**Math1024 Answer to Homework 5**

Exercise 3.6.2 (4)

$$\begin{aligned}\int \frac{x^3+3x^2+3x+1}{x^3+1}dx &= \int \left[1+\frac{3x}{x^2-x+1}\right]dx\\ &= x+\frac{3}{2}\int\frac{2x-1+1}{x^2-x+1}dx\\ &= x+\frac{3}{2}\int\frac{2x-1}{x^2-x+1}dx+\frac{3}{2}\int\frac{1}{x^2-x+1}dx\\ &= x+\frac{3}{2}\log|x^2-x+1|+\sqrt{3}\arctan\frac{2x-1}{\sqrt{3}}+C\end{aligned}$$

Exercise 3.6.2 (10)

$$\int\frac{x^2dx}{(x^2+4x+6)^2}=\int\frac{dx}{x^2+4x+6}-\int\frac{(4x+6)dx}{(x^2+4x+6)^2}.$$

Then

$$\begin{aligned}\int\frac{(4x+6)dx}{(x^2+4x+6)^2} &= \int\frac{2(x^2+4x+6)'-2}{(x^2+4x+6)^2}dx\\ &= -\frac{2}{x^2+4x+6}-\int\frac{2}{(x^2+4x+6)^2}dx\end{aligned}$$

Here you can follow Example 3.5.14. But I don't recommend to remember the formula. The key is to reduce $\int\frac{dx}{(x^2+4x+6)^2}$ to $\int\frac{dx}{x^2+4x+6}$ by integration by part.

$$\begin{aligned}\int\frac{dx}{x^2+4x+6} &= \frac{x}{x^2+4x+6}+\int\frac{x(2x+4)}{(x^2+4x+6)^2}dx\\ &= \frac{x}{x^2+4x+6}+\int\frac{2}{x^2+4x+6}dx-\int\frac{2(2x+4)+4}{(x^2+4x+6)^2}dx\\ &= \frac{x}{x^2+4x+6}+\int\frac{2}{x^2+4x+6}dx+\frac{2}{x^2+4x+6}-\int\frac{4}{(x^2+4x+6)^2}dx\end{aligned}$$

Therefore

$$\begin{aligned}\int\frac{x^2dx}{(x^2+4x+6)^2} &= \frac{x-2}{2(x^2+4x+6)}+\frac{1}{2}\int\frac{dx}{x^2+4x+6}\\ &= \frac{x-2}{2(x^2+4x+6)}+\frac{1}{2\sqrt{2}}\arctan\frac{x+2}{\sqrt{2}}+C.\end{aligned}$$

Exercise 3.6.3 (6)

Let $t = \sqrt[4]{\dfrac{a-x}{x}}$. Then $x = \dfrac{a}{t^4+1}$, $dx = -\dfrac{-4at^3dt}{(t^4+1)^2}$, and for $x > 0$,

$$\int \frac{dx}{\sqrt[4]{x^3(a-x)}} = \int \frac{\dfrac{-4at^3dt}{(t^4+1)^2}}{\dfrac{a}{t^4+1}t} = -4\int \frac{t^2dt}{t^4+1} = \sqrt{2}\int \left(\frac{1}{t^2+\sqrt{2}t+1} - \frac{1}{t^2-\sqrt{2}t+1}\right) dt$$

$$=\sqrt{2}\int \frac{dt}{\left(t+\dfrac{1}{\sqrt{2}}\right)^2+\dfrac{1}{2}} - \sqrt{2}\int \frac{dt}{\left(t-\dfrac{1}{\sqrt{2}}\right)^2+\dfrac{1}{2}}$$

$$=2\arctan(\sqrt{2}t+1) - 2\arctan(\sqrt{2}t-1) + C$$

$$=2\arctan\left(\sqrt[4]{4\frac{a-x}{x}}+1\right) - 2\arctan\left(\sqrt[4]{4\frac{a-x}{x}}-1\right) + C.$$

Exercise 3.6.3 (11)

Using the same change of variable as in Example 3.5.23, we get

$$\int \frac{dx}{2\sqrt{x}+\sqrt{x+1}+1} = \int \frac{\dfrac{y}{2}\left(1-\dfrac{1}{y^4}\right)dy}{y-\dfrac{1}{y}+\dfrac{1}{2}\left(y+\dfrac{1}{y}\right)+1} = \int \frac{(y^4-1)dy}{y^2(3y-1)(y+1)}$$

$$= \int \frac{1}{3}\left(1-\frac{2y^2-3y+3}{y^2(3y-1)}\right)dy = \int \left(\frac{1}{3}+\frac{2y+1}{y^2}-\frac{20}{3(3y-1)}\right)dy$$

$$= \frac{1}{3}y + 2\log y - \frac{1}{y} - \frac{20}{9}\log|3y-1| + C$$

$$= \frac{2}{3}(2\sqrt{x}-\sqrt{x+1}) + 2\log(\sqrt{x}+\sqrt{x+1}) - \frac{20}{9}\log\left|\sqrt{x}+\sqrt{x+1}-\frac{1}{3}\right| + C.$$

Exercise 3.6.7 (1)

Let $t = \tan\dfrac{x}{2}$. Then $\cos x = \dfrac{1-t^2}{1+t^2}$, $dx = \dfrac{2dt}{1+t^2}$, and

$$\int \frac{1-r^2}{1-2r\cos x+r^2}dx = \int \frac{1-r^2}{1-2r\dfrac{1-t^2}{1+t^2}+r^2}\frac{2dt}{1+t^2} = \int \frac{2(1-r^2)dt}{(1+r)^2dt^2+(1-r)^2}$$

$$=2\arctan\frac{1+r}{1-r}t + C = 2\arctan\left(\frac{1+r}{1-r}\tan\frac{x}{2}\right) + C$$

Exercise 3.6.7 (5)

$$\begin{aligned}\int\frac{dx}{\sin x+\tan x}&=\int\frac{dx}{\sin x\left(1+\dfrac{1}{\cos x}\right)}=\int\frac{-d\cos x}{(1-\cos^2 x)\left(1+\dfrac{1}{\cos x}\right)}\\&=\int\frac{tdt}{(t^2-1)(t+1)}=\int\left(\frac{1}{4(t-1)}-\frac{1}{4(t+1)}+\frac{1}{2(t+1)^2}\right)dt\\&=\frac{1}{4}\log\left|\frac{t-1}{t+1}\right|-\frac{1}{2(t+1)}+C=\frac{1}{4}\log\frac{1-\cos x}{1+\cos x}-\frac{1}{2(1+\cos x)}+C.\end{aligned}$$

EXERCISE 3.7.1 (2)

The integral is improper at $0^+$ and $1^-$. For $p\neq 1$ and $0<\epsilon<b<1$, we have

$$\int_\epsilon^b\frac{dx}{x(-\log x)^p}=-\int_\epsilon^b\frac{d(-\log x)}{(-\log x)^p}=-\int_{-\log\epsilon}^{-\log b}\frac{dy}{y^p}=\frac{1}{p-1}\left(\frac{1}{(-\log b)^{p-1}}-\frac{1}{(-\log\epsilon)^{p-1}}\right).$$

Since $\lim_{\epsilon\to 0^+}-\log\epsilon=+\infty$ and $\lim_{b\to 1^-}-\log b=0^+$, the limit of right always diverges. So the improper integral diverges.

For $p=1$, we have

$$\int_\epsilon^b\frac{dx}{x(-\log x)^p}=-\int_{-\log\epsilon}^{-\log b}\frac{dy}{y}=-\log(-\log b)+\log(-\log\epsilon).$$

The improper integral also diverges.

EXERCISE 3.7.1 (4)

The integral is improper at $-\infty$. For any $b>0$, we have

$$\int_{-b}^0 a^x dx=\begin{cases}\dfrac{1-a^{-b}}{\log a} & \text{if } a\neq 1\\ b & \text{if } a=1\end{cases}.$$

The limit as $b\to+\infty$ exists if and only if $a>1$. Thus the improper integral converges if and only if $a>1$, and $\displaystyle\int_{-\infty}^0 a^x dx=\frac{1}{\log a}$.

EXERCISE 3.7.1 (8)

The integral is improper at $\frac{\pi}{2}^-$. We have

$$\int_0^{\frac{\pi}{2}}\tan x dx=\lim_{a\to\frac{\pi}{2}^-}\int_0^a\tan x dx=\lim_{a\to\frac{\pi}{2}^-}-\log\cos a=-\infty.$$

The improper integral diverges.

EXERCISE 3.7.2 (8)

The integral is improper at $9^-$. We have $\int_1^{9-\epsilon} \frac{dx}{\sqrt[3]{x-9}} = \frac{3}{2}\left(\epsilon^{\frac{2}{3}} - 8^{\frac{2}{3}}\right)$. As $\epsilon \to 0^+$, the integral converges to $\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = -6$.

Exercise 3.7.2 (16,17)

The integrals are improper at $+\infty$. By $|e^{-ax}\cos bx| \le e^{-ax}$, $|e^{-ax}\sin bx| \le e^{-ax}$, the convergence of $\int_0^{+\infty} e^{-ax}dx$ and the comparison test, we see that both integrals converge. Then we may apply the integration by parts to the improper integrals. Let

$$I = \int_0^{+\infty} e^{-ax}\cos bx dx, \quad J = \int_0^{+\infty} e^{-ax}\sin bx dx.$$

We have

$$I = -a^{-1}\int_0^{+\infty} \cos bx de^{-ax} = -a^{-1}e^{-ax}\cos bx|_0^{+\infty} - a^{-1}b\int_0^{+\infty} e^{-ax}\sin bx dx = a^{-1} - a^{-1}bJ,$$
$$J = -a^{-1}\int_0^{+\infty} \sin bx de^{-ax} = -a^{-1}e^{-ax}\sin bx|_0^{+\infty} + a^{-1}b\int_0^{+\infty} e^{-ax}\cos bx dx = a^{-1}bI.$$

Solving the system, we get $I = \frac{a}{a^2+b^2}$, $J = \frac{b}{a^2+b^2}$.

Exercise 3.7.5

$F(x) = \int_a^x f(t)dt$ is increasing on $[a, +\infty)$. Recall that $\lim_{x\to+\infty} F(x)$ exists $\Leftrightarrow$ $F(x)$ is bounded on $[a, +\infty)$.

It is similar for the case of the integral $\int_a^x f(t)dt$ is improper at $a^+$. Here consider $F(x) = \int_x^b f(t)dt$, which is a decreasing function.

Exercise 3.7.6

By $|f(x)g(x)| \le \frac{1}{2}(f(x)^2 + g(x)^2)$ and $(f(x)+g(x))^2 \le 2(f(x)^2 + g(x)^2)$ and the comparison test, if $\int_a^{+\infty} f(x)^2dx$ and $\int_a^{+\infty} g(x)^2dx$ converge, then $\int_a^{+\infty} f(x)g(x)dx$ and $\int_a^{+\infty}(f(x) + g(x))^2dx$ converge.

Exercise 3.7.7 (1)

The integral $\int_2^{+\infty} \frac{dx}{x^p(\log x)^q}$ is improper at $+\infty$.

For $p < 1$, the limit $\lim_{x\to+\infty} \frac{x^p(\log x)^q}{x} = \lim_{x\to+\infty} \frac{(\log x)^q}{x^{1-p}} = 0$ implies $\frac{1}{x^p(\log x)^q} \ge \frac{1}{x} > 0$ for big $x$. By the divergence of $\int_2^{+\infty} \frac{dx}{x}$, the integral diverges.

For $p = 1$, take $y = \log x$. We have $\int_2^{+\infty} \frac{dx}{x(\log x)^q} = \int_{\log 2}^{+\infty} \frac{dy}{y^q}$, which converges if and only if $q > 1$.

For $p > 1$, pick $r$ satisfying $p > r > 1$. Then $\lim_{x\to+\infty} \frac{x^r}{x^p(\log x)^q} = \lim_{x\to+\infty} \frac{(\log x)^{-q}}{x^{p-r}} = 0$ implies $0 < \frac{1}{x^p(\log x)^q} \leq \frac{1}{x^r}$ for big $x$. By the convergence of $\int_0^{+\infty} \frac{dx}{x^r}$, the integral converges.

In conclusion, the integral converges if and only if $p > 1$, or $p = 1$ and $q > 1$.

Exercise 3.7.7 (8)

The integral $\int_0^1 \frac{dx}{x^p(1-x^q)^r}, q > 0$ is improper at 0 if $p > 0$ and 1 if $r > 0$.

Then from the limit $\lim_{x\to 0} \frac{x^t}{x^p(1-x^q)^r} = 0$ when $p < t$, we have that $\int_0^1 \frac{dx}{x^p(1-x^q)^r}$ converges when $p < 1$ due to the convergence of $\int_0^1 \frac{1}{x^t}dx, t < 1$. While $\int_0^1 \frac{dx}{x^p(1-x^q)^r}$ diverges when $p \geq 1$.

As for $\int_0^1 \frac{dx}{(1-x^q)^r} =_{y=1-x^q} \frac{1}{q}\int_0^1 \frac{(1-y)^{\frac{1}{q}-1}}{y^r}dy = \frac{1}{q}\int_0^{\frac{1}{2}} \frac{(1-y)^{\frac{1}{q}-1}}{y^r}dy + \frac{1}{q}\int_{\frac{1}{2}}^1 \frac{(1-y)^{\frac{1}{q}-1}}{y^r}dy$ converges if $r < 1$ and $\frac{1}{q} - 1 > -1$, which is $r < 1$ and $q > 0$.

Exercise 3.7.7 (11)

The integral $\int_1^{+\infty} x^p \log(1+x^q)dx$ is improper at $+\infty$.

If $q > 0$, then $\lim_{x\to+\infty} \frac{x^p \log(1+x^q)}{x^p \log x} = q$. Therefore $\int_1^{+\infty} x^p \log(1+x^q)dx$ converges if and only if $\int_1^{+\infty} x^p \log x dx$ converges. The condition is $p < -1$.

If $q = 0$, then $\int_1^{+\infty} x^p \log(1+x^q)dx = (\log 2)\int_1^{+\infty} x^p dx$ converges if and only if $p < -1$.

If $q < 0$, then $\lim_{x\to+\infty} \frac{x^p \log(1+x^q)}{x^{p+q}} = 1$. Therefore $\int_1^{+\infty} x^p \log(1+x^q)dx$ converges if and only if $\int_1^{+\infty} x^{p+q}dx$ converges. The condition is $p + q < -1$.

We conclude that $\int_1^{+\infty} x^p \log(1+x^q)dx$ converges if and only if $q \geq 0$ and $p < -1$, or $q < 0$ and $p + q < -1$.

Exercise 3.7.8 (2)

The integral is improper at $1^-$. We have

$$\lim_{x\to 1^-} \frac{\frac{x}{\sqrt{x^5 - 2x^2 + 1}}}{\frac{1}{\sqrt{1-x}}} = 1.$$

By the convergence of $\int_0^1 \frac{1}{\sqrt{1-x}}dx$ and the comparison test, the integral converges.

Exercise 3.7.8 (4)

The integral is improper at $1^-$. We have

$$\lim_{x\to 1^-} \frac{\dfrac{x\sin x}{\sqrt{x^5-2x^2+1}}}{\dfrac{1}{\sqrt{1-x}}} = \sin 1.$$

By the convergence of $\int_0^1 \frac{1}{\sqrt{1-x}}dx$ and the comparison test, the integral converges.

Exercise 3.7.9 (2)

The integral is improper at $1^-$, $1^+$, $3^-$, $3^+$. We have

$$\lim_{x\to 1} \frac{\sqrt{|x-1|}}{\sqrt{|x^2-4x+3|}} = \frac{1}{\sqrt{2}}, \quad \lim_{x\to 3} \frac{\sqrt{|x-3|}}{\sqrt{|x^2-4x+3|}} = \frac{1}{\sqrt{2}}.$$

Since $\int_0^1 \frac{dx}{\sqrt{x-1}}$ and $\int_0^1 \frac{dx}{\sqrt{x-3}}$ converges, by the comparison test, $\int_0^{10} \frac{dx}{\sqrt{|x^2-4x+3|}}$ converges.

Exercise 3.7.9 (4)

The integral may be improper at $0^+$ and $1^-$.

We have $\lim_{x\to 0^+} \sqrt{x}[(1-x)^p|\log x|^q] = 0$. By the comparison test and the convergence of $\int_0^1 \frac{dx}{\sqrt{x}}$, $\int_0^{\frac{1}{2}} (1-x)^p|\log x|^q dx$ always converges.

We also have $\lim_{x\to 1^-} \frac{(1-x)^p|\log x|^q}{(1-x)^{p+q}} = 1$. So $\int_{\frac{1}{2}}^1 (1-x)^p|\log x|^q dx$ converges if and only if $\int_0^1 (1-x)^{p+q}dx$ converges. This means $p+q>-1$.

In conclusion, $\int_0^1 (1-x)^p|\log x|^q dx$ converges if and only if $p+q>-1$.

Exercise 3.7.10 (2)

The integral $\int_0^{\frac{\pi}{2}} \frac{dx}{x^p\sin^q x}$ is improper at $0^+$.

As $\lim_{x\to 0^+} \frac{x^{p+q}}{x^p\sin^q x} = 1$, by the comparison test, the integral converges if and only if $\int_0^{\frac{\pi}{2}} \frac{dx}{x^{p+q}}$ converges, which means $p+q<1$.

Exercise 3.7.10 (6)

As $\int_0^{\frac{\pi}{4}} \frac{dx}{|\sin x - \cos x|^p} = \int_0^{\frac{\pi}{4}} \frac{dx}{|\sqrt{2}\sin(x-\frac{\pi}{4})|^p}$, thus might be improper at $\frac{\pi}{4}^-$. From $\lim_{x\to\frac{\pi}{4}^-} \frac{(x-\frac{\pi}{4})^p}{\sin^p(x-\frac{\pi}{4})} = 1$, the integral converges if and only if $\int_0^{\frac{\pi}{4}} \frac{dx}{(x-\frac{\pi}{4})^p}$ converges, which means $p<1$.

Exercise 3.7.10 (12)

As $e^{-\sqrt{x}}\cos bx^2 \le e^{-\sqrt{x}}$, and from the convergence of $\int_0^{+\infty} e^{-\sqrt{x}}dx = 2\int_0^{+\infty} ye^{-y}dy$, we can get $\int_0^{+\infty} e^{-\sqrt{x}}\cos bx^2 dx$ converges.

Exercise 3.7.11

The integral is improper at $+\infty$. For big $x$, we have

$$\frac{1}{\sqrt{x^2+1}} - \frac{1}{x} = \frac{1}{x}\left[\left(1+\frac{1}{x}\right)^{-\frac{1}{2}} - 1\right] = -\frac{1}{2x^2} + o\left(\frac{1}{x^2}\right),$$

and

$$\frac{a}{x+1} - \frac{a}{x} = -\frac{a}{x(x+1)}.$$

Since $\int_1^{+\infty}\left[-\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right]dx$ and $\int_1^{+\infty} -\frac{a}{x(x+1)}dx$ are integrable, the convergence of $\int_1^{+\infty}\left(\frac{1}{\sqrt{x^2+1}}+\frac{a}{x+1}\right)dx$ is the same as the convergence of $\int_1^{+\infty}\left(\frac{1}{x}+\frac{a}{x}\right)dx = \int_1^{+\infty}\frac{1+a}{x}dx$. This happens only when $a=-1$.

Exercise 3.7.12 (2)

The integral may be improper at $0^+$ and $+\infty$.

Suppose $q \ne 0$. Then the change of variable $y = x^q$ gives us

$$\int_0^{+\infty}\frac{\cos x^q}{x^p}dx = \text{sign}(q)\int_0^{+\infty}\frac{\cos y}{y^{\frac{p}{q}}}d(y^{\frac{1}{q}}) = \frac{1}{|q|}\int_0^{+\infty}\frac{\cos y}{y^{\frac{p}{q}-\frac{1}{q}+1}}dy.$$

The right side converges near $0^+$ if and only if $\frac{p}{q}-\frac{1}{q}+1<1$, and converges near $+\infty$ if and only if $\frac{p}{q}-\frac{1}{q}+1>0$ by the Dirichlet Test. Therefore $\int_0^{+\infty}\frac{\sin x^q}{x^p}dx$ converges if and only if $-1<\frac{p}{q}-\frac{1}{q}<0$.

The argument for the absolute value integration is similar. We have

$$\int_0^{+\infty}\frac{|\cos x^q|}{x^p}dx = \text{sign}(q)\int_0^{+\infty}\frac{|\cos y|}{y^{\frac{p}{q}}}d(y^{\frac{1}{q}}) = \frac{1}{|q|}\int_0^{+\infty}\frac{|\cos y|}{y^{\frac{p}{q}-\frac{1}{q}+1}}dy.$$

The right side converges near $0^+$ if and only if $\frac{p}{q} - \frac{1}{q} + 1 < 1$, and converges near $+\infty$ if and only if $\frac{p}{q} - \frac{1}{q} > 0$, which means the integral can not absolutely converge.

Therefore for $q \neq 0$, $\int_0^{+\infty} \frac{\cos x^q}{x^p} dx$ conditionally converges for $-1 < \frac{p}{q} - \frac{1}{q} < 0$, and diverges otherwise.

For $q = 0$, $\int_0^{+\infty} \frac{\cos x^q}{x^p} dx = (\cos 1) \int_0^{+\infty} \frac{1}{x^p} dx$ always diverges.

Exercise 3.7.12 (8)

The integral is improper at $+\infty$. Since $|\sin x| \geq \frac{1}{\sqrt{2}}$ on $[k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4}]$, we have

$$\left|\frac{\sin^3 x}{x}\right| \geq \frac{1}{2\sqrt{2}} \frac{1}{x} \geq \frac{1}{2\sqrt{2}(k+1)\pi}, \text{ for } x \in [k\pi + \frac{\pi}{4}, k\pi + \frac{3\pi}{4}].$$

Then

$$\begin{aligned} \int_0^{n\pi} \left|\frac{\sin^3 x}{x}\right| dx &= \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin^3 x}{x}\right| dx \geq \sum_{k=0}^{n-1} \int_{k\pi+\frac{\pi}{4}}^{k\pi+\frac{3\pi}{4}} \left|\frac{\sin^3 x}{x}\right| dx \\ &\geq \sum_{k=0}^{n-1} \frac{1}{2\sqrt{2}(k+1)\pi} \frac{\pi}{2} = \frac{1}{4\sqrt{2}} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right). \end{aligned}$$

Since the right side diverges to $+\infty$ as $n \to \infty$, we see that $\int_0^{+\infty} \frac{\sin^3 x}{x} dx$ diverges.

Exercise 3.7.14 (1)

The integral is improper at $0^+$. By $y = \log x$, we have $\int_0^1 (\log x)^n dx = \int_{-\infty}^0 y^n e^y dy$, which is improper at $-\infty$. By integration by parts,

$$I_n = \int_{-\infty}^0 y^n e^y dy = \int_{-\infty}^0 y^n de^y = y^n e^y \Big|_{-\infty}^0 - n \int_{-\infty}^0 y^{n-1} e^y dy = y^n e^y \Big|_{-\infty}^0 - n I_{n-1}.$$

Note that since

$$y^n e^y \Big|_{-\infty}^0 = 0^n e^0 - \lim_{b \to -\infty} b^n e^b = 0$$

converges, we see that $I_n$ converges if and only if $I_{n-1}$ converges, and we have

$$I_n = -n I_{n-1}.$$

On the other hand, the improper integral

$$I_0 = \int_{-\infty}^0 e^y dy = e^0 - \lim_{b \to -\infty} e^b = 1$$

converges. By induction, $I_n$ also converges, and

$$I_n = (-n)(-(n-1))(-(n-2)) \cdots (-1) I_0 = (-1)^n n!.$$