

EXERCISE 3.4.10 (4)(6)

You can check it by differentiating the right hand side. Here I show how to go from the left hand to the right hand side, pretending that we don't know the shape of the right hand side.

$$\begin{aligned}\int \sqrt{a^2 - x^2} &\overset{x:=a\sin t}{=\!=\!=} a^2 \int \cos^2 t dt \\ &= a^2 \int \frac{1+\cos 2t}{2} \\ &= a^2\left(\frac{t}{2} + \frac{\sin 2t}{4} + C\right) \\ &\overset{t:=\arcsin\frac{x}{a}}{=\!=\!=} \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2-x^2} + C\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+a}} &\xlongequal{\sqrt{x^2+a}:=t-x} \int \frac{d\left(\frac{t^2-a}{2t}\right)}{\frac{t^2+a}{2t}} \\ &= \int \frac{dt}{t} \\ &= \log|t| + C \\ &\xlongequal{t:=x+\sqrt{x^2+a}} \log|x+\sqrt{x^2+a}| + C\end{aligned}$$

Exercise 3.4.11 (4)

If $\int f(x)dx = F(x) + C$, then $F'(x) = f(x)$, so that $F(ax+b)' = F'(ax+b)\cdot(ax+b)' = af(ax+b)$. Therefore

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C.$$

By Example 3.4.8, we have

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$$

Then we get

$$\int \frac{dx}{\sqrt{1-(x-1)^2}} = \arcsin(x-1) + C.$$

Exercise 3.4.11 (5)

As

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dx}{\sqrt{\frac{1}{2^2}-\left(x-\frac{1}{2}\right)^2}}.$$

For $a > 0$, we have

$$\int \frac{dx}{\sqrt{a^2-(bx+c)^2}} = \int \frac{dx}{a\sqrt{1-\left(\frac{bx+c}{a}\right)^2}} = \frac{1}{a}\frac{a}{b}\arcsin\frac{bx+c}{a} + C = \frac{1}{b}\arcsin\frac{bx+c}{a} + C.$$

Therefore

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{dx}{\sqrt{\frac{1}{2^2}-\left(x-\frac{1}{2}\right)^2}} = \arcsin\frac{x-\frac{1}{2}}{\frac{1}{2}} + C = \arcsin(2x-1) + C.$$

Exercise 3.4.14 (1)

By $\int x^2dx = \frac{1}{3}x^3 + C$ and $\int \sin x dx = -\cos x + C$.

Besides the constant C here should be the same, we also require $\int f(x)dx = 1 - \cos x$ when $x > 0$ because $\frac{1}{3}x^3|_{x=0} = 0$. So we have

$$\int f(x)dx = \begin{cases} \frac{1}{3}x^3, & \text{if} x \le 0 \\ 1 - \cos x, & \text{if } x > 0 \end{cases} + C.$$

EXERCISE 3.5.1 (3)

When $p \neq -1, -2, -3$:

$$\begin{aligned}
\int x^2(ax+b)^p dx &= \int \frac{1}{a^2}((ax+b)^2 - 2abx - b^2)(ax+b)^p dx \\
&= \int (\frac{1}{a^2}(ax+b)^{p+2} - \frac{b^2}{a^2}(ax+b)^p - \frac{2b}{a^2}(ax+b-b)(ax+b)^p)dx \\
&= \frac{1}{a^2}\int (ax+b)^{p+2}dx - \frac{2b}{a^2}\int (ax+b)^{p+1}dx + \frac{b^2}{a^2}\int (ax+b)^p dx \\
&= \frac{1}{a^2(p+3)}(ax+b)^{p+3} - \frac{2b}{a^2(p+2)}(ax+b)^{p+2} + \frac{b^2}{a^2(p+1)}(ax+b)^{p+1} + C.
\end{aligned}$$

Then for $p = -1, -2, -3$:

$$\begin{aligned}
\int x^2(ax+b)^{-1}dx &= \frac{1}{a^2}\int (ax+b)dx - \frac{2b}{a^2}\int 1dx + \frac{b^2}{a^2}\int (ax+b)^{-1}dx \\
&= \frac{1}{2a^2}(ax+b)^2 - \frac{2b}{a^2}x + \frac{b^2}{a^3}\log|ax+b| + C
\end{aligned}$$

$$\begin{aligned}
\int x^2(ax+b)^{-2}dx &= \frac{1}{a^2}\int 1dx - \frac{2b}{a^2}\int (ax+b)^{-1}dx + \frac{b^2}{a^2}\int (ax+b)^{-2}dx \\
&= \frac{1}{a^2}x - \frac{2b}{a^3}\log|ax+b| - \frac{b^2}{a^3}(ax+b)^{-1} + C.
\end{aligned}$$

$$\begin{aligned}
\int x^2(ax+b)^{-3}dx &= \frac{1}{a^2}\int (ax+b)^{-1}dx - \frac{2b}{a^2}\int (ax+b)^{-2}dx + \frac{b^2}{a^2}\int (ax+b)^{-3}dx \\
&= \frac{1}{a^3}\log|ax+b| + \frac{2b}{a^3}(ax+b)^{-1} - \frac{b^2}{2a^3}(ax+b)^{-2} + C.
\end{aligned}$$

EXERCISE 3.5.1 (7)

$$\begin{aligned}
\int \frac{x^2-x+1}{(x+1)^{10}}dx &= \int \frac{(x+1)^2 - 3(x+1) + 3}{(x+1)^{10}}dx \\
&= \int \frac{1}{(x+1)^8}dx - 3\int \frac{1}{(x+1)^9}dx + 3\int \frac{1}{(x+1)^{10}}dx \\
&= -\frac{1}{7(x+1)^7} + \frac{3}{8(x+1)^8} - \frac{1}{3(x+1)^9} + C.
\end{aligned}$$

EXERCISE 3.5.1 (8)

$$\begin{aligned}\int \frac{x-1}{\sqrt{x}}dx &= \int \sqrt{x} - \frac{1}{\sqrt{x}}dx \\ &= \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + C.\end{aligned}$$

EXERCISE 3.5.3 (2)

$$\int (2^x + 3^x)^2 dx = \int (4^x + 2\cdot 6^x + 9^x)dx = \frac{4^x}{\log 4} + 2\frac{6^x}{\log 6} + \frac{9^x}{\log 9} + C.$$

EXERCISE 3.5.4 (6)

$$\begin{aligned}\int \frac{x^2}{x^2+3x+2}dx &= \int \frac{x^2-1+1}{(x+1)(x+2)}dx \\ &= \int \frac{x-1}{x+2}dx + \int \frac{1}{(x+1)(x+2)}dx \\ &= \int (1 - \frac{3}{x+2})dx + \int (\frac{1}{x+1} - \frac{1}{x+2})dx \\ &= x - 4\log|x+2| + \log|x+1| + C.\end{aligned}$$

EXERCISE 3.5.4 (8)

$$\begin{aligned}\int \frac{dx}{(x^2+1)(x^2+4)} &= \int \frac{1}{3}(\frac{1}{x^2+1} - \frac{1}{x^2+4})dx \\ &= \frac{1}{3}\text{arctan}x - \frac{1}{6}\text{arctan}\frac{x}{2} + C.\end{aligned}$$

EXERCISE 3.5.6 (8)

$$\begin{aligned}\int_0^{\pi} |\sin x \cos 2x| dx &= \int_0^{\frac{\pi}{4}} \sin x \cos 2x dx - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \cos 2x dx + \int_{\frac{3\pi}{4}}^{\pi} \sin x \cos 2x dx \\ &= \frac{1}{2}(\int_0^{\frac{\pi}{4}} (\sin 3x - \sin x)dx - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin 3x - \sin x)dx + \int_{\frac{3\pi}{4}}^{\pi} (\sin 3x - \sin x)dx) \\ &= \frac{2}{6} + \frac{1}{2}(4\sqrt{2} - 2) = 2\sqrt{2} - \frac{2}{3}\end{aligned}$$

EXERCISE 3.5.7 (3)

Now the function is $f(x) = \int_{-x^2}^{0} \frac{e^t - 1}{t}dt$.

According to the 3-th order Taylor expansion of $e^t - 1$ at 0 which is $P(t) = t + \frac{1}{2}t^2 + \frac{1}{6}t^3$, then for any $\epsilon > 0$, there is $\delta > 0$ such that

$$|x| < \sqrt{\delta} \to |t| < \delta \to |e^t - 1 - P(t)| \leq \epsilon |t|^3$$

Therefore we have

$$\begin{aligned} |f(x) - \int_{-x^2}^{0} (1 + \frac{1}{2}t + \frac{1}{6}t^2) dt| &= |\int_{-x^2}^{0} (\frac{e^t - 1}{t} - \frac{P(t)}{t}) dt| \\ &\leq \epsilon \int_{-x^2}^{0} |t|^2 dt = \frac{\epsilon}{3} x^6 \end{aligned}$$

which means

$$f(x) = x^2 - \frac{1}{4}x^4 + \frac{1}{18}x^6 + o(x^6).$$

This just shows that the integration of 5-th order approximation is 6-th order approximation and the above polynomial is the high order approximation of $f(x)$ at 0.

Exercise 3.5.9

As usual, we start with the partition that evenly divides the interval

$$a = x_0 < x_1 = a + h < \cdots < x_k = a + kh < \cdots < x_n = b = a + nh.$$

where $h = \frac{b-a}{n}$.

According to the estimation in Example 3.5.8, we use the linear approximation at the middle point $c_k = \frac{x_k + x_{k+1}}{2}$ with the Lagrange form of the remainder to estimate $f(x)$ on $[x_k, x_{k+1}]$ which has interval length $\frac{b-a}{n}$.

Then we would have the following error bound for $f(x)$ on $[x_k, x_{k+1}]$ if $f''(x)$ is bounded by K_2:

$$\Big| \int_{x_k}^{x_{k+1}} f(x) dx - f(c_k)(\frac{b-a}{n}) \Big| \leq \frac{K_2}{24} (\frac{b-a}{n})^3$$

Thus

$$\begin{aligned} \Big| \int_a^b f(x) dx - M_n \Big| &= \Big| \sum_{k=0}^{k=n-1} [\int_{x_k}^{x_{k+1}} f(x) dx - f(c_k)(\frac{b-a}{n})] \Big| \\ &\leq \sum_{k=0}^{k=n-1} \Big| \int_{x_k}^{x_{k+1}} f(x) dx - f(c_k)(\frac{b-a}{n}) \Big| \\ &\leq n \frac{K_2}{24} (\frac{b-a}{n})^3 \\ &= \frac{K_2}{24} \frac{(b-a)^3}{n^2} \end{aligned}$$