Math1024 Answer to Homework 7

EXERCISE 4.1.1

$a_n = \frac{1}{(2n-1)(2n+1)}$.

EXERCISE 4.1.2

$1.\overline{23} = 1 + \frac{23}{100} + \frac{23}{10000} + \frac{23}{1000000} + \ldots = \frac{122}{99}$

$1.\overline{230} = 1 + \frac{230}{1000} + \frac{230}{1000000} + \frac{230}{1000000000} + \ldots = \frac{1229}{999}$

$1.\overline{023} = 1 + \frac{23}{1000} + \frac{23}{1000000} + \frac{23}{1000000000} + \ldots = \frac{1022}{999}$

EXERCISE 4.1.6

Consider the area between $y = x^{n-1}$ and $y = x^n$ on the inteval $[0, 1]$,

$$A = \int_0^1 (x^{n-1} - x^n)dx = \frac{1}{n(n+1)}$$

So

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \int_0^1 \sum_{n=1}^{\infty} (x^{n-1} - x^n)dx \\ &= \int_0^1 \left(\frac{1}{1-x} - \frac{x}{1-x} \right) dx = 1\end{aligned}$$

EXERCISE 4.1.7(3)

$$\begin{aligned}S_n &= \sum_{k=0}^{n} \frac{1}{d} \left(\frac{1}{a+kd} - \frac{1}{a+(k+1)d} \right) \\ &= \frac{1}{d} \left(\frac{1}{a} - \frac{1}{a+(n+1)d} \right)\end{aligned}$$

When $n \to \infty$, $S_n = 1/ad$.

EXERCISE 4.1.8

For $n \geqslant 2$, let

$$a_n = \frac{x_n}{(1+x_1)+\ldots+(1+x_n)} = \frac{1}{(1+x_1)+\ldots+(1+x_{n-1})} - \frac{1}{(1+x_1)+\ldots+(1+x_n)}.$$

So the sum of the series is

$$\sum_{1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[1 - \frac{1}{(1+x_1)+\ldots+(1+x_n)} \right].$$

Here S_n is an increasing sequence with an upper bound 1, so the limit exists.

Exercise 4.1.9

$\forall n > 0$, $a_n > 0$, divide $a_{n+1} = a_n + a_{n-1}$ by $a_{n+1}a_na_{n-1}$, then get the first equation.

When n is sufficiently large, $a_n > n$, so $\lim_{n\to\infty} a_n = \infty$. Then by the first equation proved above

$$\sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = \lim_{n\to\infty} \left(\frac{1}{a_0a_1} - \frac{1}{a_na_{n+1}} \right) = 1.$$

The first equation can be transformed into

$$\frac{a_n}{a_{n-1}a_{n+1}} = \frac{1}{a_{n-1}} - \frac{1}{a_{n+1}}.$$

So

$$\sum_{n=1}^{\infty} \frac{a_n}{a_{n-1}a_{n+1}} = \lim_{n\to\infty} \left(\frac{1}{a_0} + \frac{1}{a_1} - \frac{1}{a_n} - \frac{1}{a_{n+1}} \right) = 2.$$

Exercise 4.1.11

1. The series is divergent. Let's check it by Cauchy criterion.

Select $0 < \epsilon < \frac{1}{6}$,

$$\begin{aligned} |S_{6n} - S_{3n}| &= \left| \frac{1}{3n+1} + \frac{1}{3n+2} - \frac{1}{3n+3} + \ldots + \frac{1}{6n-2} + \frac{1}{6n-1} - \frac{1}{6n} \right| \\ &> \frac{1}{3n+3} + \frac{1}{3n+3} - \frac{1}{3n+3} + \ldots + \frac{1}{6n} + \frac{1}{6n} - \frac{1}{6n} \\ &= \frac{1}{3n+3} + \ldots + \frac{1}{6n} = \frac{1}{3} \left(\frac{1}{n+1} + \ldots + \frac{1}{2n} \right) \\ &> \frac{1}{3} \left(\frac{1}{2n} + \ldots + \frac{1}{2n} \right) = \frac{1}{6} = \epsilon. \end{aligned}$$

Hence, the series is divergent.

2. Consider partial sum S_{3n}, which is also the partial of $\sum \left(\frac{1}{3n+1} + \frac{1}{3n+2} - \frac{2}{3n+3} \right)$. Let's show the series is convergent.

$$\begin{aligned} \frac{1}{3n+1} + \frac{1}{3n+2} - \frac{2}{3n+3} &< \frac{2}{3n+1} - \frac{2}{3n+3} \\ &< \frac{2}{3n+1} - \frac{2}{3n+4} \end{aligned}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{2}{3n+1} - \frac{2}{3n+4} \right) = 2$$

so $\sum \left(\frac{2}{3n+1} - \frac{2}{3n+4} \right)$ converges and $\lim_{n\to\infty} S_{3n}$ exists.

Since

$$S_{3n+1} = S_{3n} + \frac{1}{3n+1}, S_{3n+2} = S_{3n+1} - \frac{2}{3n+3}$$

Then

$$\lim_{n\to\infty} S_{3n+2} = \lim_{n\to\infty} S_{3n+1} = \lim_{n\to\infty} S_{3n}$$

Hence, the series converges.

3. $\forall n > 1$,

$$\frac{1}{n^n} < \frac{1}{n^2} < \frac{1}{n(n-1)}$$

so the series converges.

4.

$$\frac{1}{\sqrt{2n(2n-1)}} > \frac{1}{2n}$$

so the series diverges.

Exercise 4.1.13

Since $|(\cos x)^{(n)}| \leqslant 1$ and $|(\sin x)^{(n)}| \leqslant 1$, the remainder term $R_n \to 0$ when $n \to \infty$.

Exercise 4.1.14

For $\sum r^n$, the partial sum is

$$S_n = \frac{1-r^n}{1-r} < \frac{1}{1-r}$$

For the second series, the partial sum is

$$\begin{aligned}
S_n &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \\
&< S_n = 2 + \frac{1}{2} + \frac{1}{2\cdots 3} + \frac{1}{3\cdots 4} + \ldots + \frac{1}{(n-1)\cdots n} \\
&= 3 - \frac{1}{n} < 3
\end{aligned}$$

Exercise 4.2.2

Denote the partial sum of $\sum(-1)^n \frac{1}{\sqrt{n}}$ as S_n, the partial sum of $\sum \frac{1}{\sqrt{n}}$ as A_n. Then

$$\begin{aligned}
S_{2n} &= \left(1 + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{2n-1}}\right) - \left(\frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{2n}}\right) \\
&= \left(1 + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{2n-1}}\right) - \frac{1}{\sqrt{2}}\left(1 + \ldots + \frac{1}{\sqrt{n}}\right) \\
&= A_{2n} - \frac{1}{\sqrt{2}}A_n - \frac{1}{\sqrt{2}}A_n \\
&= A_{2n} - \sqrt{2}A_n
\end{aligned}$$

We have known that

$$A_n = \int_1^n \frac{1}{\sqrt{x}}dx + \gamma + \alpha_n = 2\sqrt{n} - 2 + \gamma + \alpha_n$$

where $\alpha_n \to 0$ when $n \to \infty$. Then

$$S_{2n} = 2\sqrt{2n} - 2 + \gamma + \alpha_n - \sqrt{2}(2\sqrt{n} - 2 + \gamma + \alpha_n) = (1-\sqrt{2})\gamma + (1-\sqrt{2})\alpha_n.$$

Let $n \to \infty$, then $\sum(-1)^n \frac{1}{\sqrt{n}} = (1-\sqrt{2})\gamma$.

Exercise 4.2.3

First to calculate the estimation of the remainder.

$$\int_n^\infty \frac{1}{x^{\frac{3}{2}}}dx = \frac{2}{\sqrt{n}}.$$

To find n such that

$$\frac{2}{\sqrt{n}} < 0.01$$

so $n > 40000$. Just to calculate $\sum_{k=1}^{40001} \frac{1}{n^{\frac{3}{2}}}$.

Exercise 4.2.4

Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$.

By compare test,

$$\lim_{n\to\infty} \frac{a_n^2}{a_n} = \lim_{n\to\infty} a_n = 0$$

shows that $\sum a_n^2$ converges.

The converse statement is not true. For instance, let $a_n = \frac{1}{n}$.

Exercise 4.2.7(3)

$$\lim_{n\to\infty} \frac{|a_n|}{\frac{1}{n^2}} = \frac{1}{2}$$

so it is convergent.

Exercise 4.2.8(2)

The best method is the integral test.

It suffices to consider the improper integral

$$\int_2^\infty \frac{dt}{t^p(\log t)^q}$$

When $p = 1$, you can directly integrate it and know that when $q > 1$, the integral converges.

For the case $p > 1$, you can select $\eta > 0$ such that $p - \eta > 1$,

$$\lim_{t\to\infty} t^{p-\eta}\frac{1}{t^p(\log t)^q} = \lim_{t\to\infty} \frac{1}{t^\eta(\log t)^q} = 0$$

so the improper integral converges. For the case $p < 1$, you can select $\tau > 0$ such that $p+\tau < 1$,

$$\lim_{t\to\infty} t^{p+\tau}\frac{1}{t^p(\log t)^q} = \lim_{t\to\infty}\frac{t^\tau}{(\log t)^q} = \infty$$

so the improper integral diverges.

In conclusion, the original series converges if and only if $p \geqslant 1$ with the conditions: when $p > 1$, q is arbitrary; when $p = 1$, $q > 1$.

Exercise 4.2.8 (4)

As

$$\sum \frac{n^r(\log n)^s}{n^p + (\log n)^q} = \sum \frac{(\log n)^s}{n^{p-r} + \frac{(\log n)^q}{n^r}}$$

Since we have $\lim_{n\to\infty} \dfrac{(\log n)^q}{n^r} = 0$ when $r > 0$.

If $p - r > 1$, then pick some t satisfying $p - r > t > 1$. We have $\dfrac{\frac{(\log n)^s}{n^{p-r}}}{\frac{1}{n^t}} = \dfrac{(\log n)^s}{n^{p-r-t}} \to 0.$ By the convergence of $\sum \dfrac{1}{n^t}$ and the comparison test, $\sum \dfrac{n^r(\log n)^s}{n^p + (\log n)^q}$ converges.

If $p - r < 1$, then pick some t satisfying $p - r < t < 1$. We have $\dfrac{\frac{(\log n)^s}{n^{p-r}}}{\frac{1}{n^t}} = \dfrac{(\log n)^s}{n^{p-r-t}} \to \infty.$ By the divergence of $\sum \dfrac{1}{n^t}$ and the comparison test, $\sum \dfrac{n^r(\log n)^s}{n^p + (\log n)^q}$ diverges.

If $p - r = 1$, series become $\sum \dfrac{(\log n)^s}{n + \frac{(\log n)^q}{n^r}}$ which diverges when $s > 0$ by comparing it to series $\sum \dfrac{1}{n}$.

In conclusion, $\sum \dfrac{n^r(\log n)^s}{n^p + (\log n)^q}$ converges if and only if $p - r > 1$.

Exercise 4.2.10 (1)

$$\sum((n^p + a)^r - (n^p + b)^r) = \sum((n^p + a)^r - (n^p + a + b - a)^r)$$

Exercise 4.2.11 (1)

By $\dfrac{1}{n\sqrt{n}} \le \dfrac{1}{n^2}$ for $n > 4$ and the convergence of $\sum \dfrac{1}{n^2}$, the series converges.

Exercise 4.2.11 (4)

By $\dfrac{1}{(\log n)^n} < \dfrac{1}{2^n}$ for $n > 9$ and the convergence of $\sum \dfrac{1}{2^n}$, the series $\sum \dfrac{1}{(\log n)^n}$ converges.

Exercise 4.2.11 (8)

We have $\dfrac{n^{\log n}}{(\log n)^n} = e^{(\log n)^2 - n\log(\log n)}$. By $(\log n)^2 < n$ and $n\log(\log n) > 2n$, we get $\dfrac{n^{\log n}}{(\log n)^n} < e^{-n}$. By the convergence of $\sum e^{-n}$ and the comparison test, the series $\sum \dfrac{n^{\log n}}{(\log n)^n}$ converges.

Exercise 4.2.12 (2)

By $\dfrac{a_n}{\frac{1}{n^{p+q}}} = \dfrac{\sin\frac{1}{n^q}}{\frac{1}{n^q}} \to 1$, $\sum \dfrac{1}{n^p}\sin\dfrac{1}{n^q}$ converges if and only if $\sum \dfrac{1}{n^{p+q}}$ converges. This means $p+q>1$.

Exercise 4.2.12 (5)

By $\dfrac{a_n}{\frac{1}{n^{2p}}} = \dfrac{\cos\frac{1}{n^p} - 1}{\frac{1}{n^{2p}}} \to -\dfrac{1}{2}$, $\sum(\cos\dfrac{1}{n^p} - 1)$ converges if and only if $\sum \dfrac{1}{n^{2p}}$ converges. This means $2p > 1$.

Exercise 4.2.15 (1)

If $|r| \geq 1$, then a_n does not converge to 0, so that the series diverges.

If $|r| < 1$, then by $n^2 > n$, we have $|r^{n^2}| < |r|^n$. By the convergence of $\sum |r|^n$ and the comparison test, $\sum r^{n^2}$ converges.

Exercise 4.2.15 (4)

If $r \geq 1$, then a_n does not converge to 0, so that the series diverges.

If $0 \leq r < 1$, then $\lim_{x\to+\infty} x^p r^x = 0$ for any p. Therefore by taking $x = \sqrt{n}$ and $p = 6$, we get $\lim n^3 r^{\sqrt{n}} = 0$. This means $\lim \dfrac{nr^{\sqrt{n}}}{\frac{1}{n^2}} = 0$. By the convergence of $\sum \dfrac{1}{n^2}$ and the comparison test, we conclude that $\sum nr^{\sqrt{n}}$ converges.

Exercise 4.2.16 (2)

Since here we have $a > 0$ to make each term of the series exist, and when $a = 0$, the series diverges.

Then $a^{\frac{1}{n}} - 1 = e^{\frac{1}{n}\log a} - 1 = \dfrac{1}{n}\log a + o(\frac{1}{n})$ for big n. By the divergence of $\sum \dfrac{1}{n}$ and the comparison test, we conclude that $\sum(a^{\frac{1}{n}} - 1)$ diverges.

Exercise 4.2.16 (6)

As $(\dfrac{an+b}{cn+d})^n = (\dfrac{a+\frac{b}{n}}{c+\frac{d}{n}})^n$ for big n, we know from the convergence of the geometric series $\sum r^n$ when $|r| < 1$. Thus series $\sum(\dfrac{an+b}{cn+d})^n$ converges when $|\dfrac{a}{c}| < 1$.

Exercise 4.2.16 (13)

As $2\sqrt[n]{a} - \sqrt[n]{b} - \sqrt[n]{c} = (\sqrt[n]{a} - \sqrt[n]{b}) + (\sqrt[n]{a} - \sqrt[n]{c})$, the series would converge when $a = b = c$.

Assuming that $a \neq b$ firstly, we have $\sqrt[n]{a} - \sqrt[n]{b} = e^{\frac{1}{n}\log a} - e^{\frac{1}{n}\log b} = 1 + \dfrac{1}{n}\log a + o(\frac{1}{n}) - 1 - \dfrac{1}{n}\log b - o(\frac{1}{n}) = \dfrac{1}{n}(\log a - \log b) + o(\frac{1}{n})$ for big n. Thus the original series diverges except $a = b = c$.

Exercise 4.2.17 (2)

Since $\sin n\pi = 0$, here we might discuss about $\sum \int_n^{n+1} \dfrac{\sin \pi x}{x^p} dx$ which is just $\int_1^\infty \dfrac{\sin \pi x}{x^p} dx$, an improper integral could be judged to converge by Dirichlet Test when $p > 0$.

Exercise 4.2.18

The partial sum of $\sum (a_n - a_{n+1})$ is bounded, and $\dfrac{1}{n}$ decreases and converges to 0. The claim follows from the Dirichlet test.