Math1024 Answer to Homework 6

Exercise 3.8.1 (2)

By $y = \frac{x^2}{2p}$, the length is

$$\int_0^a \sqrt{1+\left(\frac{x}{p}\right)^2}dx = \frac{1}{p}\int_0^a \sqrt{x^2+p^2}dx = \left(\frac{1}{2p}x\sqrt{x^2+p^2}+\frac{p}{2}\log(\sqrt{x^2+p^2}+x)\right)_0^a$$
$$= \frac{a}{2p}\sqrt{a^2+p^2}+\frac{p}{2}\log(\sqrt{a^2+p^2}+a)-\frac{p}{2}\log p.$$

Exercise 3.8.1 (7)

The length is

$$\int_0^{\frac{\pi}{4}} \sqrt{1+((\log\sec x)')^2}dx = \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}dx = \int_0^{\frac{\pi}{4}} \sec x dx$$
$$= \Big(\log|\sec x+\tan x|\Big)_0^{\frac{\pi}{4}}$$
$$= \log(\sqrt{2}+1).$$

Exercise 3.8.2 (1)

The curve is parametrized by $x = \sin^4 t$, $y = \cos^4 t$, $t \in [0, \frac{\pi}{2}]$, and the length is

$$\int_0^{\frac{\pi}{2}} \sqrt{(4\sin^3 t\cos t)^2+(-4\cos^3 t\sin t)^2}dt = 4\int_0^{\frac{\pi}{2}} \sin t\cos t\sqrt{\sin^4 t+\cos^4 t}dt$$
$$= 2\int_0^{\frac{\pi}{2}} \sin 2t\sqrt{(\sin^2 t+\cos^2 t)^2-\frac{1}{2}\sin^2 2t}dt = \int_0^{\pi} \sin u\sqrt{1-\frac{1}{2}(1-\cos^2 u)}du$$
$$= -\int_1^{-1} \frac{1}{\sqrt{2}}\sqrt{1+z^2}dz = 1+\frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$$

Exercise 3.8.5

The point on the circle is $r(\cos t, \sin t)$. The direction of the tangent line is $(\sin t, -\cos t)$, and from the definition of the involute of circle, the tangent line has length rt. Therefore the point on the curve is

$$(x,y) = r(\cos t,\sin t)+rt(\sin t,-\cos t) = r(\cos t+t\sin t,\sin t-t\cos t).$$

The arc length when unwrapped by half the circle is

$$\int_0^{\pi} \sqrt{x'^2+y'^2}dt = \int_0^{\pi} \sqrt{(rt\cos t)^2+(rt\sin t)^2}dt = \int_0^{\pi} rtdt = \frac{1}{2}r\pi^2.$$

Exercise 3.8.6 (7)

The region is between $x = e^y$ and $x = y^2-2$ for $y \in [-1,1]$. Thus the area is

$$\int_{-1}^1 [e^y-(y^2-2)]dy = e-e^{-1}-\frac{2}{3}+4 = e-e^{-1}+\frac{10}{3}.$$

Exercise 3.8.12 (1)

The ellipse is parameterized as $x = a\cos t,\ y = b\sin t,\ t \in [0, 2\pi]$. The parameterization is counterclockwise, and the area is

$$-\int_0^{2\pi} b\sin t d(a\cos t) = ab\int_0^{2\pi} \sin^2 t dt = ab\frac{1}{2}\int_0^{2\pi}(1-\cos 2t)dt = ab\frac{1}{2}2\pi = \pi ab.$$

Exercise 3.8.12 (6)

The cycloid is parameterized by $x = rt - r\sin t, y = r - r\cos t, t \in [0, 2\pi]$. However, the parameterization is clockwise. So the area is

$$\int_0^{2\pi} r(1-\cos t)d(r(t-\sin t)) = \int_0^{2\pi} r^2(1-\cos t)^2 dt = 3\pi r^2.$$

Exercise 3.8.14 (9)

Take the parametrization $x = a\cos t, y = b\sin t, t \in [0, \pi]$. The area of the surface of revolution is

$$\begin{aligned}
\int_0^{\pi} 2\pi y\sqrt{x'^2 + y'^2}dt &= 2\pi\int_0^{\pi} b\sin t\sqrt{a^2\sin^2 t + b^2\cos^2 t}dt \\
&= -2\pi b\int_0^{\pi}\sqrt{a^2(1-\cos^2 t) + b^2\cos^2 t}d(\cos t) \\
&= 2\pi b\int_{-1}^{1}\sqrt{a^2 + (b^2 - a^2)u^2}du \\
&= 2\pi b\left(\frac{1}{2}u\sqrt{a^2 + (b^2-a^2)u^2}\Big|_{-1}^{1} + \frac{a^2}{2}\int_{-1}^{1}\frac{du}{\sqrt{a^2 + (b^2-a^2)u^2}}\right) \\
&= 2\pi b^2 + \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}}\arcsin\dfrac{\sqrt{a^2-b^2}}{a}, & \text{if } a > b, \\ \dfrac{2}{\sqrt{b^2-a^2}}\log\dfrac{b+\sqrt{b^2-a^2}}{a}, & \text{if } a < b. \end{cases}
\end{aligned}$$

Exercise 3.8.23 (9)

Take the parametrization $x = a\cos t, y = b\sin t, t \in [0, \pi]$. The volume of the solid of revolution is

$$\begin{aligned}
-\int_{t=0}^{t=\pi}\pi y^2 dx &= -\pi\int_0^{\pi}(b\sin t)^2 a d\cos t = -\pi ab^2\int_0^{\pi}(1-\cos^2 t)d\cos t \\
&= -\pi ab^2\int_1^{-1}(1-u^2)du = \frac{4}{3}\pi ab^2.
\end{aligned}$$

Exercise 3.8.24 (5)

The distance from $y^2 = x+1$ to the line $x+y=1$ is $\dfrac{-y^2+1-y+1}{\sqrt{2}}$ cause the region is on the positive side of the line $-x-y+1=0$. Then we choose $(-1,1)$ to be the direction of the line, then the progression of the curve $y^2 = x+1$ in the direction of $(-1,1)$ is $\dfrac{(-1,1)}{\sqrt{2}} \cdot d(y^2-1, y) = \dfrac{1-2y}{\sqrt{2}}dy$. Since $x+y=1$ is just the revolving line, it does not produce volume for the solid of revolution. The volume is

$$\int_{y=-2}^{y=1} \pi \left(\frac{-y^2+1-y+1}{\sqrt{2}} \right)^2 \frac{1-2y}{\sqrt{2}} dy = \frac{81}{10\sqrt{2}}\pi.$$

Exercise 3.8.24 (12)

We parameterize the ellipse by $x = a\cos t, y = b\sin t, t \in [0, 2\pi]$. The distance from the ellipse to the line $bx+ay-2ab=0$ is $-\dfrac{b(a\cos t)+a(b\sin t)-2ab}{\sqrt{a^2+b^2}} = \dfrac{ab}{\sqrt{a^2+b^2}}(2-\cos t-\sin t)$. If we choose $(a,-b)$ to be the direction of $bx+ay-2ab=0$, then the progression of the ellipse in the direction of $(a,-b)$ is $\dfrac{(a,-b)}{\sqrt{a^2+b^2}} \cdot d(a\cos t, b\sin t) = \dfrac{-a^2\sin t - b^2\cos t}{\sqrt{a^2+b^2}}dt$. Then volume of the solid of revolution is

$$\int_{t=0}^{t=2\pi} \pi \left(\frac{ab}{\sqrt{a^2+b^2}}(2-\cos t-\sin t) \right)^2 \frac{-a^2\sin t - b^2\cos t}{\sqrt{a^2+b^2}} dt = \frac{\pi a^2 b^2}{(a^2+b^2)^{\frac{3}{2}}} I.$$

By using the fact that the integral of $\sin nx$ and $\cos nx$ on $[0, 2\pi]$ is always 0, we get

$$\begin{aligned} I &= \int_0^{2\pi} (2-\cos t-\sin t)^2(-a^2\sin t - b^2\cos t)dt \\ &= \int_0^{2\pi} (4-4(\cos t+\sin t)+(\cos t+\sin t)^2)(-a^2\sin t - b^2\cos t)dt \\ &= \int_0^{2\pi} (-4(\cos t+\sin t))(-a^2\sin t - b^2\cos t)dt \\ &= 4\int_0^{2\pi} (a^2\sin^2 t + b^2\cos^2 t + (a^2+b^2)\sin t\cos t)dt \\ &= 2\int_0^{2\pi} (a^2+b^2)dt = 4\pi(a^2+b^2). \end{aligned}$$

Therefore the volume is $\dfrac{4\pi^2 a^2 b^2}{\sqrt{a^2+b^2}}$.

Exercise 3.8.27 (2)

When we fix the value of z, we would get an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{z^2}{c^2}$. The area of the ellipse is $\pi ab\left(1+\dfrac{z^2}{c^2}\right)$. Then the volume of the solid is

$$\int_{-c}^{c} \pi ab\left(1+\frac{z^2}{c^2}\right) dz = \frac{8}{3}\pi abc.$$

Exercise 3.8.27 (4)

[By bounded by three coordinate planes, we really mean inside the first quadrant]

We cut the solid at $z \in [0,1]$ and get a triangle bounded by $x+y=1-z^2$, the x-axis, and the y-axis. The triangle is half of the square of side length $1-z^2$ and has area $(1-z^2)^2$. The volume of the solid is

$$\int_0^1 (1-z^2)^2 dz = \frac{8}{15}.$$

Exercise 3.8.28 (3)

Let a be the distance between the centers of the two cuts. Let r be the radius of the cylinder. Let L be the line that form angle α with the cylinder. Then we cut the cylinder by planes parallel to the cylinder and perpendicular to L. The planes are parameterized by the distance $x \in [-r,r]$ to the center line of the cylinder. The section at distance x is a rectangle of base $2\sqrt{r^2-x^2}$ and height $a - x\cot\alpha$. The area of the rectangle is $2\sqrt{r^2-x^2}(a-x\cot\alpha)$, and the volume is

$$\int_{-r}^{r} 2\sqrt{r^2-x^2}(a-x\cot\alpha)dx = \int_{-r}^{r} 2\sqrt{r^2-x^2}adx = \pi r^2 a.$$

Exercise 3.8.31 (9)

Using the idea in Example 3.8.27, we can get the volume of the solid of revolution which is formed by revolving region bounded by $y=\cos x$ and $y=\sin x$ around $y=1$ over $[0,2\pi]$ (the volume of the revolution is the same over each $[k\pi,(k+2)\pi]$ due to the periodicity of the $\sin x$ and $\cos x$)is

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} 2\pi x[(1-\sin x)-(1-\cos x)]dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 2\pi x[(1-\cos x)-(1-\sin x)]dx \\
&+ \int_{\frac{5\pi}{4}}^{2\pi} 2\pi x[(1-\sin x)-(1-\cos x)]dx \\
&= (\frac{11\sqrt{2}}{2}+4)\pi^2 + \frac{\sqrt{2}}{2}\pi.
\end{aligned}$$

Exercise 3.8.33

The answer for this problem depends on the specific setting you choose, so long as the setting can reveal that two types of methods can lead to the same answer. I will show my setting and corresponding answer as follows.

Since $f(x)$ is defined on $[0,a]$, continuously differentiable and invertible, $f(x)$ is monotonic on $[0,a]$. We have known that $f(a)=0$. Here I let $f(0)=0$, then $f(x)\geqslant 0$ for $x\in[0,a]$. Now let the region surrounded by two coordinate axes and the graph of $f(x)$ be S and revolve it around the $x-$axis, we get a revolution solid and want to calculate its volume V.

By the method of Section 3.8.4,

$$V = \pi \int_0^a f^2(x)dx$$

By the method of Example 3.8.27, I slide the solid into pieces as cylindrical surfaces and add up the area of the surfaces, then get the volume. Specifically, take a point $(x, f(x))$, then the line segment from $(0, f(x))$ to $(x, f(x))$ revolve around the $x-$axis and we get a cylindrical surface. As the point $(x, f(x))$ move on the graph of $f(x)$, we slide the solid into pieces. Note that these cylindrical surface are parametrized by y from 0 to $f(a)$, where $y = f(x)$. For the piece at y, the area is

$$2\pi y \cdot x$$

and

$$V = 2\pi \int_0^{f(a)} xydy = \pi \int_0^{f(a)} xdy^2 = xf^2(x)\Big|_0^a - \pi \int_0^a f^2(x)dx = \pi \int_0^a f^2(x)dx$$

Hence, these two method can lead to the same answer.

Exercise 3.8.34

1. B_n is inside the $\mathbf{R}^n$, described by

$$x_1^2 + x_2^2 + \ldots + x_n^2 \leqslant 1.$$

Consider the hyperplane $x_n = t$, $-1 \leqslant t \leqslant 1$. It cuts the ball B_n into pieces as disks when t moves between -1 and 1. Then

$$\begin{aligned} \beta_n &= \beta_{n-1} \int_{-1}^1 (\sqrt{1-t^2})^{n-1} dt \\ &= 2\beta_{n-1} \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta \end{aligned}$$

Therefore

$$\beta_n = \begin{cases} \dfrac{\pi^k}{k!} & n = 2k, k = 1, 2, \ldots \\ \dfrac{2(2\pi)^k}{(2k+1)!!} & n = 2k+1, k = 0, 1, 2, \ldots \end{cases}$$

2. B_n is the union of spheres

$$x_1^2 + x_2^2 + \ldots + x_n^2 = r^2, 0 \leqslant r \leqslant 1.$$

So just need to add up the area of all these spheres. Then

$$\beta_n = \alpha_{n-1} \int r^{n-1} dr = \frac{\alpha_{n-1}}{n},$$

Exercise 3.9.1 (2)

$x^2 + y^2 = 4.$

Exercise 3.9.1 (6)

$x^2 + y^2 = x + y.$

Exercise 3.9.2 (3)

$r\cos\theta + r\sin\theta = 1.$

Exercise 3.9.2 (4)

$r = \cot\theta\csc\theta.$

Exercise 3.9.3

Let C_1 and C_2 be these two curve respectively. Rotate C_1 clockwise by angle π, then the central symmetric image is the C_2.

Exercise 3.9.5

Rotate the curve by angle α, then the equation becomes

$$r = f(\theta + \alpha).$$

1. Of course here you can see that it is a circle by the equation in Cartesian coordinate. But here we think in the way as indicated by the statement of this problem.

The curve is invariant under rotation by angle 2π. Rotate it by angle $\pi/2$, the equation becomes

$$r = \cos\theta.$$

In the lecture note, you know this is a circle, so the curve above is also a circle.

2. $r = \sin(\theta + \alpha)$, where α is a parameter.

Exercise 3.9.8

By the symmetry of the curve $r = a + \cos\theta$, we only need to consider the case of $a > 0$, the length is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(a+\cos\theta)^2 + \sin^2\theta d\theta} \\ &= 2(a+1)\int_0^{\pi} \sqrt{1 - \frac{4a}{(1+a)^2}\sin^2\theta d\theta} \end{aligned}$$

This is an elliptic integral, which cannot be expressed by elementary functions except for the case of $a = 1$.

Next we calculate the area of the region enclosed by the curve. When $a = 1$, the curve is a circle.

When $a > 1$, the curve doesn't intersect itself. So the area is

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (a+\cos\theta)^2 d\theta \\ &= \left(a^2 + \frac{1}{2}\right)\pi \end{aligned}$$

If $0 < a < 1$, let $\phi = \arccos(-a) \in (\pi/2, \pi)$.

$$\begin{aligned} A &= \int_0^{\phi} (a + \cos\theta)^2 d\theta - \int_{\phi}^{\pi} (a + \cos\theta)^2 d\theta \\ &= (2a^2 + 1)\arccos(-a) - \left(a^2 + \frac{1}{2}\right)\pi - 5a\sqrt{1 - a^2} \end{aligned}$$

Exercise 3.9.11

$$\begin{aligned} A &= \int_0^{\pi} \sqrt{\theta^4 + 4\theta^2} d\theta \\ &= \int_0^{\pi} \theta\sqrt{\theta^2 + 4} d\theta \\ &= \frac{1}{2}\int_4^{\pi^2+4} \sqrt{y} dy \\ &= \frac{1}{3}[(\pi^2 + 4)^{\frac{3}{2}} - 8] \end{aligned}$$

Exercise 3.9.12(7)

By the symmetry of the two curves, only need to consider the case of $0 < c < 1$, then these curve intersect at $\theta = -\pi/4$ and $\theta = \pi/4$. Since $0 < c < 1$, when$\theta = -\pi/4$, $r > 0$. Then

$$\begin{aligned} A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + c\sin\theta)^2 d\theta \\ &= \frac{\pi}{2} + \left(\frac{\pi}{4} - \frac{1}{2}\right)c^2 \end{aligned}$$