Math1024 Answer to Homework 4

EXERCISE 3.5.10 (3)

When $p \neq -2, -1$, then

$$\begin{aligned}\int (x-1)(x+1)^p dx &= \int [(x+1)^{p+1} - 2(x+1)^p] dx \\ &= \frac{1}{p+2}(x+1)^{p+2} - \frac{2}{p+1}(x+1)^{p+1} + C.\end{aligned}$$

When $p = -2, -1$, we have

$$\int (x-1)(x+1)^{-2} dx = \log|x+1| + \frac{2}{x+1} + C,$$
$$\int (x-1)(x+1)^{-1} dx = x - 2\log|x+1| + C.$$

EXERCISE 3.5.11 (2)

$$\begin{aligned}\int (x+a^x)^2 dx &= \int (x^2 + 2xa^x + a^{2x}) dx \\ &= \frac{1}{3}x^3 + \frac{a^{2x}}{2\log a} + 2\int xa^x dx \\ &= \frac{1}{3}x^3 + \frac{a^{2x}}{2\log a} + \frac{2}{\log a}\int x da^x \\ &= \frac{1}{3}x^3 + \frac{a^{2x}}{2\log a} + \frac{2}{\log a} xa^x - \frac{2}{\log a}\int a^x dx \\ &= \frac{1}{3}x^3 + \frac{a^{2x}}{2\log a} + \frac{2}{\log a} xa^x - \frac{2}{(\log a)^2} a^x + C.\end{aligned}$$

EXERCISE 3.5.11 (3)

$$\begin{aligned}\int \frac{xe^x dx}{(x+1)^2} &= -\int xe^x d\frac{1}{x+1} = -\frac{1}{x+1}xe^x + \int \frac{1}{x+1} d(xe^x) \\ &= -\frac{1}{x+1}xe^x + \int \frac{1}{x+1}(x+1)e^x dx = -\frac{1}{x+1}xe^x + e^x + C = \frac{e^x}{x+1} + C.\end{aligned}$$

EXERCISE 3.5.14

Using integration by parts, we know

$$\int_0^1 x^n a^x dx = \frac{1}{\log a}\int_0^1 x^n da^x = \frac{1}{\log a}(a - \int_0^1 a^x n x^{n-1} dx) = \frac{a}{\log a} - \frac{n}{\log a}\int_0^1 x^{n-1} a^x dx$$

Denote $b_n = \int_0^1 x^n a^x dx$, Then

$$b_n = \frac{a}{\log a} - \frac{n}{\log a} b_{n-1}$$

Thus we have,

$$b_n = \frac{a}{\log a} + \sum_{k=1}^{n-1} (-1)^k \frac{n \cdots (n-k+1)}{(\log a)^{k+1}} a + (-1)^n \frac{n!}{(\log a)^n} b_0, \quad n \geq 1$$

where $b_0 = a/\log a - 1/\log a$.

EXERCISE 3.5.16 (3)

$$\begin{aligned}
\int \Big(\frac{\log(x+a)}{x+b} + \frac{\log(x+b)}{x+a}\Big) dx &= \int \log(x+a) d(\log|x+b|) + \int \frac{\log(x+b)}{x+a} dx \\
&= \log(x+a)\log(x+b) - \int \frac{\log(x+b)}{x+a} dx + \int \frac{\log(x+b)}{x+a} dx \\
&= \log(x+a)\log(x+b) + C.
\end{aligned}$$

EXERCISE 3.5.16 (6)

$$\begin{aligned}
\int \left(\log(x+\sqrt{1+x^2})\right)^2 dx &= x\left(\log(x+\sqrt{1+x^2})\right)^2 - \int 2x[\log(x+\sqrt{1+x^2})] \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} dx \\
&= x\left(\log(x+\sqrt{1+x^2})\right)^2 - 2\int \frac{x\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} dx \\
&= x\left(\log(x+\sqrt{1+x^2})\right)^2 - 2\int \log(x+\sqrt{1+x^2}) d\sqrt{1+x^2} \\
&= x\left(\log(x+\sqrt{1+x^2})\right)^2 - 2\sqrt{1+x^2}\log(x+\sqrt{1+x^2}) + 2x + C.
\end{aligned}$$

EXERCISE 3.5.17

(1) From integration by parts,

$$\begin{aligned}
\int_a^b ((x+A)^2 + B) df'(x) &= \Big(f'(x)((x+A)^2+B)\Big)\Big|_a^b - \int_a^b 2(x+A)f'(x)dx \\
&= \Big(f'(x)((x+A)^2+B) - 2(x+A)f(x)\Big)_a^b + 2\int_a^b f(x)dx
\end{aligned}$$

Then by removing terms, we have the stated result.

(2)First, we find A such that

$$(b+A)f(b) - (a+A)f(a) = f(a)\frac{b-a}{2} + f(b)\frac{b-a}{2},$$

we deduce $A = -(a+b)/2$.
Then we take $B = -(b-a)^2/4$ to make

$$f'(x)((x+A)^2+B)|_a^b = 0.$$

With these values, we get

$$\int_a^b f(x)dx = \frac{f(a)+f(b)}{2}(b-a) + \frac{1}{2}\int_a^b ((x+A)^2+B)f''(x)dx.$$

Direct computation shows that

$$\int_a^b |(x+A)^2+B|dx = -\int_a^b [(x+A)^2+B]dx = \frac{(b-a)^3}{6}$$

(3) From the result of (2),

$$\begin{aligned}
|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)| &= |\frac{1}{2}\int_a^b ((x+A)^2+B)f''(x)dx| \\
&\le \frac{1}{2}\sup_{x\in(a,b)} f''(x)\int_a^b |(x+A)^2+B|dx \\
&= \sup_{x\in(a,b)} f''(x)\frac{(b-a)^3}{12}
\end{aligned}$$

Thus we can get the error formula for the trapezoidal rule in Theorem 3.3.1 by applying this error bound to n sub intervals with the same interval length $(b-a)/n$.

Exercise 3.5.23 (1)

$$\begin{aligned}
\int x^2\sqrt{a^2-x^2}dx &= \int -\frac{1}{3}xd(a^2-x^2)^{\frac{3}{2}} \\
&= -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} + \int \frac{1}{3}(a^2-x^2)^{\frac{3}{2}}dx \\
&= -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} - \frac{1}{8}x(2x^2-5a^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a} + C.
\end{aligned}$$

Exercise 3.5.23 (2)

$$\begin{aligned}
\int x\sqrt{a^2-x^2}dx &= \int -\frac{1}{3}d(a^2-x^2)^{\frac{3}{2}} \\
&= -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}} + C.
\end{aligned}$$

Exercise 3.5.26 (1)

Let $y = 2x - 1$. Then $x = (y+1)/2$ and

$$\begin{aligned}\int (x^2+1)(2x-1)^{10}dx &= \int (\frac{(y+1)^2}{4}+1)y^{10}\frac{1}{2}dy \\ &= \frac{1}{2}\int (\frac{1}{4}y^{12}+\frac{1}{2}y^{11}+\frac{5}{4}y^{10})dy \\ &= \frac{1}{104}y^{13}+\frac{1}{192}y^{12}+\frac{5}{352}y^{11}+C.\end{aligned}$$

EXERCISE 3.5.27 (2)

$$\begin{aligned}\int \frac{bx+c}{x^2+a^2}dx &= \int \frac{b}{2}\frac{1}{x^2+1}d(x^2+1)+\int \frac{c}{x^2+a^2}dx \\ &= \frac{b}{2}\log(x^2+1)+\frac{c}{a}\arctan\frac{x}{a}+C.\end{aligned}$$

EXERCISE 3.5.27 (6)

$$\begin{aligned}\int \frac{x^3}{\sqrt[3]{x^2+a^2}}dx &= \int \frac{3}{4}x^2 d(x^2+a^2)^{\frac{2}{3}} \\ &= \frac{3}{4}x^2(x^2+a^2)^{\frac{2}{3}} - \frac{3}{4}\int \frac{3}{5}d(x^2+a^2)^{\frac{5}{3}} \\ &= \frac{3}{4}x^2(x^2+a^2)^{\frac{2}{3}} - \frac{9}{20}x^2(x^2+a^2)^{\frac{5}{3}}+C.\end{aligned}$$

EXERCISE 3.5.28 (6)

$$\begin{aligned}\int \sin(\log x)dx &=_{y=\log x} \int e^y \sin y dy \\ &= \frac{1}{2}e^y(\sin y - \cos y)+C \\ &= \frac{1}{2}x(\sin(\log x)-\cos(\log x))+C.\end{aligned}$$

EXERCISE 3.5.28 (9)

$$\begin{aligned}\int \sin 2x\sqrt{a+\cos^2 x}dx &= -\int \sqrt{a+\cos^2 x}d(a+\cos^2 x) \\ &= -\frac{2}{3}(a+\cos^2 x)^{\frac{3}{2}}+C.\end{aligned}$$

Exercise 3.5.29 (9)

$$\begin{aligned}\int (\arccos x)^2 dx &= \int y^2 d(\cos y) = y^2\cos y - 2\int y\cos y dy = y^2\cos y - 2\int y d(\sin y)\\ &= y^2\cos y - 2y\sin y + 2\int \sin y dy = y^2\cos y - 2y\sin y - 2\cos y + C\\ &= x((\arccos x)^2 - 2) - 2\sqrt{1-x^2}\arccos x + C.\end{aligned}$$

Exercise 3.5.30 (2)

$$\int e^{\sqrt{x}}dx = \int e^y 2ydy = 2\int yde^y = 2ye^y - 2\int e^y dy = 2ye^y - 2e^y + C = 2(\sqrt{x}-1)e^{\sqrt{x}} + C.$$

Exercise 3.5.30 (9)

Let $y = \sqrt{e^x + a}$. Then $x = \log(y^2 - a)$, $dx = \dfrac{2ydy}{y^2 - a}$, and

$$\begin{aligned}\int \sqrt{e^x + a}dx &= \int y\frac{2ydy}{y^2 - a} = \int \left(2 + \frac{2a}{y^2 - a}\right) dy\\ &= \begin{cases} 2y + \sqrt{a}\log\left|\dfrac{y-\sqrt{a}}{y+\sqrt{a}}\right| + C, & \text{if } a > 0\\ 2y + 2\sqrt{-a}\arctan\dfrac{y}{\sqrt{-a}} + C, & \text{if } a < 0\\ 2y - \dfrac{2a}{y} + C, & \text{if } a = 0\end{cases}\\ &= \begin{cases} 2\sqrt{e^x + a} + \sqrt{a}\log\left|\dfrac{\sqrt{e^x+a}-\sqrt{a}}{\sqrt{e^x+a}+\sqrt{a}}\right| + C, & \text{if } a > 0\\ 2\sqrt{e^x + a} + 2\sqrt{-a}\arctan\dfrac{\sqrt{e^x+a}}{\sqrt{-a}} + C, & \text{if } a < 0\\ 2\sqrt{e^x} + C, & \text{if } a = 0\end{cases}\\ &= \begin{cases} 2\sqrt{e^x + a} + \sqrt{a}x - 2\sqrt{a}\log(\sqrt{e^x+a}+\sqrt{a}) + C, & \text{if } a > 0\\ 2\sqrt{e^x + a} + 2\sqrt{-a}\arctan\sqrt{-\dfrac{e^x}{a} - 1} + C, & \text{if } a < 0\\ 2e^{\frac{1}{2}x} + C, & \text{if } a = 0\end{cases}\end{aligned}$$

Exercise 3.5.33 (3)

$$\int 2^{f(x)}f'(x)dx = \int 2^{f(x)}df(x) = \frac{2^{f(x)}}{\log 2} + C.$$

Exercise 3.5.34 (2)

Let $x := \pi - y$, then

$$\begin{aligned} I = \int_0^\pi x f(\sin x) dx &= \int_\pi^0 (\pi - y) f(\sin(\pi - y)) d(\pi - y) \\ &= \int_0^\pi (\pi - y) f(\sin y) dy \\ &= \pi \int_0^\pi f(\sin y) dy - \int_0^\pi y f(\sin y) dy \\ &= \pi \int_0^\pi f(\sin y) dy - I \end{aligned}$$

so

$$I = \frac{\pi}{2} \int_0^\pi f(\sin y) dy$$

Exercise 3.5.35

Because $\phi(x) = \frac{1}{x}$ is not differentiable at $0 \in (-1, 1)$.

Without change of variable,

$$\int_{-1}^1 \frac{dx}{1 + x^2} = \arctan x|_{-1}^1 = \frac{\pi}{2}.$$

But if you insist on using the change of variable $y = \frac{1}{x}$, then

$$\int_{-1}^1 \frac{-\frac{1}{y^2}}{1 + \frac{1}{y^2}} dy = -\int_{-1}^1 \frac{dy}{1 + y^2} = -\frac{\pi}{2},$$

which is a wrong answer.

Exercise 3.5.36

$$\begin{aligned} \int_a^b f(x + t) dx &= \int_a^b f(x + t) d(x + t) \\ &= \int_{a+t}^{b+t} f(y) dy \\ &= \int_0^{b+t} f(y) dy - \int_0^{a+t} f(y) dy \end{aligned}$$

then

$$\frac{d}{dt} \int_a^b f(x + t) dx = f(b + t) - f(a + t)$$

Exercise 3.5.38

If $f(x)$ is even, then $f(-x) = f(x)$.

$$
\begin{aligned}
\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\
&= -\int_{-a}^{0} f(-x)d(-x) + \int_{0}^{a} f(x)dx \\
&= -\int_{a}^{0} f(y)dy + \int_{0}^{a} f(x)dx \\
&= \int_{0}^{a} f(y)dy + \int_{0}^{a} f(x)dx.
\end{aligned}
$$

If $f(x)$ is odd, then $f(-x) = -f(x)$.

$$
\begin{aligned}
\int_{-a}^{a} f(x)dx &= \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \\
&= \int_{-a}^{0} -f(x)d(-x) + \int_{0}^{a} f(x)dx \\
&= \int_{-a}^{0} f(-x)d(-x) + \int_{0}^{a} f(x)dx \\
&= \int_{a}^{0} f(y)dy + \int_{0}^{a} f(x)dx \\
&= -\int_{0}^{a} f(y)dy + \int_{0}^{a} f(x)dx = 0.
\end{aligned}
$$

EXERCISE 3.5.40 (1)

If $p \neq -1, -2$,

$$\begin{aligned}
\int (1+\sqrt{x})^p dx &= \int (1+y)^p dy^2 \\
&= \int 2y(1+y)^p dy \\
&= \int 2yd\frac{(1+y)^{p+1}}{p+1} \\
&= \frac{2y(1+y)^{p+1}}{p+1} - 2\int \frac{(1+y)^{p+1}}{p+1} d(y+1) \\
&= \frac{2}{p+1}(1+y)^{p+1}\left(\frac{p+1}{p+2}y - \frac{1}{p+2}\right) + C \\
&= \frac{2}{(p+1)(p+2)}(1+y)^{p+1}[(p+1)y-1] + C \\
&= \frac{2}{(p+1)(p+2)}(1+\sqrt{x})^{p+1}[(p+1)\sqrt{x}-1] + C
\end{aligned}$$

If $p = -2$,

$$\begin{aligned}
\int (1+\sqrt{x})^p dx &= \int (1+y)^{-2} dy^2 \\
&= \int 2y(1+y)^{-2} dy \\
&= -\int 2yd(1+y)^{-1} \\
&= -\frac{2y}{1+y} + 2\int \frac{1}{1+y} d(y+1) \\
&= -\frac{2y}{1+y} + 2\log|1+y| + C \\
&= -\frac{2\sqrt{x}}{1+\sqrt{x}} + 2\log|1+\sqrt{x}| + C
\end{aligned}$$

If $p = -1$,

$$\begin{aligned}\int (1+\sqrt{x})^p dx &= \int (1+y)^{-1} dy^2 \\ &= \int 2y(1+y)^{-1} dy \\ &= 2\int \frac{1+y-1}{1+y} dy \\ &= 2\int \left(1 - \frac{1}{1+y}\right) dy \\ &= 2y - 2\log|1+y| + C\end{aligned}$$

Exercise 3.5.42 (5)

Recall that

$$\csc^2 x = 1 + \cot^2 x.$$

So

$$\begin{aligned}\int \cot^6 x \csc^4 x dx &= -\int \cot^6 x \csc^2 x d(\cot^x) \\ &= -\int \cot^6 x(1+\cot^2 x) d(\cot^x) \\ &= -\frac{1}{7}\cot^7 x - \frac{1}{9}\cot^9 x + C.\end{aligned}$$

Exercise 3.5.43 (6)

Recall that $a\sin x + b\cos x = \sin(x+\phi)$, where

$$\cos\phi = \frac{a}{\sqrt{a^2+b^2}}, \sin\phi = \frac{b}{\sqrt{a^2+b^2}}.$$

Then

$$\begin{aligned}\int \frac{dx}{a\sin x + b\cos x} &= \frac{a}{\sqrt{a^2+b^2}} \int \frac{dx}{\sin(x+\phi)} \\ &= \frac{a}{\sqrt{a^2+b^2}} \int \frac{dx}{2\sin\frac{x+\phi}{2}\cos\frac{x+\phi}{2}} \\ &= \frac{a}{\sqrt{a^2+b^2}} \int \frac{dx}{2\tan\frac{x+\phi}{2}\cos^2\frac{x+\phi}{2}} \\ &= \frac{a}{\sqrt{a^2+b^2}} \int \frac{d\tan\frac{x+\phi}{2}}{\tan\frac{x+\phi}{2}} \\ &= \frac{a}{\sqrt{a^2+b^2}} \log\left|\tan\frac{x+\phi}{2}\right| + C\end{aligned}$$

Exercise 3.5.43 (7)

$$\begin{aligned}
\int \frac{1+\sin x}{1+\cos x}dx &= \int \frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2}}{2\cos^2\frac{x}{2}}dx \\
&= \int \left(\frac{1}{2}-\frac{1}{2}\tan^2\frac{x}{2}+\tan\frac{x}{2}\right)dx \\
&= \int \left(\frac{1}{2}-\frac{1}{2}\tan^2\frac{x}{2}+\tan\frac{x}{2}\right)\cos^2\frac{x}{2}\frac{1}{2\cos^2\frac{x}{2}}dx \\
&= \int \left(\frac{1}{2}-\frac{1}{2}\tan^2\frac{x}{2}+\tan\frac{x}{2}\right)\frac{1}{\left(1+\tan^2\frac{x}{2}\right)}\frac{1}{\cos^2\frac{x}{2}}dx \\
&= \int \left(1-\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right)\frac{1}{\left(1+\tan^2\frac{x}{2}\right)}d\tan\frac{x}{2} \\
&= \int \frac{1-t^2+2t}{1+t^2}dt \\
&= \int \left(-1+\frac{2}{1+t^2}+\frac{2t}{1+t^2}\right)dt \\
&= -t+2\arctan t+\log|1+t^2|+C \\
&= -\tan\frac{x}{2}+x+\log\left|1+\tan^2\frac{x}{2}\right|+C
\end{aligned}$$

Exercise 3.5.45

We want to find A, B, C, such that

$$\int \left[\frac{1}{(a+b\cos x)^n}-\frac{B}{(a+b\cos x)^{n-1}}-\frac{C}{(a+b\cos x)^{n-2}}\right]dx = \frac{A\sin x}{(a+b\cos x)^{n-1}}+\text{const.}$$

Differentiate both side, then

$$\frac{1}{(a+b\cos x)^n}-\frac{B}{(a+b\cos x)^{n-1}}-\frac{C}{(a+b\cos x)^{n-2}} = \left[\frac{A\sin x}{(a+b\cos x)^{n-1}}\right]'.$$

Transform all trigonometric function into $\cos x$, we have

$$[(2-n)Ab+Cb^2]\cos^2 x+(Aa+Bb+2abC)\cos x+Ba-1+(n-1)Ab+Ca^2=0.$$

Let $y := \cos x$, then

$$[(2-n)Ab+Cb^2]y^2+(Aa+Bb+2abC)y+Ba-1+(n-1)Ab+Ca^2=0,$$

therefore,

$$\begin{cases} b(2-n)A+b^2C=0 \\ aA+bB+2abC=0 \\ aB+b(n-1)A+a^2C=1. \end{cases}$$

If $b \neq 0$, then you will find that $n \neq 1$ and

$$A = \frac{b^2}{(b^2-a^2)(n-1)}, B = \frac{(3-2n)ab}{(b^2-a^2)(n-1)}, C = \frac{(n-2)b}{(b^2-a^2)(n-1)}.$$

If $b = 0$, then $a \neq 0$, since $|a| \neq |b|$. So $A = 0$ and $aB + a^2C = 1$. That is, for all B and C satisfying $aB + a^2C = 1$, we still have the required equation. You can directly check this situation by letting $b = 0$ in the required equation.

EXERCISE 3.5.47 (3)

$$\begin{aligned}
\int \frac{dx}{\sin x - \sin a} &= \frac{1}{\cos a}\int \frac{\cos\left(\frac{x+a}{2} - \frac{x-a}{2}\right)}{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}dx \\
&= \frac{1}{\cos a}\int \frac{\cos\frac{x+a}{2}\cos\frac{x-a}{2} + \sin\frac{x+a}{2}\sin\frac{x-a}{2}}{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}dx \\
&= \frac{1}{2\cos a}\int \left(\frac{\cos\frac{x-a}{2}}{\sin\frac{x-a}{2}} + \frac{\sin\frac{x+a}{2}}{\cos\frac{x+a}{2}}\right)dx \\
&= \frac{1}{2\cos a}\left(\int \frac{\cos\frac{x-a}{2}}{\sin\frac{x-a}{2}}dx + \int \frac{\sin\frac{x+a}{2}}{\cos\frac{x+a}{2}}dx\right) \\
&= \frac{1}{\cos a}\log\left|\frac{\sin\frac{x-a}{2}}{\cos\frac{x+a}{2}}\right| + C
\end{aligned}$$

EXERCISE 3.5.48 (9)

$$\begin{aligned}
\int (x(x+1))^{\frac{3}{2}}dx &= \int \left(\sqrt{x^2+x}\right)^3 dx \\
&= \int \left(\frac{1}{2}\sqrt{(2x+1)^2-1)}\right)^3 dx \\
&= \frac{1}{16}\int (\sqrt{y^2-1})^3 dy \\
&= \frac{1}{16}y(\sqrt{y^2-1})^3 - \frac{3}{64}\int \sqrt{y^2-1}dy^2 \\
&= \frac{1}{16}y(\sqrt{y^2-1})^3 - \frac{3}{64}\log|y^2-1| + C \\
&= \frac{1}{16}(2x+1)(\sqrt{(2x+1)^2-1})^3 - \frac{3}{64}\log|(2x+1)^2-1| + C \\
&= \frac{1}{2}(2x+1)(x^2+x)^{\frac{3}{2}} - \frac{3}{64}\log(4x^2+4x) + C \\
&= \frac{1}{2}(2x+1)(x^2+x)^{\frac{3}{2}} - \frac{3}{64}\log(x^2+x) + C
\end{aligned}$$

Exercise 3.5.49 (2)

$$\begin{aligned}
\int \arctan\sqrt{x}dx &= \int \arctan t dt^2 \\
&= t^2 \arctan t - \int t^2 d(\arctan t) \\
&= t^2 \arctan t - \int \frac{t^2+1-1}{t^2+1} dt \\
&= t^2 \arctan t - \int \left(1 - \frac{1}{t^2+1}\right) dt \\
&= t^2 \arctan t - t + \arctan t + C \\
&= (\sqrt{x}+1)\arctan\sqrt{x} - \sqrt{x} + C
\end{aligned}$$